1 Quadratic functions

In this chapter you will learn:

- about the shape and main features of graphs of quadratic functions
- about the uses of different forms of a quadratic function
- how to solve quadratic equations and simultaneous equations
- how to identify the number of solutions of a quadratic equation
- how to use quadratic functions to solve practical problems.

Introductory problem

A small dairy farmer wants to sell a new type of luxury cheese. After a fixed set-up cost of \$250, he can produce the cheese at a cost of \$9 per kilogram. He is able to produce up to 400 kg, but he plans to take advance orders and produce only what he can sell. His market research suggests that the amount he would be able to sell depends on the price in the following way: the amount decreases proportionally with the price; if he charged \$20 per kg he would not sell any, and if the cheese was free he would 'sell' the maximum 400 kg that he could produce. What price per kilogram should the farmer set in order to maximise his profit?

Problems like this, where we have to maximise or minimise a certain quantity, are known as optimisation problems. They are common in economics and business (e.g. minimising costs and maximising profits), biology (e.g. finding the maximum possible size of a population) and physics (e.g. electrons moving to the lowest energy state). The quadratic function is the simplest function with a maximum or minimum point, so it is often used to model such situations. Quadratic functions are also found in many natural phenomena, such as the motion of a projectile or the dependence of power on voltage in an electric circuit.

1A The quadratic form $y = ax^2 + bx + c$

A **quadratic function** has the general form $y = ax^2 + bx + c$ (where $a \neq 0$). In this chapter we will investigate graphs of quadratic functions and, in particular, how features of the graphs relate to the **coefficients** a, b and c.

A function is a rule that tells you what to do with any value you put in. We will study functions in general in chapter 4, but before then you will learn about some particular types of functions.

$x \in \mathbb{R}$ means that x can be any real number.
See Prior Learning section G on the CD-ROM for the meaning of such statements.

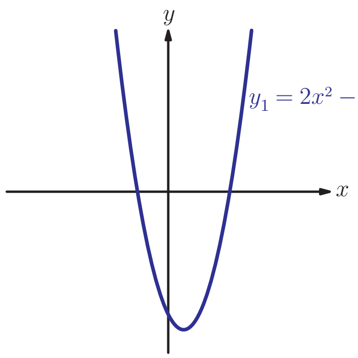
Let us look at two examples of quadratic functions:

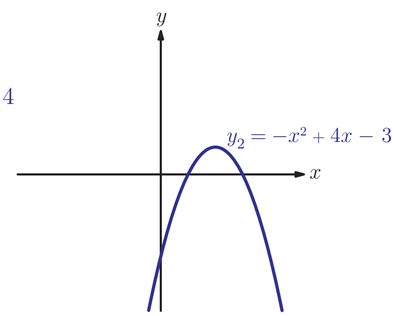
$y_1 = 2x^2 - 2x - 4$ and $y_2 = -x^2 + 4x - 3$ $(x \in \mathbb{R})$

EXAM HINT

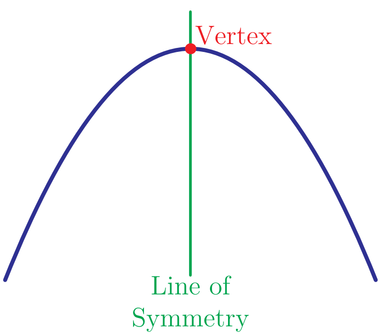
See Calculator Skills sheets 2 and 4 on the CD-ROM for how to sketch and analyse graphs on a graphic display calculator.

You can use your calculator to plot the two graphs:





The word 'quadratic' indicates that the term with the highest power in the equation is x^2. It comes from the Latin *quadratus*, meaning 'square'.



These two graphs have a similar shape, called a **parabola**. A parabola has a single turning point (called its vertex) and a vertical line of symmetry passing through the vertex. The most obvious difference between the two graphs above is that the first one has a minimum point whereas the second has a maximum point. This is due to the different signs of the x^2 term in y_1 and in y_2.

You can use your calculator to find the position of the vertex of a parabola. For the graphs above you should find that the coordinates of the vertices are (0.5, –4.5) and (2, 1); the lines of symmetry therefore have equations $x = 0.5$ and $x = 2$.

KEY POINT 1.1

For a quadratic function $f(x) = ax^2 + bx + c$:

If $a > 0$, $f(x)$ is a *positive* quadratic. The graph has a *minimum* point and goes *up* on both sides.

If $a < 0$, $f(x)$ is a *negative* quadratic. The graph has a *maximum* point and goes *down* on both sides.

The constant coefficient (denoted by c here) gives the position of the y-intercept of the graph, that is, where the curve crosses the y-axis.

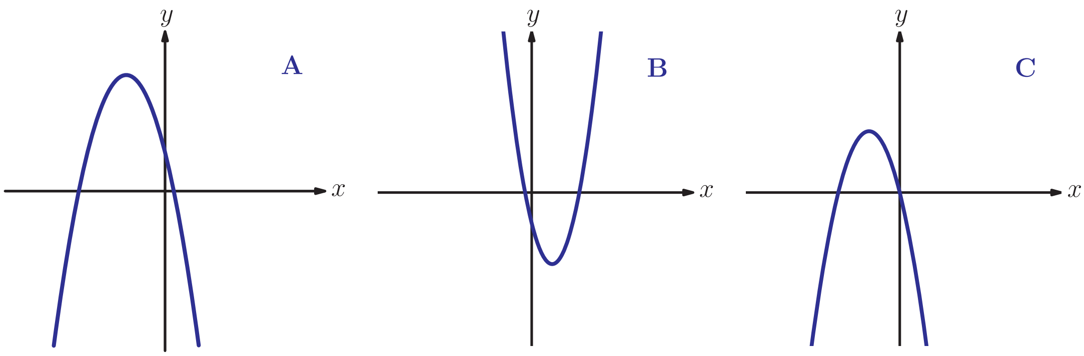
Worked example 1.1

Match each equation to the corresponding graph, explaining your reasons.

(a) $y = 3x^2 - 4x - 1$

(b) $y = -2x^2 - 4x$

(c) $y = -x^2 - 4x + 2$



Graph B is the only positive quadratic.

Graph B shows a positive quadratic, so graph B corresponds to equation (a).

We can distinguish between the other two graphs by their y-intercepts.

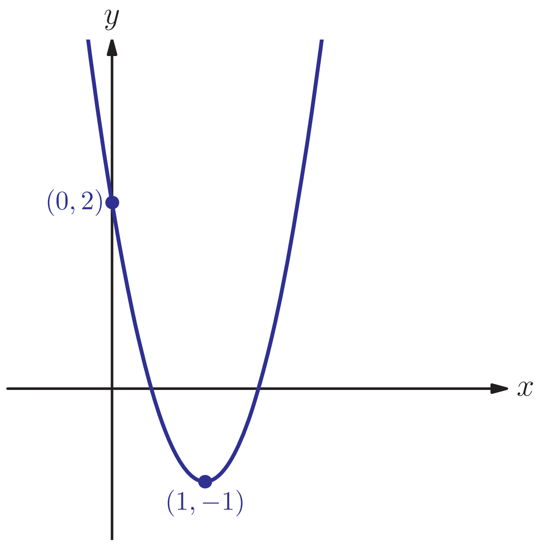
Graph A has a positive y-intercept, so graph A corresponds to equation (c). Graph C corresponds to equation (b).

Although we are mainly concerned with investigating how the features of a graph are determined by the coefficients in the equation, it is often useful to be able to do the reverse. In other words, given a graph, can we find the coefficients? The following example illustrates how to tackle this type of problem.

Finding the equation of a given graph is important in *mathematical modelling*, where often a graph is generated from experimental data and we seek an equation to describe it.

Worked example 1.2

The graph shown below has the equation $y = ax^2 - 6x + c$.



Find the values of c and a.

c is the y-intercept.

The y-intercept of the graph is $(0, 2)$, so $c = 2$

The coordinates of the vertex need to satisfy the equation of the graph.

The vertex is at $x = 1$ and $y = -1$, so

$-1 = a(1)^2 - 6(1) + 2$

$-1 = a - 4$

$a = 3$

The shape of the graph and the position of the y-intercept are the only two features we can read directly from the quadratic equation. We may also be interested in other properties, such as

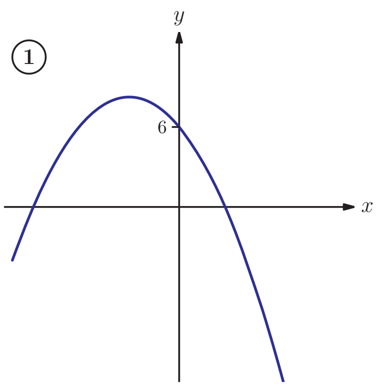
- the position of the line of symmetry
- the coordinates of the vertex
- the x-intercepts.

