3.6 SKETCHING POLYNOMIALS

3.6.1 GRAPHICAL SIGNIFICANCE OF ROOTS

We have already been making use of the graphs of polynomial functions to help us during this chapter. We are now in a position where we can sketch the graphs of polynomial functions as well as give meaning to the geometrical relationship between the polynomial expression and its graph. In particular we are interested in the geometrical significance of the roots of a polynomial.

The relationship between the roots of a polynomial and its graph can be summarised as follows:

If the polynomial $P(x)$ is factorised into unique (single) factors, $(x-a)$, $(x-b)$, $(x-c)$, . . . so that

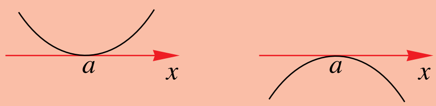
$$P(x) = (x-a)(x-b)(x-c)\ldots, \text{ where } a \neq b \neq c \neq \ldots,$$

the curve will **cut the *x*–axis** at each of the points $x = a, x = b, x = c, \ldots$.
That is, at each of these points the curve will look like one of

If the polynomial $P(x)$ is factorised and has a repeated (squared) factor, $(x-a)^2$, and unique factors $(x-b)$, $(x-c)$, . . . so that

$$P(x) = (x-a)^2(x-b)(x-c)\ldots, \text{ where } a \neq b \neq c \neq \ldots,$$

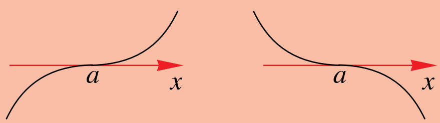
the curve will **touch the *x*–axis** at $x = a$ and cut the x–axis at each of the other points $x = b, x = c, \ldots$.
That is, at $x = a$ the curve will look like one of



If the polynomial $P(x)$ is factorised and has a repeated (cubed) factor, $(x-a)^3$, and unique factors $(x-b)$, $(x-c)$, . . . so that

$$P(x) = (x-a)^3(x-b)(x-c)\ldots, \text{ where } a \neq b \neq c \neq \ldots,$$

the curve will **cut the *x*–axis** at $x = a$ but with a change in concavity, i.e., there will be a **stationary point of inflection** at $x = a$ and it will cut the x–axis at each of the other points $x = b, x = c, \ldots$.
That is, at $x = a$ the curve will look like one of

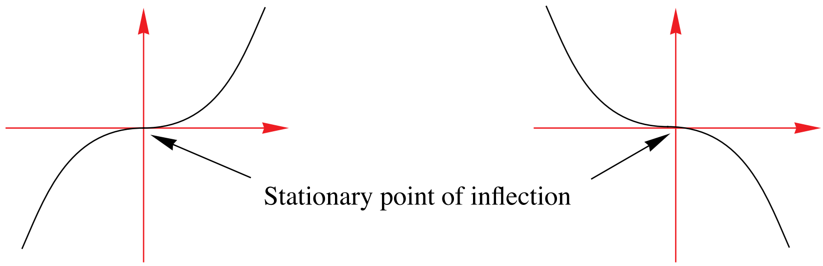


3.6.2 CUBIC FUNCTIONS

A cubic function has the general form $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0,\ a, b, c, d \in \mathbb{R}$.

We first consider the polynomial $f(x) = ax^3$.

For $a > 0$ we have: For $a < 0$ we have:

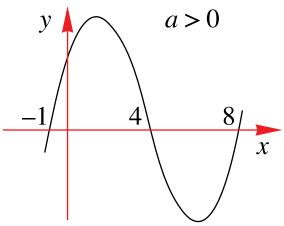


All other cubic polynomials with real coefficients can be factorised into one of the following forms:

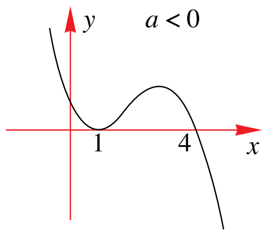
$P(x) = a(x-k)^3$	i.e., 3 identical real zeros, meaning three equal roots, and so, a stationary point of inflection at $x = k$.
$P(x) = a(x-k)^2(x-m)$	i.e., 2 identical real zeros and one other real zero, meaning two equal roots and a third different root, so that at $x = k$ there is a turning point on the x–axis.
$P(x) = a(x-k)(x-m)(x-n)$	i.e., 3 distinct real zeros, meaning three different roots, and so the curve will cut the x–axis at three different points on the x–axis.
$P(x) = a(x-k)(x^2+px+q)$	i.e., 1 real zero and an irreducible real quadratic, meaning that there is only one root and so the curve cuts the x–axis at only one point, $x = k$.

