3 The long journey of mathematics

CHAPTER OBJECTIVES:

1.5 Complex numbers: the number $i = \sqrt{-1}$; the terms real part, imaginary part, conjugate, modulus and argument; cartesian form $z = a + ib$; sums, products and quotients of complex numbers

1.6 The complex plane

1.7 Powers of complex numbers; *n*th roots of a complex number

1.8 Conjugate roots of polynomial equations with real coefficients

1.9 Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases where there is a unique solution, an infinity of solutions or no solution

2.5 Polynomial functions and their graphs; the factor and remainder theorems; the fundamental theorem of algebra

2.6 Solving quadratic equations using the quadratic formula; use of the discriminant $\Delta = b^2 - 4ac$ to determine the nature of the roots; solving polynomial equations both graphically and algebraically; sum and product of the roots of polynomial equations

2.7 Solutions of $g(x) \geq f(x)$: graphical or algebraic methods, for simple polynomials up to degree 3; use of technology for these and other functions

Before you start

You should know how to:

1 Solve quadratic equations by factorization.

e.g. $x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0$

$\Rightarrow x = 4$ or $x = -1$

2 Find a linear combination of two polynomials.

e.g. $f(x) = x^2 - 3x + 1$ and $g(x) = x^3 + 7x - 3$

$5f(x) + 2g(x) = 5(x^2 - 3x + 1) + 2(x^3 + 7x - 3)$

$= 2x^3 + 5x^2 - x - 1$

Skills check

1 Solve these quadratic equations:

a $x^2 + 2x - 3 = 0$ **b** $x^2 - 11x + 10 = 0$

c $2x^2 + x - 3 = 0$

2 Given the polynomials $f(x) = x^2 - 3x + 1$, $g(x) = 2x^3 - x^2 + 3x - 4$ and $h(x) = 3x^4 - 2x^2 - 5$, find:

a $f(x) + g(x)$

b $2h(x) - 4g(x) + 5f(x)$

c $\frac{1}{2}h(x) - \frac{2}{5}g(x)$

Important problems that challenged great minds

The Italian mathematician Leonardo of Pisa, best known as Fibonacci, most important contribution to mathematics was spreading the use of the Hindu-Arabic numeral system throughout Europe. In the next centuries, first in Italy, and then in other parts of Europe, bursts of mathematical creativity lead to incredible developments and discoveries in mathematics and science in general.

Over the centuries generations of mathematicians have helped the scientific community to achieve great insight into nature, moving us forward in our understanding of the world and allowing the remarkable development of science and technology. Throughout this history, scientific progress has always been related to revolutions in mathematical thought.

In this chapter we are going to take a close look at the evolution of the most fundamental mathematical concept – the concept of number. Using modern methods we are going to discover and explore the properties of a new set of numbers. These are the set of complex numbers.

3.1 Introduction to complex numbers

Solving quadratic equations using the quadratic formula

Zero is in many ways a mysterious number. Medieval mathematicians could not decide whether or not it really was a number! Nowadays, however, zero has high status in mathematics due to its algebraic properties. One is the zero factor property, that can be used to solve some polynomial equations.

→ Zero factor property: $a \times b = 0 \Rightarrow a = 0$ or $b = 0$

A quadratic equation has the form, $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$, and $a \neq 0$. When one of the coefficients is zero there is a special case that you can solve without using the general quadratic formula.

If a function vanishes for a particular value of its argument, $f(x) = 0$, then x is called a zero or root of $f(x)$.

Special cases:

i $b = 0, c \neq 0 \Rightarrow ax^2 + c = 0$

$$\Rightarrow x^2 = -\frac{c}{a}$$

$$\Rightarrow x = \pm\sqrt{-\frac{c}{a}} \Rightarrow x = -\sqrt{-\frac{c}{a}} \quad \text{or} \quad x = \sqrt{-\frac{c}{a}}$$

The solutions are real and opposite if $-\frac{c}{a} > 0$.