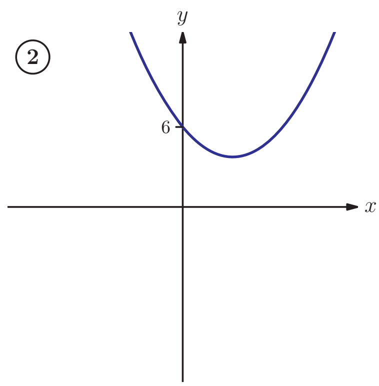
In the next two sections we will see how rewriting the equation of the graph in different forms allows us to identify these features. In some of the questions below you will need to find them using your calculator.

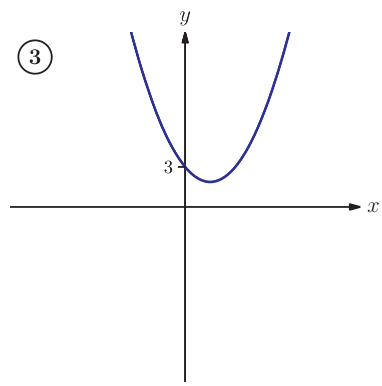
Exercise 1A

1. Match the equations to their corresponding graphs.

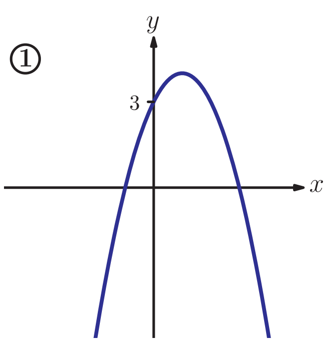
(i) A: $y=-x^2-3x+6$ B: $y=2x^2-3x+3$ C: $y=x^2-3x+6$

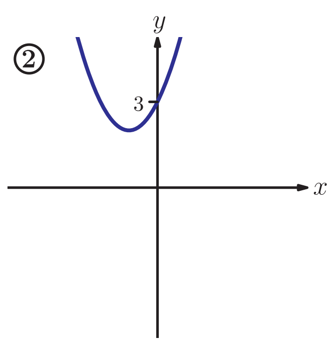


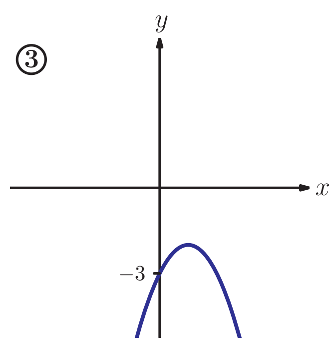




(ii) A: $y=-x^2+2x-3$ B: $y=-x^2+2x+3$ C: $y=x^2+2x+3$







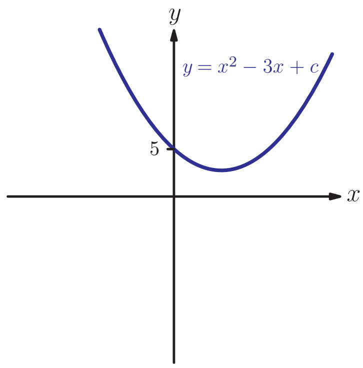
2. Write the following quadratic expressions in the form ax^2+bx+c.
   - (a) (i) $2(x-1)(x+5)$ (ii) $5(x-1)(x-3)$
   - (b) (i) $-4(x+2)(4-x)$ (ii) $-(1-x)(2-x)$
   - (c) (i) $3(x-1)^2+3$ (ii) $4(x+2)^2-5$
   - (d) (i) $-4(x-1)^2-1$ (ii) $-2(x+2)^2-3$

See Prior Learning section K on the CD-ROM if you need to review the technique of expanding brackets.

3. Find the y-intercept of the graph of each equation.
   - (a) (i) $y=2(x-1)(x+3)$ (ii) $y=3(x+1)(x-1)$
   - (b) (i) $y=-3x(x-2)$ (ii) $y=-5x(x-1)$
   - (c) (i) $y=-(x-1)(x+2)$ (ii) $y=-3(x-1)(x+2)$
   - (d) (i) $y=2(x-3)^2+1$ (ii) $y=5(x-1)^2-3$

4. The diagrams show quadratic graphs and their equations. Find the value of c in each case.

(a) (i)



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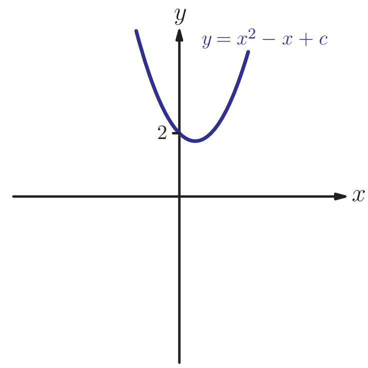
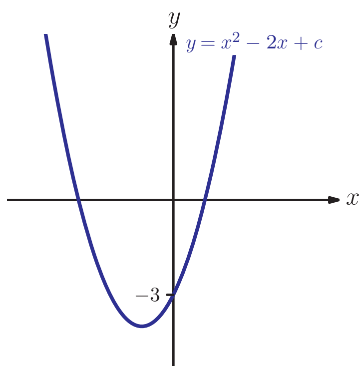


(b) (i)



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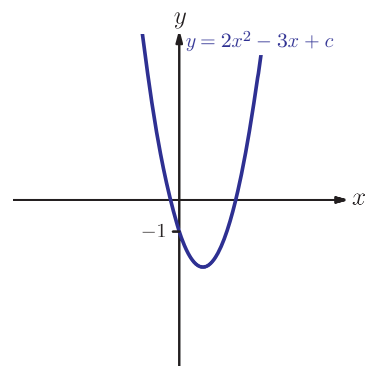
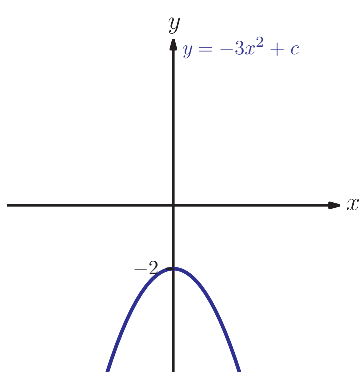


(c) (i)



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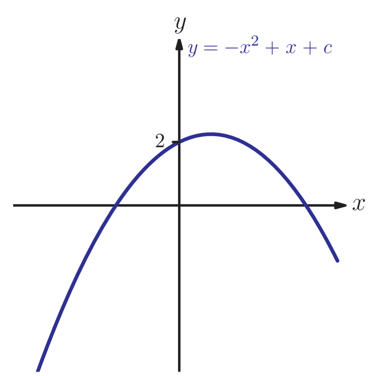
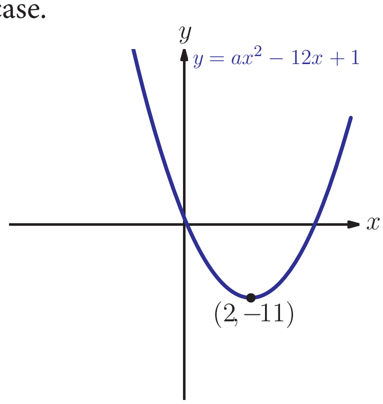


5. The diagrams show quadratic graphs and their equations. Find the value of a in each case.

(a) (i)



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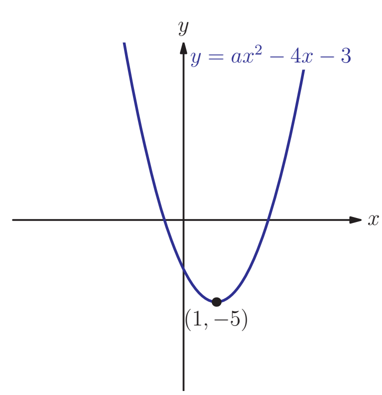
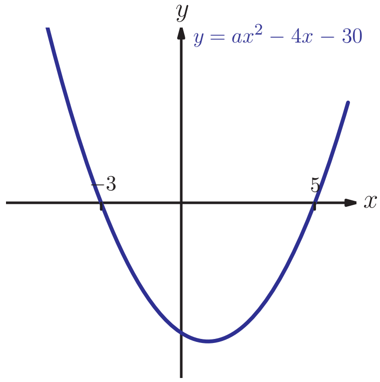


(b) (i)



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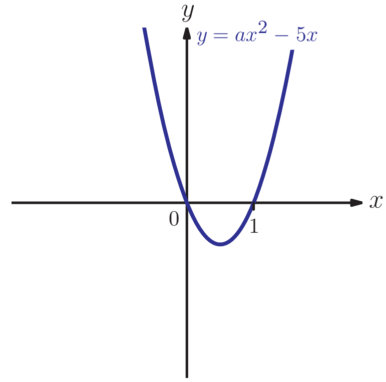
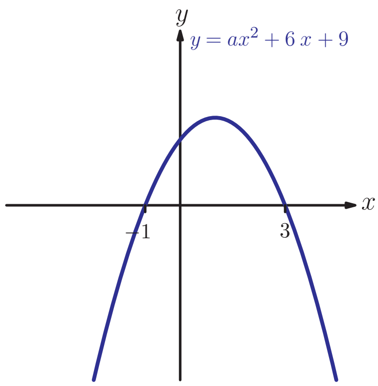