Some examples are shown below.

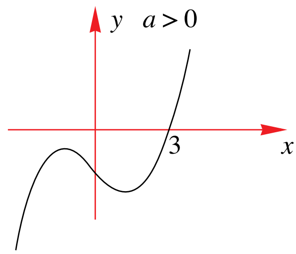
$P(x) = a(x-4)(x-8)(x+1)$



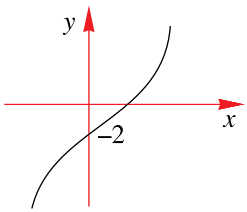
$P(x) = a(x-1)^2(x-4)$



$P(x) = a(x-3)(x^2+4x+5)$



$P(x) = x^3 + 4x - 2$



Notice that in this case we cannot factorise the cubic into rational linear factors. The x–intercept can be found using the TI–83 ($x = 0.4735$ to 4 d.p.)

The key to sketching polynomials is to first express them (where possible) in factored form. Once that is done we can use the results of §3.6.1. Of course, although we have only looked at the cubic function in detail, the results of §3.6.1 hold for polynomials of higher order than three.

EXAMPLE 3.19

Sketch the graph of (a) $P(x) = (x-1)(x-3)(x+1)$.

(b) $P(x) = (x-1)^2(x+3)$.

(c) $P(x) = (2-x)^3$

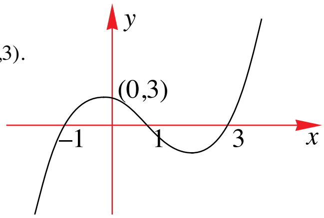
Solution

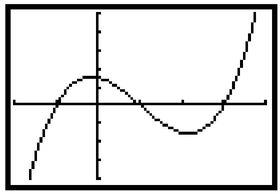
(a) The polynomial $P(x) = (x-1)(x-3)(x+1)$ is already in factored form. As the factors are unique, there are three distinct roots and so the curve cuts the x–axis at $x = 1$, $x = -1$ and $x = 3$.

As the leading coefficient is positive the graph has the basic shape: 

The y–intercept occurs when $x = 0$, i.e., $P(0) = (-1)(-3)(1) = 3$.

Note: Turning point does not occur at (0,3).

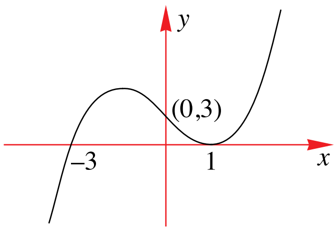


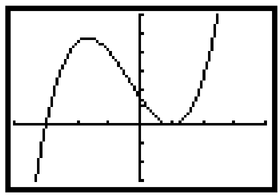


(b) The polynomial $P(x) = (x-1)^2(x+3)$ is in factored form and has a repeated factor $(x-1)^2$ and a unique factor $(x+3)$. That is, it has a **double root** at $x = 1$ and a single root at $x = -3$. This means the curve will have a **turning point** on the x–axis at $x = 1$ and will cut the x–axis at $x = -3$.

As the leading coefficient is positive the graph has the basic shape: 

The y–intercept occurs when $x = 0$, i.e., $P(0) = (-1)^2(3) = 3$.

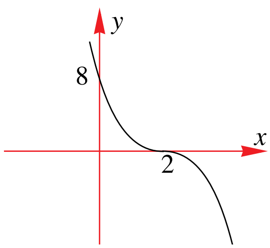


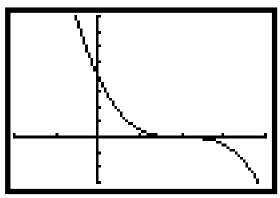


(c) The polynomial $P(x) = (2-x)^3$ is in factored form with repeated factor $(2-x)^3$. This means that there is a **treble root** at $x = 2$ and so, there is a **stationary point of inflection** on the x–axis at $x = 2$.

As the leading coefficient is negative the graph has the basic shape: 

The y–intercept occurs when $x = 0$, i.e., $P(0) = (2)^3 = 8$.





Note: We leave the general discussion of the cubic polynomial function $f(x) = a(x-k)^3 + h$ to Chapter 5, except to state that this curve would look exactly like $f(x) = ax^3$ but with its stationary point of inflection now located at (k, h).