When $-\frac{c}{a} < 0$ the solutions are not real.

ii $b \neq 0, c = 0 \Rightarrow ax^2 + bx = 0$ *Factorize and apply the zero product property.*

$$\Rightarrow x(ax + b) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -\frac{b}{a}$$

The solutions are always real and distinct and one is always zero.

iii $b = 0, c = 0 \Rightarrow ax^2 = 0$

$$\Rightarrow x^2 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = 0$$

This is the only case where there is only one (double) real solution – which is zero.

The method for finding a **general formula for the solutions of a quadratic equation** is called 'completing the square'. This method can be used directly as in case **i**, or again by factorization.

Method I: Completing the square

$ax^2 + bx + c = 0$ *Divide the equation by a.*

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow x^2 + 2 \cdot x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Add $\left(\frac{b}{2a}\right)^2$ *to both sides in order to apply the formula* $(A \pm B)^2 = A^2 \pm 2AB + B^2$

See Chapter 14, section 2.2

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Factorize and simplify.

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Take the square root of both sides and simplify.

$$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Method II: Completing the square and factorization

$ax^2 + bx + c = 0$

$\Rightarrow a^2x^2 + abx + ac = 0$

Multiply the equation by a.

$$\Rightarrow \overbrace{(ax)^2 + 2 \cdot ax \cdot \frac{b}{2} + \left(\frac{b}{2}\right)^2}^{\text{perfect square}} - \left(\frac{b}{2}\right)^2 + ac = 0$$

Add and subtract $\left(\frac{b}{2}\right)^2$ *in order to apply the formula*

$$\Rightarrow \left(ax + \frac{b}{2}\right)^2 - \frac{b^2 - 4ac}{4} = 0$$

$(A \pm B)^2 = A^2 \pm 2AB + B^2$

$$\Rightarrow \left(ax + \frac{b}{2}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2}\right)^2 = 0$$

Apply $A^2 - B^2 = (A - B)(A + B)$.

$$\Rightarrow \left(ax + \frac{b}{2} - \frac{\sqrt{b^2 - 4ac}}{2}\right)\left(ax + \frac{b}{2} + \frac{\sqrt{b^2 - 4ac}}{2}\right) = 0$$

Apply the zero product property.

$$\text{either } ax + \frac{b}{2} - \frac{\sqrt{b^2 - 4ac}}{2} = 0$$

$$\Rightarrow ax = -\frac{b}{2} + \frac{\sqrt{b^2 - 4ac}}{2} \Rightarrow x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Solve for x and simplify.

$$\text{or } ax + \frac{b}{2} + \frac{\sqrt{b^2 - 4ac}}{2} = 0$$

$$\Rightarrow ax = -\frac{b}{2} - \frac{\sqrt{b^2 - 4ac}}{2} \Rightarrow x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Solve for x and simplify.

→ You can use the **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the **solutions** or **roots** of a quadratic equation.

Example 1

Use the quadratic formula to solve these equations. Check your answers with a GDC.

a $3x^2 + 11x + 6 = 0$

b $5x^2 - 9x - 3 = 0$

c $3px^2 + (p - 6)x - 2 = 0$

Answers

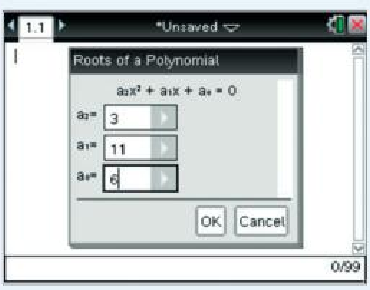
a $\underbrace{3}_{a}x^2 + \underbrace{11}_{b}x + \underbrace{6}_{c} = 0 \quad \Rightarrow x = \dfrac{-11 \pm \sqrt{11^2 - 4\cdot 3\cdot 6}}{2\cdot 3}$

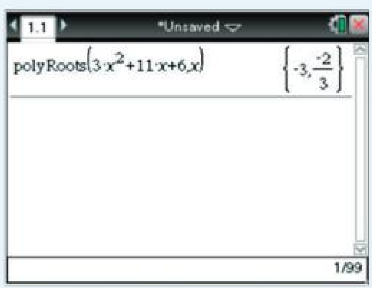
$$= \frac{-11 \pm \sqrt{121 - 72}}{6}$$

$$= \frac{-11 \pm \sqrt{49}}{6}$$

$$= \frac{-11 \pm 7}{6}$$

$$\Rightarrow x = \frac{-11 - 7}{6} = -3 \text{ or } x = \frac{-11 + 7}{6} = -\frac{2}{3}$$