(c) (i)



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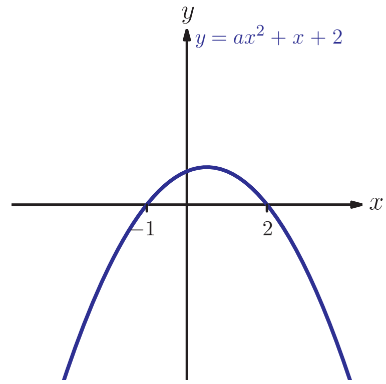
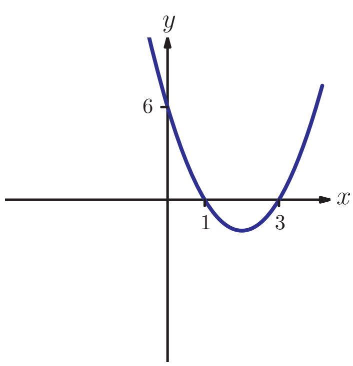


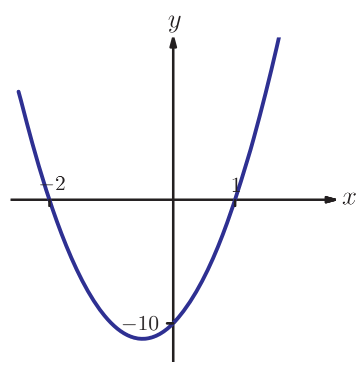
6. The diagrams show graphs of quadratic functions of the form $y = ax^2 + bx + c$. Write down the value of c and then find the values of a and b.

In this question you may need to solve simultaneous equations. See Prior Learning section Q on the CD-ROM if you need a reminder.

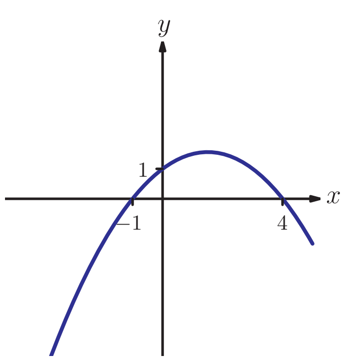
(a) (i)



(ii)



(b) (i)



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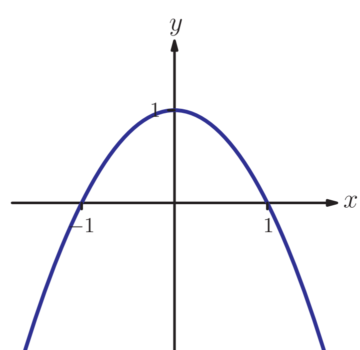


7. For each of the following quadratic functions, find the coordinates of the vertex of the graph.

(a) (i) $y=3x^2-4x+1$ (ii) $y=2x^2+x-4$

(b) (i) $y=-5x^2+2x+10$ (ii) $y=-x^2+4x-5$

8. Find the x-values for which $y=0$.

(a) (i) $y=3x^2-4x-3$ (ii) $y=4x^2+x-3$

(b) (i) $y=4x+2-x^2$ (ii) $y=x+5-2x^2$

(c) (i) $y=-2x^2+12x-18$ (ii) $y=2x^2-6x+4.5$

9. Find the equation of the line of symmetry of the parabolas.

(a) (i) $y=x^2-4x+6$ (ii) $y=2x^2+x+5$

(b) (i) $y=4+3x-2x^2$ (ii) $y=2-x+3x^2$

10. Find the values of x for which

(a) (i) $3x^2+4x-7=15$ (ii) $x^2+x-1=3$

(b) (i) $4x+2=3x^2$ (ii) $3-5x=x^2+2$

1B The completed square form $y=a(x-h)^2+k$

It is often useful to write a quadratic function in a different form.

Worked example 1.3 below shows how to find the values of h and k.

Every quadratic function can be written in the form $y=a(x-h)^2+k$. For example, you can check by multiplying out the brackets that $2x^2-2x-4=2\left(x-\frac{1}{2}\right)^2-\frac{9}{2}$. This second form of a quadratic equation, called the **completed square form**, allows us to find the position of the line of symmetry of the graph and the coordinates of the vertex. It can also be used to solve equations because x only appears once, in the squared term.

We know that squares are always positive, so $(x-h)^2\geq 0$. It follows that for $y=a(x-h)^2+k$:

- if $a>0$, then $a(x-h)^2\geq 0$ and so $y\geq k$; moreover, $y=k$ only when $x=h$
- if $a<0$, then $a(x-h)^2\leq 0$ and so $y\leq k$; moreover, $y=k$ only when $x=h$.

Hence the completed square form gives the extreme (maximum or minimum) value of the quadratic function, namely k, as

well as the value of x at which that extreme value occurs, h.
The point at which the extreme value occurs is called a **turning point** or **vertex**.

KEY POINT 1.2

> A quadratic function $y = a(x-h)^2 + k$ has turning point (h, k) and line of symmetry $x = h$.
>
> For $a > 0$, $y \geq k$ for all x.
>
> For $a < 0$, $y \leq k$ for all x.

The next example shows how the functions y_1 and y_2 from the previous section can be rearranged into completed square form.

Worked example 1.3

(a) Write $2x^2 - 2x - 4$ in the form $a(x-h)^2 + k$

(b) Hence write down the coordinates of the vertex and the equation of the line of symmetry of the graph $y_1 = 2x^2 - 2x - 4$.

Expand the brackets.
Compare coefficients with the given expression.

(a) $a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k$

Comparing coefficients of x^2: $a = 2$

Comparing coefficients of x: $-2ah = -2$

But $a = 2$:

$\therefore -4h = -2$

$\Leftrightarrow h = \frac{1}{2}$

Comparing constants:

$ah^2 + k = -4$

But $a = 2, h = \frac{1}{2}$:

$\therefore \frac{1}{2} + k = -4$

$\Leftrightarrow k = -\frac{9}{2}$

Extract information from the turning point form (h, k).

(b) Line of symmetry is $x = \frac{1}{2}$ and the vertex is $\left(\frac{1}{2}, -\frac{9}{2}\right)$

We can also use the completed square form to solve equations. This is illustrated in the next example, which also shows you how to deal with negative coefficients.

Worked example 1.4

(a) Write $-x^2 + 4x - 3$ in the form $a(x - h)^2 + k$

(b) Hence solve the equation $y_2 = -8$

Expand the required form and compare coefficients

(a) $a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k$

Comparing the coefficients of x^2: $a = -1$

Comparing the coefficients of x:

$$4 = -2ah$$
$$4 = 2h$$
$$\Leftrightarrow h = 2$$

Comparing constants

$$-3 = ah^2 + k$$
$$-3 = -4 + k$$
$$\Leftrightarrow k = 1$$

Therefore $-x^2 + 4x - 3 = -(x-2)^2 + 1$

Isolate the term containing x

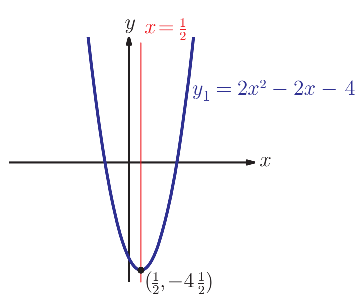
(b) $y_2 = -(x-2)^2 + 1$

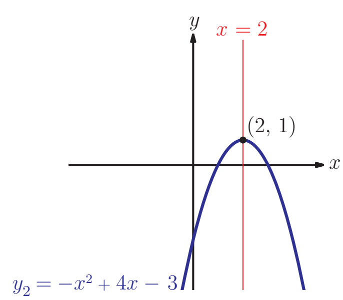
$$-(x-2)^2 + 1 = -8$$
$$\Leftrightarrow -(x-2)^2 = -9$$
$$\Leftrightarrow (x-2)^2 = 9$$
$$\Leftrightarrow x - 2 = \pm 3$$
$$\therefore x = 5 \text{ or } x = -1$$

EXAM HINT

The line of symmetry (and the x-coordinate of the vertex) can also be found using the formula $x = -\dfrac{b}{2a}$, which is given in the Formula booklet. We will explain in section 1D where this formula comes from.

We can now label the lines of symmetry and the coordinates of the turning points on the graphs of y_1 and y_2.

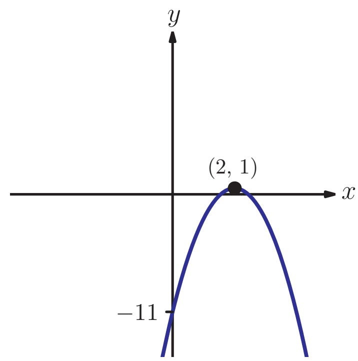




We can also reverse the process to find the equation of a given graph. If the coordinates of the maximum or minimum point are apparent from the graph, then it is easiest to write down the equation in completed square form.

