Trigonometric identities

You should know the identities for the sine and cosine of a sum or a difference of two angles.

Identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
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Easy consequences of the above identities:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

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If you don't think that the above are easy consequences, let me know and I will explain in class.

Identities

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$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Proof

Let's put $\alpha + \beta$ on the unit circle.

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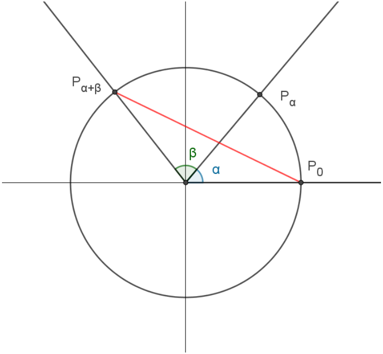

We then have from the definition of cosine that $\cos(\alpha + \beta)$ is the x-coordinate of point $P_{\alpha+\beta}$. We will calculate the length of $P_0 P_{\alpha+\beta}$.

Proof

Using Pythagorean Theorem:

$$|P_0 P_{\alpha+\beta}|^2 = (1 - \cos(\alpha + \beta))^2 + \sin^2(\alpha + \beta)$$

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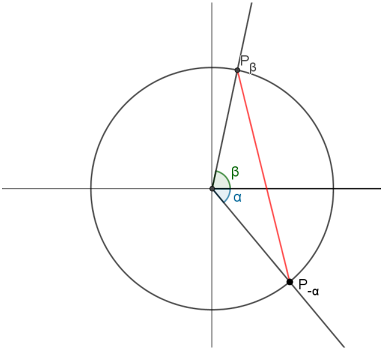
We get:

$$|P_0 P_{\alpha+\beta}|^2 = 1 - 2\cos(\alpha+\beta) + \cos^2(\alpha+\beta) + \sin^2(\alpha+\beta) = 2 - 2\cos(\alpha+\beta)$$

Proof

Now if we rotate our triangle we get:



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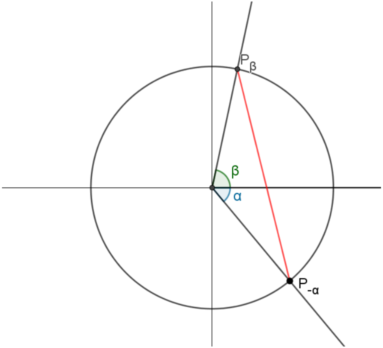

Of course the triangle did not change, so the lenght of the red segment is the same:

$$|P_0P_{\alpha+\beta}| = |P_{-\alpha}P_\beta|$$

Proof

We will calculate $|P_{-\alpha}P_{\beta}|^2$.

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We get:

$$\begin{aligned} |P_{-\alpha}P_\beta|^2 =& \cos^2\beta - 2\cos\beta\cos(-\alpha) + \cos^2(-\alpha) + \\ &+ \sin^2(-\alpha) - 2\sin(-\alpha)\sin\beta) + \sin^2\beta = \\ =& 2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha \end{aligned}$$

Proof

So finally we got:

$$2 - 2\cos(\alpha + \beta) = 2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha$$

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So:

$$\cos(\alpha + \beta) = \cos\beta\cos\alpha - \sin\beta\sin\alpha$$

Summary

Please try and understand this proof. If you think some details are missing, let me know.

Applications

Let's calculate $\sin 105^\circ$.

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$$\begin{aligned}\sin(105^\circ) = \sin(45^\circ + 60^\circ) = \\ &= \sin 45^\circ \cos 60^\circ + \sin 60^\circ \cos 45^\circ = \\ &= \frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

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We could've predicted the result, since:

$$\sin 105^\circ = \sin \frac{7\pi}{12} = \sin(\frac{\pi}{2} + \frac{\pi}{12}) = \cos \frac{\pi}{12}$$

Short test

Short test will be similar to examples above.

In case of any questions, you can email me at T.J.Lechowski@gmail.com.