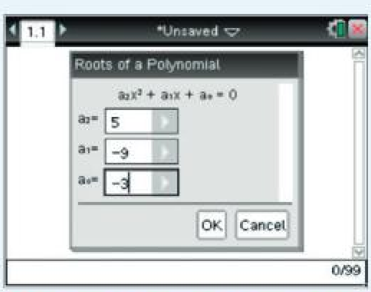


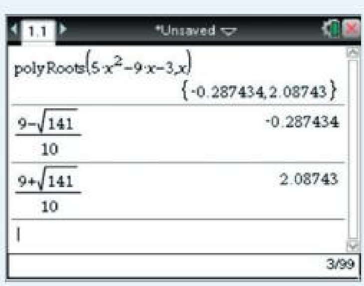
b $\underbrace{5}_{a}x^2 \underbrace{- 9}_{b}x \underbrace{- 3}_{c} = 0 \quad \Rightarrow x = \dfrac{-(-9) \pm \sqrt{(-9)^2 - 4\cdot 5\cdot(-3)}}{2\cdot 5}$

$$= \frac{9 \pm \sqrt{81 + 60}}{10}$$

$$= \frac{9 \pm \sqrt{141}}{10}$$

$$\Rightarrow x = \frac{9 - \sqrt{141}}{10} \text{ or } x = \frac{9 + \sqrt{141}}{10}$$





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The Babylonians (2000–1600 BCE) knew how to solve a quadratic equation by using a quadratic formula in a slightly different form from the one we use today. They were essentially using the standard formula in two different types of quadratic equation $x^2 + bx = c$ and $x^2 + bx = c$, where b and c were positive but not necessarily integers. Why did the Babylonians need to consider two different types of quadratic equations? You may wish to explore their methods for solving these equations and their contributions to the progress of mathematics.

c $\underbrace{3p}_{a}x^2+\underbrace{(p-6)}_{b}x\underbrace{-2}_{c}=0$

$$\Rightarrow x=\frac{-(p-6)\pm\sqrt{(p-6)^2-4\cdot 3p\cdot(-2)}}{2\cdot 3p}$$

$$=\frac{-(p-6)\pm\sqrt{p^2-12p+36+24p}}{6p}$$

$$=\frac{-(p-6)\pm\sqrt{p^2+12p+36}}{6p}$$

$$=\frac{-(p-6)\pm\sqrt{(p+6)^2}}{6p}=\frac{-(p-6)\pm(p+6)}{6p}$$

$$\Rightarrow x=\frac{-p+6-p-6}{6p}=-\frac{1}{3}\quad\text{or}\quad x=\frac{-p+6+p+6}{6p}=\frac{2}{p}$$

This problem cannot be solved by a GDC because it requires a Computer Algebra System.

Exercise 3A

1 Solve these quadratic equations, giving your answers in exact form.

a $2x^2-3x=0$ **b** $3x^2-75=0$

c $5x^2-4x=0$ **d** $7+28x^2=0$

e $242x^2+2x=0$ **f** $\sqrt{2}x^2-\sqrt{8}=0$

g $\pi x^2-11x=0$ **h** $ex^2-\sqrt{3}=0$

2 Use the quadratic formula to solve these equations. Check your answers with a GDC.

a $2x^2+5x+2=0$ **b** $3x^2-10x+3=0$

c $5x^2+3x-2=0$ **d** $21x^2+5x-6=0$

e $9x^2-6x+35=0$ **f** $122x=143x^2+24$

3 Solve these equations and write the solutions in exact form. Check your answers with a GDC.

a $x^2+4x+2=0$ **b** $5x^2-6x-1=0$

c $3x^2-x-3=0$ **d** $2x^2+11x+13=0$

e $11x^2=23x-7$ **f** $29x=5x^2-41$

4 Solve for x:

a $x^2+px-2p^2=0$ **b** $kx^2+(k+2)x+2=0$

c $2ax^2+6=ax+12x$ **d** $x^2-2a^2=b^2-ax-3ab$

Discriminant of a quadratic equation

A quadratic equation can have:

- two real roots
- one repeated real root
- no real roots.