Worked example 1.5

Find the quadratic relationship that fits this graph.



Since we are given information about the maximum point, we use the completed square form.

Maximum point is at (2, 1), so the function must be of the form $y = a(x-2)^2 + 1$

Use the remaining information to find *a*.

The y-intercept is at (0, −11). According to our function, when $x = 0$,

$$y = a(0-2)^2 + 1$$
$$= 4a + 1$$

Therefore

$$4a + 1 = -11$$
$$\Leftrightarrow \quad a = -3$$

Answer the question.

So the relationship is $y = -3(x-2)^2 + 1$

Exercise 1B

1. Write down the coordinates of the vertex of each of the following quadratic functions.

(a) (i) $y = (x-3)^2 + 4$ (ii) $y = (x-5)^2 + 1$

(b) (i) $y = 2(x-2)^2 - 1$ (ii) $y = 3(x-1)^2 - 5$

(c) (i) $y = (x+1)^2 + 3$ (ii) $y = (x+7)^2 - 3$

(d) (i) $y = -5(x+2)^2 - 4$ (ii) $y = -(x+1)^2 + 5$

2. Write the following expressions in the form $a(x-h)^2+k$.

(a) (i) x^2-6x+4 (ii) $x^2-10x+21$

(b) (i) x^2+4x+1 (ii) x^2+6x-3

(c) (i) $2x^2-12x+5$ (ii) $3x^2+6x+10$

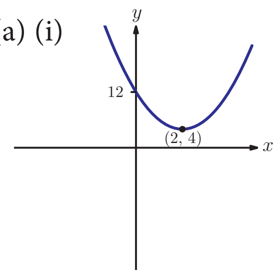
(d) (i) $-x^2+2x-5$ (ii) $-x^2-4x+1$

(e) (i) x^2+3x+1 (ii) $x^2-5x+10$

(f) (i) $2x^2+6x+15$ (ii) $2x^2-5x-1$

3. Find the equation of each graph in the form $y=a(x-h)^2+k$.

(a) (i)



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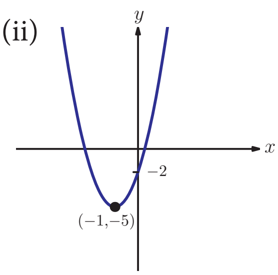
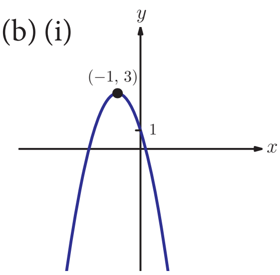


(b) (i)



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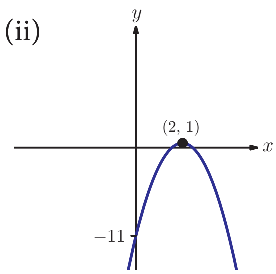


4. You are given that $y=x^2-6x+11$.

(a) Write y in the form $(x - h) + k$.

(b) Find the minimum value of y. *[3 marks]*

5. The curve $y=a(x+b)^2+c$ has a minimum point at (3, 6) and passes through the point (1, 14).

(a) Write down the values of b and c.

(b) Find the value of a. *[4 marks]*

6. (a) Write $2x^2+4x-1$ in the form $a(x-h)^2+k$.

(b) Hence write down the equation of the line of symmetry of the graph $y=2x^2+4x-1$.

(c) Find the exact solutions of the equation $2x^2+4x-1=0$.

[8 marks]

EXAM HINT

When you are asked to find *exact solutions*, leave your answer with roots and fractions rather than evaluating the decimal equivalent; this means that you should not use a calculator to find the answer (although you can still use it to check your answer).

1C The factorised form $y = a(x-p)(x-q)$

The **factorised form** is especially useful for determining another significant feature of a quadratic function, its **zeros**, defined as those values of x (if any) for which $y = 0$. Graphically, they are the values of x at which the curve crosses the x-axis. They are also called the **roots** of the equation $a(x-p)(x-q) = 0$.

For each of the functions y_1 and y_2 that we studied earlier, there are two zeros, as you can see from their graphs. We may be able to find the values of these zeros by factorising the quadratic function.

See Prior Learning section N on the CD-ROM for how to factorise quadratics.

KEY POINT 1.3

> A quadratic function $y = a(x-p)(x-q)$ has zeros at $x = p$ and $x = q$.

Check that the expression $a(x-p)(x-q)$ does equal zero when $x = p$ or when $x = q$.

Worked example 1.6

Using factorisation, find the zeros of functions y_1 and y_2.

Factorise the quadratic function, taking out the common factor 2 first.

$$\begin{aligned} y_1 &= 2x^2 - 2x - 4 \\ &= 2(x^2 - x - 2) \\ &= 2(x+1)(x-2) \end{aligned}$$

Read off the values of p and q.

The zeros of y_1 are -1 and 2.

Factorise the quadratic function. It helps to take out the factor of -1 first.

$$\begin{aligned} y_2 &= -x^2 + 4x - 3 \\ &= -(x^2 - 4x + 3) \\ &= -(x-1)(x-3) \end{aligned}$$

Read off the values of p and q.

The zeros of y_2 are 1 and 3.

EXAM HINT

The line of symmetry of the graph is half-way between the zeros. You can sometimes use this fact to help you factorise or to check your answer.

In some exam questions you may have to carry out these procedures in reverse. For example, you could be given information about the vertex or zeros of a quadratic function and asked to find the coefficients.

Worked example 1.7

A quadratic function has vertex at $(-1,-5)$, and one of its zeros is 4. Find the equation of the function in the form $y = ax^2 + bx + c$.

If we can find the other zero, then we can write down the factorised form of the function. Use the fact that the zeros are at an equal distance from the line of symmetry, on either side of it.

Line of symmetry is $x = -1$ and one zero is 4, so the other zero is -6.

Write down the factorised form; don't forget a.

$$y = a(x-4)(x+6)$$

We need to find a. To do this, we use the information about the vertex.

Vertex is at $x = -1$ and $y = -5$, so

$$-5 = a(-1-4)(-1+6)$$

$$\Leftrightarrow \quad -5 = -25a$$

$$\Leftrightarrow \quad a = \frac{1}{5}$$

Hence $y = \frac{1}{5}(x-4)(x+6)$

To put the equation in the required form, we need to multiply out the brackets.

$$y = \frac{1}{5}(x^2 + 2x - 24)$$

$$= \frac{1}{5}x^2 + \frac{2}{5}x - \frac{24}{5}$$

Exercise 1C

1. Write down the zeros of the following quadratic functions.

(a) (i) $(x+2)(x-3)$ (ii) $(x-5)(x+1)$

(b) (i) $x(x+3)$ (ii) $2x(x-2)$

(c) (i) $2(5-x)(2+x)$ (ii) $4(1-x)(1+x)$

(d) (i) $(2x-1)(3x+5)$ (ii) $(4x-3)(3x+1)$

2. By factorising, find the zeros of the following quadratic functions.

(a) (i) x^2+4x-5 (ii) x^2-6x+8

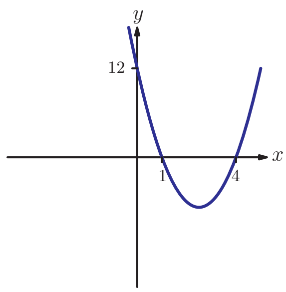
(b) (i) $2x^2+x-6$ (ii) $3x^2-x-10$

(c) (i) $6x^2-7x-3$ (ii) $8x^2-6x-5$

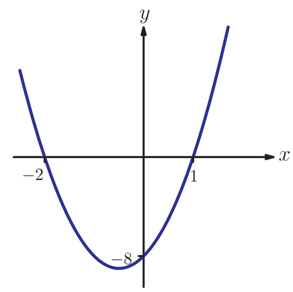
(d) (i) $12-x-x^2$ (ii) $10-3x-x^2$

3. For each graph, find its equation in the form $y=ax^2+bx+c$.

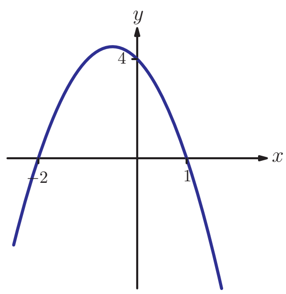
(a) (i)



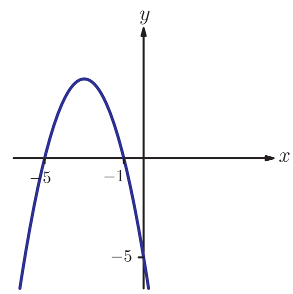
(ii)



(b) (i)



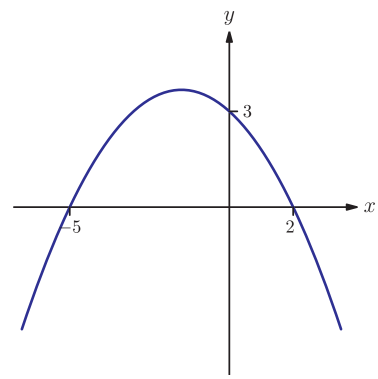
(ii)



4. (a) Factorise $2x^2+5x-12$.

