Problem Let $x, y \in \mathbb{R}$. Prove that if $y^3 + yx^2 \leqslant x^3 + xy^2$ then $y \leqslant x$.

Solution We want to prove a conditional statement. The equivalent to this statement is its contrapositive:

if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$

$y > x$ is our assumption. $y^3 + yx^2 > x^3 + xy^2$ is the conclusion we want to reach.

We start with the assumption and multiply both sides by $(x^2 + y^2)$:

$$\begin{aligned} y &> x && / \ (x^2+y^2) \\ y(x^2+y^2) &> x(x^2+y^2) \\ yx^2 + y^3 &> x^3 + xy^2 \\ y^3 + yx^2 &> x^3 + xy^2 \end{aligned}$$

Of course the crucial step is multiplying both sides by $(x^2 + y^2)$. Can we do this? We need to make sure of two things. First, that this expression is not less than 0 (otherwise we would need to reverse the inequality sign). Second, that it is not 0 (if it is, then the inequality may not be strict, i.e. it would become $\geqslant$ instead of $>$). Think why in our example $x^2 + y^2$ is positive.