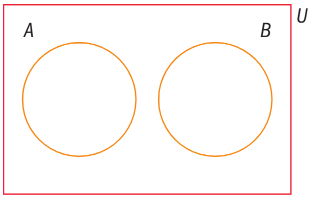
8.7 Two special cases: mutually exclusive and independent events

Two events, A and B, are **mutually exclusive** if whenever A occurs it is impossible for B to occur and, similarly, whenever B occurs it is impossible for A to occur.

Events A and A' are the most obvious example of mutually exclusive events – either one or the other must occur, but A and A' cannot occur at the same time.

For example, in tossing a coin, the events 'a head is tossed' and 'a tail is tossed' are mutually exclusive.

Here is the Venn diagram for mutually exclusive events A and B.



As the two sets do not overlap, $A \cap B = \varnothing$.

→ Events A and B are mutually exclusive if and only if $P(A \cap B) = 0$.

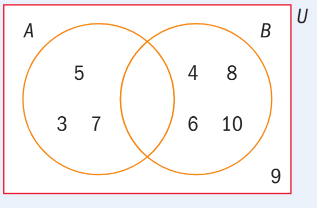
Example 13

The numbers 3, 4, 5, 6, 7, 8, 9, 10 are each written on an identical piece of card and placed in a bag. A random experiment is: a card is selected at random from the bag.
Let A be the event 'a prime number is chosen' and B the event 'an even number is chosen'.

a Draw a Venn diagram that describes the random experiment.
b Determine whether the events A and B are mutually exclusive.

Answers

a



Draw a Venn diagram to show the sets A and B.

$A \cap B = \varnothing$, so $P(A \cap B) = 0$.

b A and B are mutually exclusive.

The intersection $A \cap B$ is empty.

In 1933, the Russian Mathematician Andrey Nikolaevich Kolmogorov (1903–1987) defined probability by these axioms:

- The probability of all occurrences is 1
- Probability has a value which is greater than or equal to zero
- When occurrences cannot coincide their probabilities can be added

The mathematical properties of probability can be deduced from these axioms. Kolmogorov used his probability work to study the motion of the planets and the turbulent flow of air from a jet engine.

What is an axiom? Find out more about Euclid's axioms for geometry, written 2000 years ago.

Exercise 8L

In each experiment, determine whether the events A and B are mutually exclusive.

1 Roll an unbiased six-faced dice.
Let A be the event 'roll a square number' and let B be the event 'roll a factor of six'.

2 Roll an unbiased six-faced dice.
Let A be the event 'roll a four' and let B be the event 'roll a six'.

3 Roll an unbiased six-faced dice.
Let A be the event 'roll a prime number' and let B be the event 'roll an even number'.

4 Roll an unbiased six-faced dice.
Let A be the event 'roll a square number' and let B be the event 'roll a prime number'.

5 Each of the numbers 3, 4, 5, 6, 7, 8, 9, 10 are written on identical pieces of card and placed in a bag. A card is selected at random from the bag.
Let A be the event 'a square number is chosen' and let B be the event 'an odd number is chosen'.

6 Each of the numbers 5, 6, 7, 8, 9, 10 are written on identical pieces of card and placed in a bag. A card is selected at random from the bag.
Let A be the event 'a square number is chosen' and let B be the event 'an even number is chosen'.

7 Each of the numbers 2, 3, 4, 5, 6, 7, 8, 9 are written on identical pieces of card and placed in a bag. A card is selected at random from the bag.
Let A be the event 'an even number is chosen' and let B be the event 'a multiple of three is chosen'.

8 Two unbiased coins are tossed.
Let A be the event 'two heads show' and let B be the event 'one head shows'.

If two events, A and B, are mutually exclusive, the effect of the first event, A, on the second, B, could not be greater – if A occurs, then it is impossible that B can occur (and vice versa). The occurrence of one event completely prevents the occurrence of the other.