(b) Hence write down the coordinates of the points where the graph of $y=2x^2+5x-12$ crosses the x-axis. *[5 marks]*

5. This graph has equation $y=ax^2+bx+c$. Find the values of a, b and c. *[5 marks]*



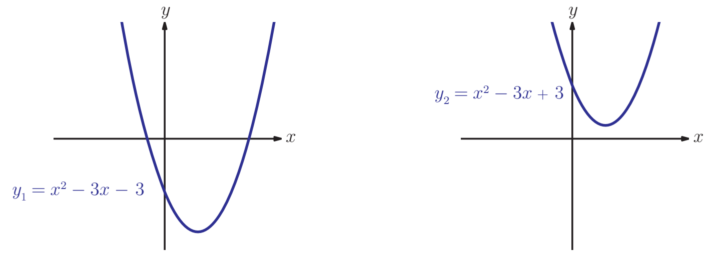
1D The quadratic formula and the discriminant

It is not always possible to find the zeros of a quadratic function by factorising. For example, try factorising $y_1=x^2-3x-3$ and $y_2=x^2-3x+3$. It seems that neither of these equations can

EXAM HINT

Remember that finding zeros of a quadratic function is the same as solving a quadratic equation.

be factorised, but their graphs reveal that the first one has two zeros while the second has none.



Instead of factorising, we can use the following formula to find zeros of a quadratic function.

KEY POINT 1.4

The zeros of the quadratic function $f(x) = ax^2 + bx + c$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$$

See Fill-in Proof sheet 1 'Proving the quadratic formula' on the CD-ROM for how to prove this formula.

EXAM HINT

Don't spend too long trying to factorise a quadratic – use the formula if you are asked to find exact solutions, and use a calculator (graph or equation solver) otherwise.

See Calculator Skills sheets 4 and 6 on the CD-ROM for how to do this.

Worked example 1.8

Use the quadratic formula to find the zeros of $x^2 - 5x - 3$.

As it is not obvious how to factorise the quadratic expression, we use the quadratic formula.

Here $a = 1$, $b = -5$, $c = -3$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times (-3)}}{2}$$

$$= \frac{5 \pm \sqrt{37}}{2}$$

The zeros are

$$\frac{5 + \sqrt{37}}{2} = 5.54 \text{ and } \frac{5 - \sqrt{37}}{2} = -0.541 \text{ (3 SF)}$$

Let us examine what happens if we try to apply the quadratic formula to find the zeros of $y_2 = x^2 - 3x + 3$:

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 3}}{2} = \frac{3 \pm \sqrt{-3}}{2}$$

As the square root of a negative number is not a real number, it follows that $x^2 - 3x + 3$ has no real zeros.

Worked example 1.8 is an example of a quadratic function with two zeros, just as the function y_1 on page 16. y_2 is an example of a quadratic function with no real zeros. It is also possible to have a quadratic function with one zero.

Looking more closely at the quadratic formula, we see that it can be separated into two parts:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

You have already met the first term, $\frac{-b}{2a}$: $x = \frac{-b}{2a}$ is the line of symmetry of the parabola.

The second term in the formula involves a root expression: $\sqrt{b^2 - 4ac}$. The expression $b^2 - 4ac$ inside the square root is called the **discriminant** of the quadratic (often symbolised by the Greek letter Δ).

As noted above, the square root of a negative number is not a real number, so if the discriminant is negative, there can be no real zeros of the function.

If the discriminant is zero, the quadratic formula gives $x = -\frac{b}{2a} \pm 0$, so there is only one root, $x = -\frac{b}{2a}$. In this case, the graph is **tangent** to the x-axis (meaning that the graph touches the x-axis rather than crossing it) at a point that lies on the line of symmetry – the vertex, in fact.

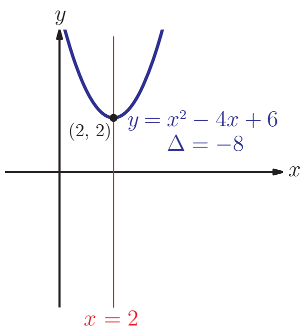
The graphs at the top of page 18 demonstrate the three possible situations when finding the zeros of quadratic functions.

Note that $\frac{\sqrt{\Delta}}{2a}$ is the distance of the zeros from the line of symmetry $x = \frac{-b}{2a}$.

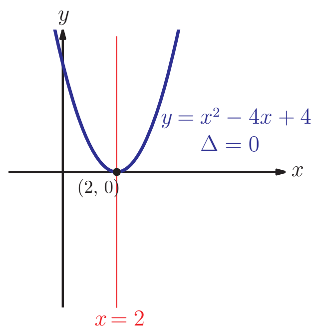
EXAM HINT

In exams you should either give exact answers (such as $\frac{5+\sqrt{37}}{2}$) or round your answers to 3 significant figures, unless you are explicitly told otherwise.
See Prior Learning section B on the CD-ROM for the rules of rounding.

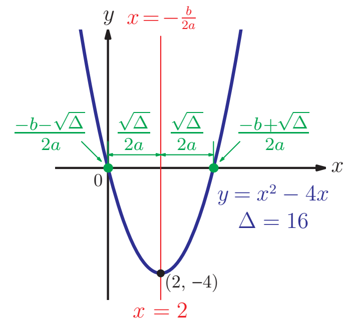
No real zeros



One real zero



Two real zeros



KEY POINT 1.5

For a quadratic function $y = ax^2 + bx + c$, the discriminant is $\Delta = b^2 - 4ac$.

- If $\Delta < 0$, the function has no real zeros.
- If $\Delta = 0$, the function has one (repeated) zero.
- If $\Delta > 0$, the function has two (distinct) real zeros.

The situation $\Delta = 0$ is often said to produce a 'repeated root' or 'equal roots' of $y = ax^2 + bx + c$, because in factorised form the function is $y = a(x-p)(x-p)$, which gives two equal root values p and q. An expression of the form $(x-p)(x-p)$ is also referred to as a 'perfect square'.

Worked example 1.9

Find the exact values of k for which the quadratic equation $kx^2 - (k+2)x + 3 = 0$ has a repeated root.

Repeated root means that $\Delta = b^2 - 4ac = 0$. So we identify a, b and c, and write down the equation $b^2 - 4ac = 0$.

$a = k$, $b = -(k+2)$, $c = 3$

$(-k-2)^2 - 4(k)(3) = 0$

$\Leftrightarrow k^2 + 4k + 4 - 12k = 0$

This is a quadratic equation in k.

$\Rightarrow \quad k^2 - 8k + 4 = 0$

It doesn't look as if we can factorise, so use the quadratic formula.

$$k = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 1 \times 4}}{2}$$

$$= \frac{8 \pm \sqrt{48}}{2}$$

$$= \frac{8 \pm 4\sqrt{3}}{2}$$

$$= 4 \pm 2\sqrt{3}$$

EXAM HINT

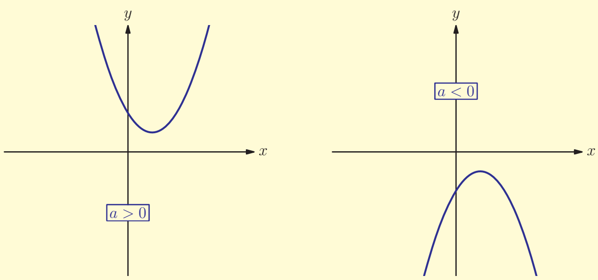
Questions of this type nearly always lead to a quadratic equation for k.

When $\Delta < 0$, the graph does not intersect the x-axis, so must lie entirely above or entirely below it. The two cases are distinguished by the value of a.

KEY POINT 1.6

For a quadratic function $y = ax^2 + bx + c$ with $\Delta < 0$:

- if $a > 0$, then $y > 0$ for all x
- if $a < 0$, then $y < 0$ for all x.



Worked example 1.10

Given the quadratic function $y = -3x^2 + kx - 12$, where $k > 0$, find the values of k such that $y < 0$ for all x.

y is a negative quadratic, so $y < 0$ means that the graph is entirely below the x-axis. This will happen when the quadratic has no real roots.