EXAMPLE 3.20

Sketch the graph of (a) $P(x) = (2x-1)(x+1)^2(x-2)$

(b) $P(x) = (1-x)(x+1)^3$

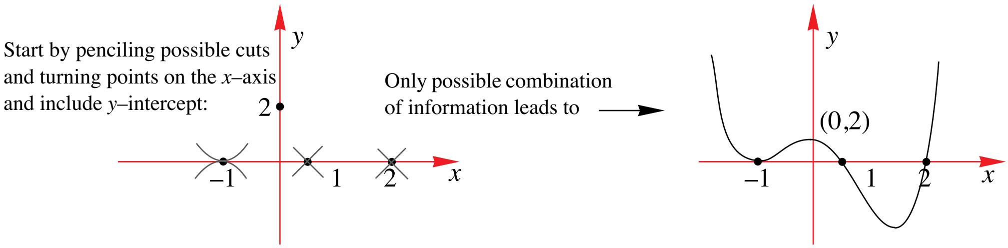
(c) $P(x) = x(x+2)^3(x-1)$.

Solution

(a) We have single roots at $x = \frac{1}{2}$ and $x = 2$ and a **double root** at $x = -1$. This means that the curve will cut the x–axis at $x = 0.5$ and $x = 2$ but will have a **turning point** at $(-1, 0)$.

The y–intercept is given by $P(0) = (-1)(1)^2(-2) = 2$.

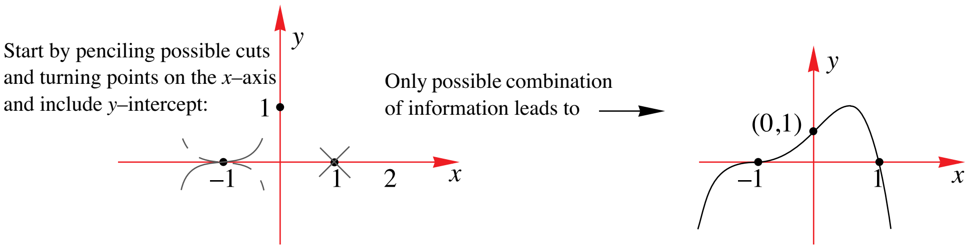
We start by filling in the information on a set of axes and then sort of 'join the dots':



(b) We have a single root at $x = 1$ and a **treble root** at $x = -1$. This means that the curve will cut the x–axis at $x = -1$ and will have a **stationary point of inflection** at $(-1, 0)$.

The y–intercept is given by $P(0) = (1)(1)^3 = 1$.

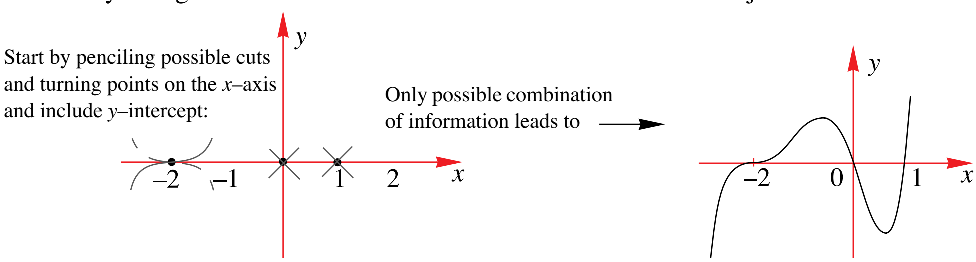
We start by filling in the information on a set of axes and then sort of 'join the dots':



(c) We have single roots at $x = 0$ and $x = 1$ and a **treble root** at $x = -2$. This means that the curve will cut the x–axis at $x = 0$ and $x = 1$ and will have a **stationary point of inflection** at $(-2, 0)$.

The y–intercept is given by $P(0) = (0)(2)^3(-1) = 0$.

We start by filling in the information on a set of axes and then sort of 'join the dots':



So far we have looked at sketching graphs of polynomials whose equations have been in factored form. So what happens when a polynomial function isn't in factored form? Well, in this case we first factorise the polynomial (if possible) and use the same process as we have used so far. We factorise the polynomial either by 'observation' or by making use of the factor theorem.

EXAMPLE 3.21

Sketch the graph of $f(x) = x^3 - 3x^2 - x + 3$.

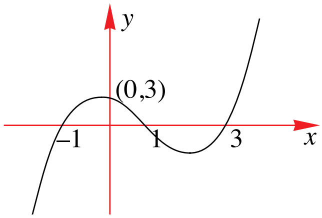
Solution

By observation, we have: $x^3 - 3x^2 - x + 3 = x^2(x-3) - (x-3)$

$$= (x^2 - 1)(x - 3)$$
$$= (x+1)(x-1)(x-3)$$

That is, $f(x) = (x+1)(x-1)(x-3)$.