The other extreme is when the occurrence of the one event does not affect in any way the occurrence of the other. Then the two events are **mathematically independent** of each other.

For example, if a one-euro coin is tossed and then a one-dollar coin is tossed, the fact that the euro coin landed 'heads' does not affect in any way whether the dollar coin lands 'heads' or 'tails'. The two events are independent of each other.

Another way to put this is that the probability that A occurs, $P(A)$, remains the same given that B has occurred. Writing this as an equation, A and B are independent whenever $P(A) = P(A \mid B)$.

The definition of $P(A \mid B)$ is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Thus whenever A and B are independent:

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

Rearranging, $P(A \cap B) = P(A) \times P(B)$

→ A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$.

If you are asked to determine whether two events are independent, this is the test you must use.

Example 14

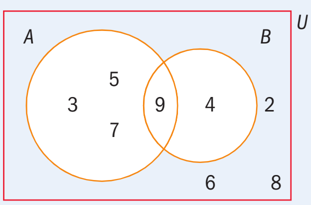
The numbers 2, 3, 4, 5, 6, 7, 8, 9 are each written on identical pieces of card and placed in a bag.
A card is selected at random from the bag.
Let A be the event 'an odd number is chosen' and let B be the event 'a square number is chosen'.

a Draw a Venn diagram to represent the experiment.

b Determine whether A and B are independent events.

Answers

a



The event $A \cap B$ is 'an odd number is chosen ***and*** *a square number is chosen' or 'an odd square number is chosen'.*

b $P(A) \times P(B) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

From the Venn diagram,

$P(A) = \frac{4}{8} = \frac{1}{2} \quad P(B) = \frac{2}{8} = \frac{1}{4}$

$P(A \cap B) = \frac{1}{8}$

$A \cap B = \{9\}$, *hence* $P(A \cap B) = \frac{1}{8}$

So A and B are independent events.

Now, consider the definition for (mathematical) independence: $P(A \cap B) = P(A) \times P(B)$.

This work links to the chi-squared test for independence that you studied in Chapter 5. Recall that to calculate the expected frequencies, the row total is multiplied by the column total and then divided by the overall total of frequencies. This is a direct consequence of the definition of mathematical independence.

Exercise 8M

For each experiment determine whether the events A and B are independent.

1 The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are each written on identical cards and placed in a bag.
A card is selected at random from the bag.

Let A be the event 'an odd number is chosen' and let B be the event 'a square number is chosen'.

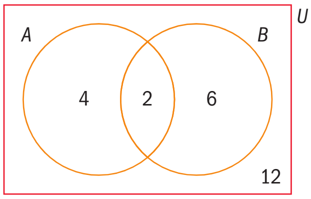
2 The numbers 1, 2, 3, 4, 5, 6 are each written on identical cards and placed in a bag.
A card is selected at random from the bag.

Let A be the event 'an even number is chosen' and let B be the event 'a square number is chosen'.

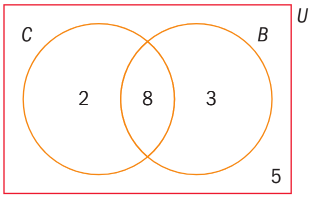
3 The numbers 2, 3, 4, 5, 6, 7, 8, 9, 10 are each written on identical cards and placed in a bag.
A card is selected at random from the bag.

Let A be the event 'a prime number is chosen' and let B be the event 'a multiple of three is chosen'.

4 The Venn diagram shows the number of students who take Art and/or Biology in a class.
Use the Venn diagram to determine whether taking Art and taking Biology are independent events.



5 The Venn diagram shows the number of students who take Chemistry and/or Biology in a class.
Use the Venn diagram to determine whether taking Chemistry and taking Biology are independent events.



6 The Venn diagram shows the number of students who take Chemistry and/or Physics in a class.
Use the Venn diagram to determine whether taking Chemistry and taking Physics are independent events.