No real roots means $\Delta < 0$:

$$b^2 - 4ac < 0$$

$$k^2 - 4(-3)(-12) < 0$$

$$k^2 < 144$$

We are told that $k > 0$.

$$\therefore 0 < k < 12$$

Exercise 1D

1. Evaluate the discriminant of each quadratic expression.

(a) (i) $x^2 + 4x - 5$ (ii) $x^2 - 6x - 8$

(b) (i) $2x^2 + x + 6$ (ii) $3x^2 - x + 10$

(c) (i) $3x^2 - 6x + 3$ (ii) $9x^2 - 6x + 1$

(d) (i) $12 - x - x^2$ (ii) $-x^2 - 3x + 10$

2. State the number of zeros of each expression from Question 1.

3. Use the quadratic formula to find the exact solutions of the following equations.

(a) (i) $x^2-3x+1=0$ (ii) $x^2-x-1=0$

(b) (i) $3x^2+x-2=0$ (ii) $2x^2-6x+1=0$

(c) (i) $4+x-3x^2=0$ (ii) $1-x-2x^2=0$

(d) (i) $x^2-3=4x$ (ii) $3-x=2x^2$

4. Find the values of k for which

(a) (i) the equation $2x^2-x+3k=0$ has two distinct real roots

(ii) the equation $3x^2+5x-k=0$ has two distinct real roots

(b) (i) the equation $5x^2-2x+(2k-1)=0$ has equal roots

(ii) the equation $2x^2+3x-(3k+1)=0$ has equal roots

(c) (i) the equation $-x^2+3x+(k-1)=0$ has real roots

(ii) the equation $-2x^2+3x-(2k+1)=0$ has real roots

(d) (i) the equation $3kx^2-3x+2=0$ has no solutions

(ii) the equation $-kx^2+5x+3=0$ has no solutions

(e) (i) the quadratic expression $(k-2)x^2+3x+1$ has a repeated zero

(ii) the quadratic expression $-4x^2+5x+(2k-5)$ has a repeated zero

(f) (i) the graph of $y=x^2-4x+(3k+1)$ is tangent to the x-axis

(ii) the graph of $y=-2kx^2+x-4$ is tangent to the x-axis

(g) (i) the expression $-3x^2+5k$ has no real zeros

(ii) the expression $2kx^2-3$ has no real zeros

5. Find the exact solutions of the equation $3x^2=4x+1$. *[3 marks]*

6. Show that the graph of $y=4x^2+x+\frac{1}{16}$ has its vertex on the x-axis. *[3 marks]*

7. Find the values of parameter m for which the quadratic equation $mx^2-4x+2m=0$ has equal roots. *[5 marks]*

8. Find the exact values of k such that the equation $-3x^2+(2k+1)x-4k=0$ is tangent to the x-axis. *[6 marks]*

9. Find the set of values of k for which the equation $x^2 - 6x + 2k = 0$ has no real solutions. *[5 marks]*

10. Find the range of values of the parameter c such that $2x^2 - 3x + (2c-1) \geq 0$ for all x. *[5 marks]*

11. Find the possible values of m such that $mx^2 + 3x - 4 < 0$ for all x. *[5 marks]*

12. The positive difference between the zeros of the quadratic expression $x^2 + kx + 3$ is $\sqrt{69}$. Find the possible values of k. *[4 marks]*

1E Intersections of graphs and simultaneous equations

Whenever we need to locate an intersection between two graphs, we are solving **simultaneous equations**. This means that we are trying to find values of x and y that satisfy both equations.

See Prior Learning section Q on the CD-ROM for revision of linear simultaneous equations.

You can always find the intersections of two graphs by using a calculator. Remember, however, that a calculator only gives approximate solutions. If exact solutions are required, then we have to use an algebraic method. In many cases the best method is substitution, where we replace every occurrence of one variable in one of the equations by an expression for it derived from the other equation.

EXAM HINT

See Calculator Skills sheet 5 on the CD-ROM for how to find coordinates of intersection points.

Worked example 1.11

Find the coordinates of the points of intersection between the line $y = 2x - 1$ and the parabola $y = x^2 - 3x + 5$.

At intersection points the y-coordinates for the two curves are equal, so we can replace y in the first equation by the expression for y from the second equation.

$$x^2 - 3x + 5 = 2x - 1$$

continued . . .

This rearranges to a quadratic equation. Try to factorise it. In this case, factorisation gives the *x*-coordinates of the intersection points.

$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$
$$\Rightarrow \quad x = 2 \text{ or } 3$$

The corresponding *y*-coordinates can be found by substituting the *x*-values back into one of the original equations (both should give the same answer). Let us pick the first equation as it is simpler.

$y = 2x - 1$

$= 2 \Rightarrow y = 2 \times 2 - 1 = 3$

$= 3 \Rightarrow y = 2 \times 3 - 1 = 5$

The coordinates of the intersection points are (2, 3) and (3, 5).

Sometimes we only want to know how many intersection points there are, rather than to find their actual coordinates. The discriminant can be used to determine the number of intersections.

Worked example 1.12

Find the value of k for which the line with equation $y = x - k$ is tangent to the parabola $y = x^2$.

The line being tangent to the parabola means that the two graphs intersect at only one point. The number of intersections between the line and the parabola will depend on the value of *k*. This makes sense, as varying *k* moves the line up and down, so sometimes it will intersect the parabola and sometimes it won't.

Let us try to find the intersections. At intersection points the *y*-coordinates for the two curves are equal, so we can replace *y* in the second equation by the expression for *y* from the first equation.

Line equation: $y = x - k$

Substitute into the parabola equation $y = x^2$ to get

$$x - k = x^2$$

This is a quadratic equation; write it in the form of quadratic expression equal to zero.

$$x^2 - x + k = 0$$

For this to have only one solution, we need $\Delta = 0$.

One solution $\Rightarrow b^2 - 4ac = 0$

$$(-1)^2 - 4(1)(k) = 0$$
$$1 - 4k = 0$$
$$k = \frac{1}{4}$$

EXAM HINT

Questions about the *number* of intersections are often (though not always) solved using the discriminant.

Worked example 1.12 illustrates how a geometrical problem (intersections of two curves) can be solved by purely algebraic methods. There is a whole branch of mathematics studying such methods, called analytic geometry. It was developed in the 17th century by the French philosopher and mathematician René Descartes. Establishing a link between geometry and algebra was a major step in the development of modern mathematics.

The parabola belongs to a family of quadratic curves called 'conic sections', which also includes the circle, the ellipse and the hyperbola. There are many fascinating and beautiful results concerning conic sections, and a lot of these can be investigated using properties of the quadratic function.

Exercise 1E

1. Find the coordinates of intersection between the given parabola and the given straight line.

(a) (i) $y = x^2 + 2x - 3$ and $y = x - 1$

(ii) $y = x^2 - 4x + 3$ and $y = 2x - 6$

(b) (i) $y = -x^2 + 3x + 9$ and $2x - y = 3$

(ii) $y = x^2 - 2x + 8$ and $x - y = 6$

2. Solve the following simultaneous equations:

(a) (i) $x - 2y = 1,\ 3xy - y^2 = 8$

(ii) $x + 2y = 3,\ y^2 + 2xy + 9 = 0$

(b) (i) $xy = 3,\ x + y = 4$

(ii) $x + y + 8 = 0,\ xy = 15$

(c) (i) $x + y = 5,\ y = x^2 - 2x + 3$

(ii) $x - y = 4,\ y = x^2 + x - 5$

3. Find the coordinates of the points of intersection of the graph of $y = x^2 - 4$ and the line $y = 8 - x$. *[6 marks]*

4. Solve the simultaneous equations

$y = 2x^2 - 3x + 2$ and $3x + 2y = 5$ *[7 marks]*

5. A circle has equation $x^2 - 6x + y^2 - 2y - 8 = 0$.

(a) Show that the x-coordinates of the points of intersection of the circle with the line $y = x - 8$ satisfy the equation $x^2 - 12x + 36 = 0$.

(b) Hence show that the line is tangent to the circle. *[5 marks]*

6. Find the exact values of m for which the line $y = mx + 3$ and the curve with equation $y = 3x^2 - x + 5$ have only one point of intersection. *[6 marks]*

1F Using quadratic functions to solve problems

Quadratic functions are very common in applications of mathematics. Many natural phenomena can be modelled using quadratic functions. For example: the motion of a projectile follows a path which is approximately a parabola; the elastic energy of a particle attached to the end of a spring is proportional to the square of the extension; electric power in a circuit is a quadratic function of the voltage. Properties of quadratic functions are also widely used in optimisation problems, where a certain quantity has to be maximised or minimised. In this section we look at some typical examples.

Worked example 1.13

A rectangle has perimeter 100 cm. What is the largest its area can be?

The area of a rectangle is length × width. Introduce variables so we can write equations.	Let x = length, y = width. Then Area $= xy$
It is impossible to see from this equation alone what the maximum possible value of the area is. However, we can proceed by writing an equation relating x and y, using the known perimeter.	Perimeter $= 2x + 2y = 100$
This means that we can express the area in terms of only one of the variables.	$2y = 100 - 2x$ $\Rightarrow\ y = 50 - x$ $\therefore$ Area $= x(50 - x)$ $= 50x - x^2$

continued . . .

The area function is a negative quadratic, so we know how to find its maximum value (the y-coordinate of the vertex).