This is in fact the same function as that in Example 3.19 (a) and so we have:



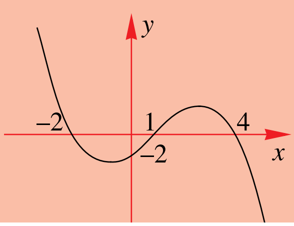
As always, we also have at our disposal the graphics calculator!

We now look at obtaining the equation of a polynomial from a given set of information. If a graph of a polynomial has sufficient information, then it is possible to determine the unique polynomial satisfying all the given information.

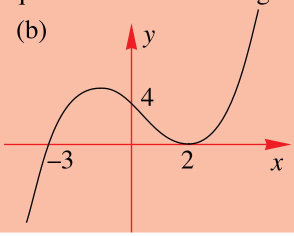
EXAMPLE 3.21

Determine the equation of the following cubic graphs

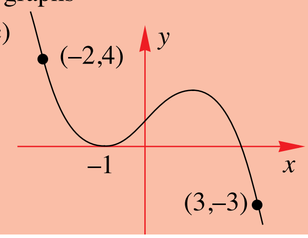
(a)



(b)



(c)



Solution

(a) In this instance we have the curve cutting the x–axis at three distinct points, $x = –2$, $x = 1$ and $x = 4$, meaning that the function will have three distinct factors, namely, $x + 2$, $x – 1$ and $x – 4$.

Therefore, we can write down the equation $f(x) = a(x+2)(x-1)(x-4)$, where a needs to be determined.

Using the point $(0, –2)$ we have $f(0) = a(0+2)(0-1)(0-4) = -2 \therefore 8a = -2 \Leftrightarrow a = -\frac{1}{4}$.

Therefore, $f(x) = -\frac{1}{4}(x+2)(x-1)(x-4)$.

(b) In this instance we have the curve cutting the x–axis at one point, $x = –3$, and having a turning point at $x = 2$. Meaning that the function will have a single linear factor, $x + 3$, and a repeated factor $(x-2)^2$.

Therefore, we can write down the equation $f(x) = a(x+3)(x-2)^2$, where a needs to be determined.

Using the point (0, 4) we have $f(0) = a(0+3)(0-2)^2 = 4 \therefore 12a = 4 \Leftrightarrow a = \frac{1}{3}$.

Therefore, $f(x) = \frac{1}{3}(x+3)(x-2)^2$.

(c) The only obvious information is that there is a turning point at $x = -1$ and so the polynomial will have a repeated factor $(x+1)^2$.

Therefore the polynomial will take on the form $f(x) = (ax+b)(x+1)^2$.
Then, to determine the values of a and b we use the coordinates (–2,4) and (3,–3).
At (–2,4): $4 = (-2a+b)(-2+1)^2 \Leftrightarrow 4 = -2a+b$ – (1)
At (3,–3): $-3 = (3a+b)(3+1)^2 \Leftrightarrow -3 = 48a+16b$ – (2)
Solving for a and b we have:

From (1) $b = 4+2a$ and substituting into (2) we have $-3 = 48a+16(4+2a) \therefore a = -\frac{67}{80}$

Substituting into $b = 4+2a$ we have $b = 4+2\left(-\frac{67}{80}\right) = \frac{93}{40}$.

Therefore, $f(x) = \left(-\frac{67}{80}x+\frac{93}{40}\right)(x+1)^2 = \frac{1}{80}(186-67x)(x+1)^2$.

EXERCISES 3.6

1. Sketch the graphs of the following polynomials

(a)	$P(x) = x(x-2)(x+2)$	(b)	$P(x) = (x-1)(x-3)(x+2)$
(c)	$T(x) = (2x-1)(x-2)(x+1)$	(d)	$P(x) = \left(\frac{x}{3}-1\right)(x+3)(x-1)$
(e)	$P(x) = (x-2)(3-x)(3x+1)$	(f)	$T(x) = (1-3x)(2-x)(2x+1)$
(g)	$P(x) = -x^2(x-4)$	(h)	$P(x) = (1-4x^2)(2x-1)$
(i)	$T(x) = (x-1)(x-3)^2$	(j)	$T(x) = \left(1-\frac{x}{2}\right)^2(x+2)$
(k)	$P(x) = x^2(x+1)(2x-3)$	(l)	$P(x) = 4x^2(x-2)^2$
(m)	$P(x) = \frac{1}{2}(x-3)(x+1)(x-2)^2$	(n)	$T(x) = -(x-2)(x+2)^3$
(o)	$P(x) = (x^2-9)(3-x)^2$	(p)	$T(x) = -2x(x-1)(x+3)(x+1)$
(q)	$P(x) = x^4+2x^3-3x^2$	(r)	$T(x) = \frac{1}{4}(4-x)(x+2)^3$
(s)	$T(x) = -x^3(x^2-4)$	(t)	$T(x) = (2x-1)\left(\frac{x}{2}-1\right)(x-1)(1-x)$