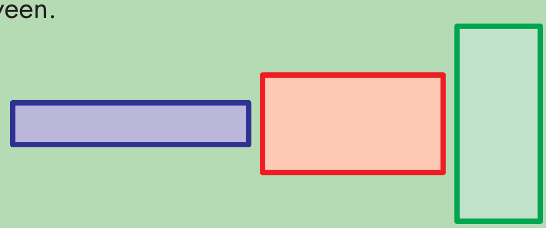
The x-coordinate of the vertex is

$$x = -\frac{b}{2a}$$
$$= -\frac{50}{2(-1)}$$
$$= 25$$

Negative quadratic, so the vertex is a maximum point with $f(25) = 50 \times 25 - 25^2 = 625$

Therefore the maximum area is $625\,\text{cm}^2$.

This is an example of *constrained optimisation*: we are trying to find a best possible solution while keeping some quantity fixed. It is intuitively clear that a short and wide or a tall and thin rectangle will have very small area, so we expect the largest area to occur somewhere in between.

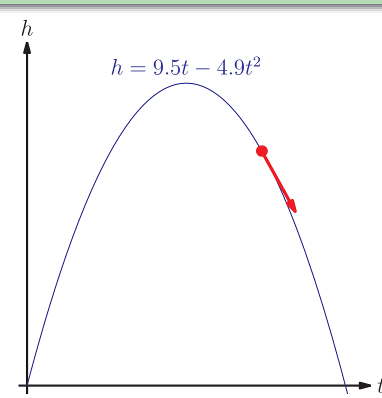


A related problem is finding the minimum possible surface area for an object of fixed volume. Examples of this can be seen in nature: snakes have evolved to be long and thin in order to maximise their surface area – they are cold-blooded reptiles and need to gain as much heat as possible from the sun through their skin; polar bears, who live in the Arctic, avoid losing too much heat through their skin by adopting a rounder shape, which minimises the surface area for a given volume.

You may have noticed that in Worked example 1.13, the rectangle with the largest area is actually a square (with length = width = 25). It turns out that of all planar shapes with a fixed perimeter, the circle has the largest possible area. This so-called 'isoperimetric problem' has several intriguing proofs and many applications.

The next example presents an application to vertical motion under gravity. If we ignore air resistance, then the height above the ground of an object thrown vertically upwards will be a quadratic function of time.

Note that the path of the projectile can be represented by a similar parabola, as we shall see in Worked example 1.15.



Worked example 1.14

A ball is thrown vertically upwards from ground level and moves freely under gravity. The height h of the ball above the ground can be modelled by the equation $h = 9.5t - 4.9t^2$, where t is the time, measured in seconds, after the ball is thrown.

(a) How long does the ball take to return to the ground?
(b) What height does the ball reach?

When the ball returns to the ground, $h = 0$, so we are looking for roots of the quadratic function. In this case, it is easy to factorise the quadratic.

(a) $9.5t - 4.9t^2 = 0$

$\Rightarrow \quad t(9.5 - 4.9t) = 0$

$\Rightarrow \quad t = 0 \text{ or } t = \dfrac{9.5}{4.9} = 1.94 \ (3\text{ SF})$

$t = 0$ is the time when the ball left the ground.

$\therefore$ the ball returns to the ground after 1.94 s.

We are now looking for the maximum point (vertex) of the quadratic function.

(b) Vertex is at

$$t = -\frac{b}{2a}$$

$$= -\frac{9.5}{2 \times (-4.9)}$$

$$= 0.969$$

$$h = 9.5(0.969) - 4.9(0.969)^2$$

$$= 4.60 \ (3\text{ SF})$$

$\therefore$ the maximum height is 4.60 m.

A related problem is to describe the path of a projectile. In the absence of air resistance, this path is a parabola. The following example illustrates how we can use this fact. It also shows how to set out your working when you are using a calculator to analyse graphs.

Worked example 1.15

A ball is projected horizontally from a window 8 m above ground and then moves freely under gravity. The height of the ball above the ground, in metres, is given by $y = 8 - \dfrac{x^2}{2u}$, where x is the horizontal distance from the window and u is the initial speed of the ball (in metres per second).

(a) The ball covers a horizontal distance of 18 m before hitting the ground. Find the value of u.

(b) Find the horizontal distance that the ball covers while it is more than 4 m above ground.

continued . . .

The ball hits the ground when $y = 0$, so we know that $y = 0$ when $x = 18$.

(a) $y = 0$ when $x = 18$:

$$0 = 8 - \frac{18^2}{2u}$$

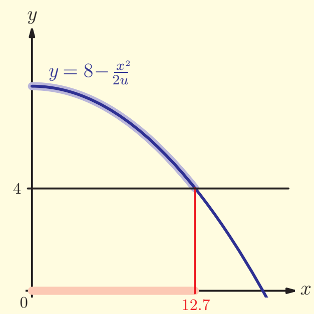
$$\Rightarrow \quad u = \frac{18^2}{16} = 20.25$$

We can draw the graph and find the values of x for which $y > 4$. To do this, we need the intersection of the graph with the horizontal line $y = 4$.

(b)

$$y = 8 - \frac{x^2}{40.5}$$

The graph is



$x = 12.7\,m$ (3 SF) (from GDC)

Exercise 1F

1. The sum of two numbers is 8 and their product is 9.75.
What are the two numbers? *[6 marks]*

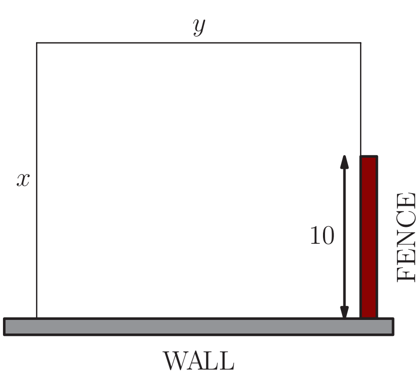
2. A rectangle has perimeter equal to 12 cm. Let x cm be the length of one of the sides. Express the area of the rectangle in terms of x, and hence find the maximum possible area of the rectangle. *[6 marks]*

3. A farmer wishes to fence off a rectangular area adjacent to a wall. There is an existing piece of fence, 10 m in length, that is perpendicular to the wall, as shown in the diagram.

The length of the new fencing is to be 200 m. Let x and y be the dimensions of the enclosure.

(a) Write down an expression for the area of the enclosure in terms of x only.

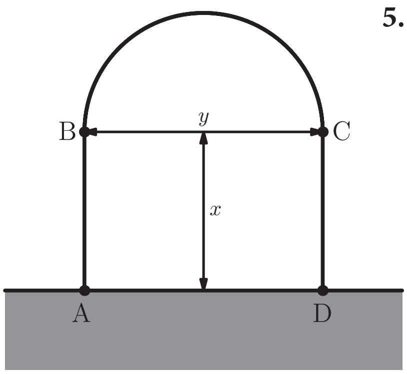
(b) Hence find the values of x and y that produce the maximum possible area. *[6 marks]*



4. A ball is thrown upwards from ground level, and its height above the ground (in metres) is given by the equation $h = 8t - 4.9t^2$, where t is the time (in seconds) after the ball is thrown.

(a) When does the ball return to the ground?

(b) Find the maximum height of the ball. *[4 marks]*



5. The enclosure ABCD consists of straight lines AB and CD together with a semi-circle BC. ABCD is a rectangle with sides x and y, as shown in the diagram. The perimeter of the enclosure is 60 m.

(a) Show that the area of the enclosure is given by

$$A = 30y - \frac{1}{8}\pi y^2.$$

(b) Show that the maximum possible area of the enclosure is when $x = 0$ (so that the enclosure is a semi-circle).

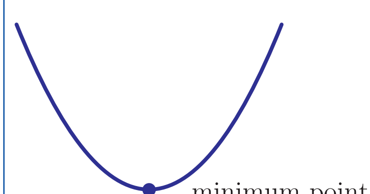
(c) Find the possible dimensions of the enclosure so that its area is 200 m^2. *[7 marks]*

6. A computer salesman finds that if he sells n computers, he can make a profit of $\$(200 - 4n)$ per computer. How many computers should he sell in order to maximise his profit? *[5 marks]*

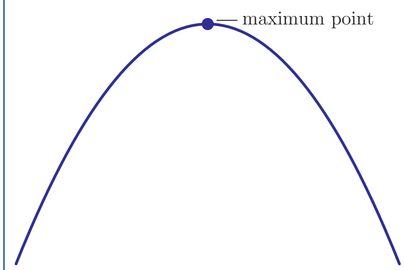
Summary

Quadratic functions have the general form $f(x) = ax^2 + bx + c$, where a $\neq 0$.

The graphs of quadratic functions are **parabolas**, which have a single **turning point** or vertex. This point is either a minimum or a maximum point. The following table summarises their main features and how these are determined by the **coefficients** a,b and c.

Feature	What to look at	Conclusion
Overall shape – does parabola open upward or downward?	Sign of a	a > 0 

continued . . .

Feature	What to look at	Conclusion
		$a < 0$ 
y-intercept	Value of c	y-intercept $(0, c)$
Vertex (or turning point)	Completed square form $y = a(x - h)^2 + k$	Vertex (h, k) for $a > 0$, $y \geq k$ for all x. for $a < 0$, $y \leq k$ for all x.
Line of symmetry	Completed square form or quadratic formula	$x = h$ or $x = -\dfrac{b}{2a}$
Zeros	Factorised form $y = a(x - p)(x - q)$ or quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Roots p and q, x-intercepts $(p, 0)$ and $(q, 0)$
The number of real roots	Discriminant $\Delta = b^2 - 4ac$	$\Delta > 0 \Rightarrow$ two distinct real roots $\Delta = 0 \Rightarrow$ one root (repeated root, equal roots) $\Delta < 0 \Rightarrow$ no real roots

- It is also possible to determine some of the coefficients from the graph: the y-intercept, the ordinate of the vertex, the zeros and the shape of the graph (positive or negative a) can be read directly from the graph.
- We can solve quadratic equations by factorising, completing the square, applying the quadratic formula or using the graphing and equation solver programs on a calculator.
- We can also solve **simultaneous equations** using the points of intersection of two graphs, or using the method of substitution.
- Quadratic functions can be used to solve problems, especially optimisation problems, if we model the situation using a quadratic function.

Introductory problem revisited

A small dairy farmer wants to sell a new type of luxury cheese. After a fixed set-up cost of \$250, he can produce the cheese at a cost of \$9 per kilogram. He is able to produce up to 400 kg, but he plans to take advance orders and produce only what he can sell. His market research suggests that the amount he would be able to sell depends on the price in the following way: the amount decreases proportionally with the price; if he charged \$20 per kg he would not sell any, and if the cheese was free he would 'sell' the maximum 400 kg that he could produce. What price per kilogram should the farmer set in order to maximise his profit?

Let x (in dollars) be the selling price per kilogram of cheese, and let m be the amount produced (and sold). Then the production cost is $\$(250+9m)$ and the amount of money earned is $\$(mx)$, giving a profit of $P = mx-(250+9m)$ dollars.

The amount m depends on x. We are told that the relationship is 'proportional', i.e. linear; when $x = 20$, $m = 0$ (no one would buy the cheese if it cost \$20/kg), and when $x = 0$, $m = 400$ (the farmer would be able to 'sell' all 400 kg of cheese that he could produce if it were free). Hence m is a linear function of x (a straight line graph) that passes through the points $(0, 400)$ and $(20, 0)$. We can then find that its equation is $m = 400-20x$. So the total profit is

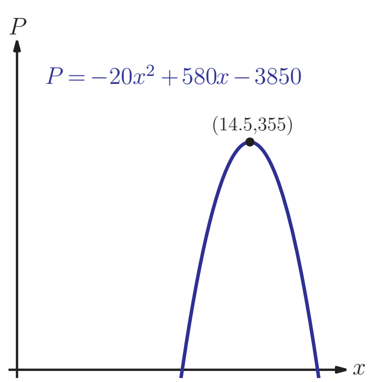
$$\begin{aligned} P &= mx-(250+9m) \\ &= (400-20x)x-\big(250+9(400-20x)\big) \\ &= -20x^2+580x-3850 \end{aligned}$$

This is a quadratic function of x. Since it is a negative quadratic, its vertex is a maximum point. The x-coordinate of the vertex is given by the equation for the line of symmetry, $x = -\frac{b}{2a}$. So the maximum profit is achieved when

$$x = -\frac{580}{2\times(-20)} = 14.5$$

Thus, to maximise profit, the farmer should sell the cheese at \$14.50 per kilogram.

We can also graph the function to see how the profit depends on the selling price. Outside the range of x-values shown on the graph, the farmer would make a loss.



Mixed examination practice 1

Short questions

1. (a) Factorise $x^2+5x-14$.

(b) Solve $x^2+5x-14=0$. *[4 marks]*

2. The quadratic function $y=(x-a)^2+b$ has a turning point at (3, 7).

(a) State whether this turning point is a maximum or a minimum point.

(b) State the values of a and b. *[3 marks]*

3. The quadratic function $y=a(x-b)^2+c$ passes through the points $(-2,\ 0)$ and $(6,\ 0)$. Its maximum y-value is 48. Find the values of a, b and c. *[6 marks]*

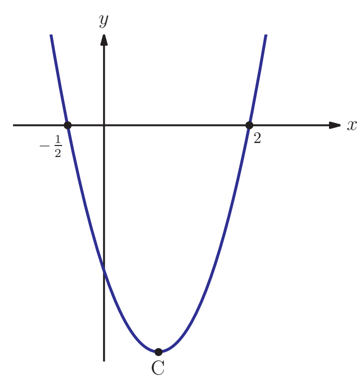
4. A quadratic function passes through the points $(k,\ 0)$ and $(k+4,\ 0)$. Find in terms of k the x-coordinate of the turning point. *[2 marks]*

5. The diagram represents the graph of the function

$$f: x \mapsto (x-p)(x-q)$$

(a) Write down the values of p and q.

(b) The function has a minimum value at the point C. Find the x-coordinate of C. *[4 marks]*



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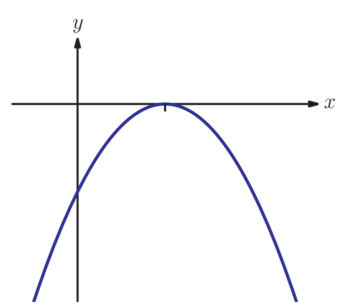
6. The diagram shows the graph of the function $y = ax^2 + bx + c$.

Complete the table below to show whether each expression is positive, negative or zero.

Expression	Positive	Negative	zero
A			
C			
b^2-4ac			
B			



[4 marks]

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7. (a) Write $x^2 - 10x + 35$ in the form $(x-p)^2 + q$.

(b) Hence, or otherwise, find the maximum value of $\frac{1}{(x^2-10x+35)^3}$.

[5 marks]

8. Find the exact values of k for which the equation $2kx + (k+1)x + 1 = 0$ has equal roots. *[6 marks]*

9. Find the range of values of k for which the equation $2x^2 + 6x + k = 0$ has no real roots. *[6 marks]*

10. Find the values of k for which the quadratic function $x^2 - (k+1)x + 3$ has only one zero. *[6 marks]*

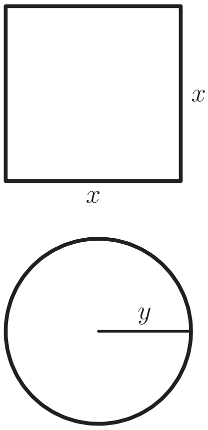
11. Let α and β denote the roots of the quadratic equation $x^2 - kx + (k-1) = 0$.

(a) Express α and β in terms of k.

(b) Given that $\alpha^2 + \beta^2 = 17$, find the possible values of k. *[6 marks]*

Long questions

1. The diagram shows a square with side x cm and a circle with radius y cm.



(a) Write down an expression for the perimeter

(i) of the square

(ii) of the circle

(b) The two shapes are made out of a piece of wire of total length 8 cm. Find an expression for x in terms of y.

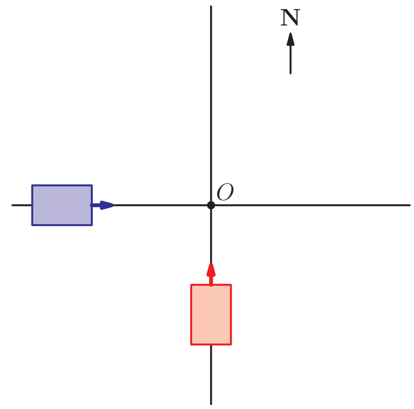
(c) Show that the total area of the two shapes is given by

$$A = \frac{\pi}{4}(\pi + 4)y^2 - 2\pi y + 4$$

(d) If the total area of the two shapes is the smallest possible, what percentage of the wire is used for the circle?

[11 marks]

2. Two cars are travelling along two straight roads which are perpendicular to each other and meet at the point O, as shown in the diagram. The first car starts 50 km west of O and travels east at a constant speed of 20 km/h. The second car starts 30 km south of O at the same time and travels north at a constant speed of 15 km/h.



(a) Show that at time t, the distance d between the two cars satisfies $d^2 = 625t^2 - 2900t + 3400$.

(b) Hence find the closest distance between the two cars. *[9 marks]*

3. **(a)** The graph of $y = x^2 - 6x + k$ has its vertex on the x-axis. Find the value of k.

(b) A second parabola has its vertex at $(-2, 5)$ and passes through the vertex of the first graph. Find the equation of the second graph in the form $y = ax^2 + bx + c$.

(c) Find the coordinates of the other point of intersection between the two graphs. *[12 marks]*