2. Sketch the graph of the following polynomials

(a) $P(x) = x^3 - 4x^2 - x + 4$ (b) $P(x) = x^3 - 6x^2 + 8x$

(c) $P(x) = 6x^3 + 19x^2 + x - 6$ (d) $P(x) = -x^3 + 12x + 16$

(e) $P(x) = x^4 - 5x^2 + 4$ (f) $P(x) = 3x^3 - 6x^2 + 6x - 12$

(g) $P(x) = -2x^4 + 3x^3 + 3x^2 - 2x$ (h) $P(x) = 2x^4 - 3x^3 - 9x^2 - x + 3$

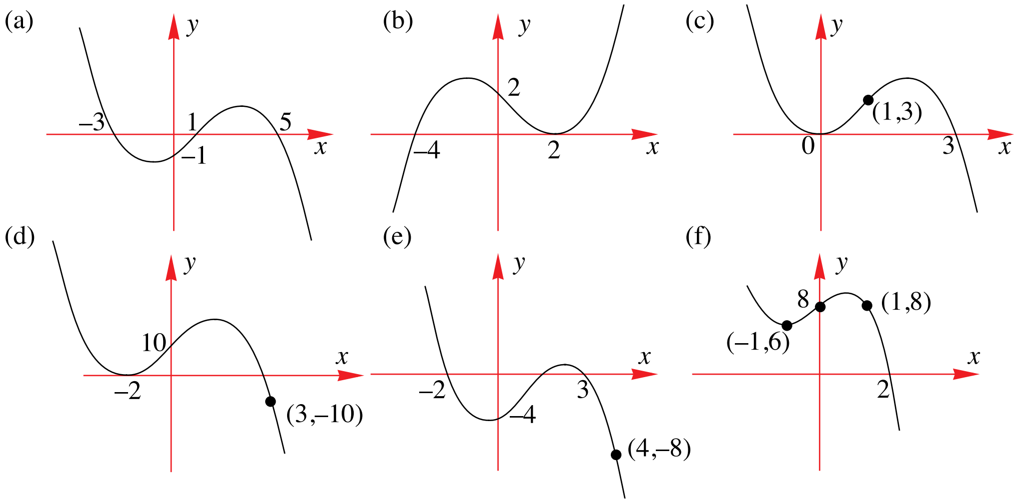
(i) $T(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$ (j) $T(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

3. Sketch the graph of

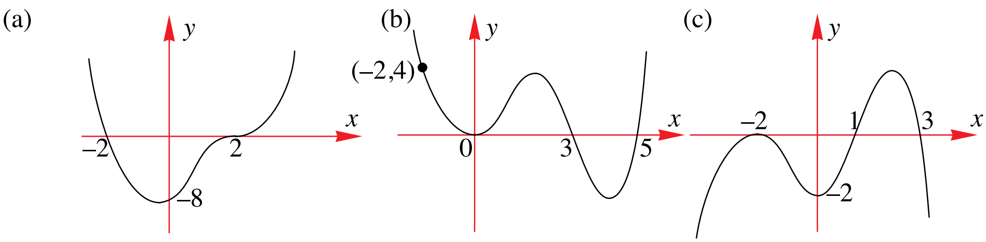
(a) $P(x) = x^3 - kx$ where i. $k = b^2$ ii. $k = -b^2$.

(b) $P(x) = x^3 - kx^2$ where i. $k = b^2$ ii. $k = -b^2$.

4. Determine the equations of the following cubic functions



5. Determine the equation of the following functions



6. Sketch a graph of $f(x) = (x - b)(ax^2 + bx + c)$ if $b > 0$ and

(a) $b^2 - 4ac = 0, a > 0, c > 0$

(b) $b^2 - 4ac > 0, a > 0, c > 0$

(c) $b^2 - 4ac < 0, a > 0, c > 0$

7. (a) On the same set of axes sketch the graphs of $f(x) = (x - a)^3$ and $g(x) = (x - a)^2$. Find $\{(x, y) : f(x) = g(x)\}$.

(b) Hence find $\{x: (x - a)^3 > (x - a)^2\}$.