Example 2

Solve these equations.

a $3x^2 + 5x - 2 = 0$

b $4x^2 + 12x + 9 = 0$

c $5x^2 + x + 4 = 0$

Answers

a $$3x^2+5x-2=0 \Rightarrow x=\frac{-5\pm\sqrt{(5)^2-4\cdot3\cdot(-2)}}{2\cdot3}=\frac{-5\pm\sqrt{25+24}}{6}=\frac{-5\pm\sqrt{49}}{6}$$

$$\Rightarrow x=\frac{-5-7}{6}=-2 \quad\text{or}\quad x=\frac{-5+7}{6}=\frac{1}{3}$$

b $$4x^2+12x+9=0 \Rightarrow x=\frac{12\pm\sqrt{(-12)^2-4\cdot4\cdot9}}{2\cdot4}$$

$$=\frac{-12\pm\sqrt{144-144}}{8}=\frac{-12\pm\sqrt{0}}{8}=-\frac{3}{2}$$

c $$5x^2+x+4=0 \Rightarrow x=\frac{-1\pm\sqrt{1^2-4\cdot5\cdot4}}{2\cdot5}=\frac{-1\pm\sqrt{1-80}}{10}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{-79}}{10}\notin\mathbb{R}$$

Investigation – the general quadratic function

A general quadratic function can be written $y = ax^2 + bx + c$, with $a, b, c \in \mathbb{R}$, $a \neq 0$. By using completing the square find the location of the minimum ($a > 0$) or maximum ($a < 0$) point on this curve. Hence, or otherwise, find the conditions on the coefficients a, b, c which determine how many solutions there are to the equation $ax^2 + bx + c = 0$.

The nature of the **roots** in Example 2 depends on the expression under the square root, that is, $b^2 - 4ac$. The expression $\Delta = b^2 - 4ac$ is called the **discriminant** because it acts to *discriminate* between the three different types of solutions.

The symbol used for the discriminant $b^2 - 4ac$ is the Greek letter Δ (delta).

→ **i** $\Delta = b^2 - 4ac > 0$

If the discriminant is positive, you can add $\sqrt{b^2 - 4ac}$ to $-b$ and subtract $\sqrt{b^2 - 4ac}$ from $-b$. In this case, you obtain two different numbers so there are **two distinct real roots**.

ii $\Delta = b^2 - 4ac = 0$

If the discriminant is equal to zero, adding zero to $-b$ and subtracting zero from $-b$ gives the same solution so there is **one repeated real root**.

iii $\Delta = b^2 - 4ac < 0$

If the discriminant is less than zero, the expression under the square root is negative, and therefore the square root is not a real number. There are **no real roots**.

Why do we use Greek letters to represent so many quantaties in mathematics. You may wish to explore the ancient Greeks' contributions to number, geometry or algebra.

Example 3

Without solving the equations, determine the nature of their roots.

a $x^2 - x + 1 = 0$
b $3x^2 + 30x - 75 = 0$
c $5x^2 + 4x - 1 = 0$

Answers

a $x^2 - x + 1 = 0 \Rightarrow$ — *Find the discriminant*
$\Delta = (-1)^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3 < 0$ — *$\Delta < 0$*
No real roots.

b $3x^2 + 30x + 75 = 0 \Rightarrow$ — *Find the discriminant*
$\Delta = 30^2 - 4 \cdot 3 \cdot 75 = 900 - 900 = 0$ — *$\Delta = 0$*
One real root.

c $5x^2 + 4x - 1 = 0 \Rightarrow$ — *Find the discriminant*
$\Delta = 4^2 - 4 \cdot 5 \cdot (-1) = 16 + 20 = 36 > 0$ — *$\Delta > 0$*
Two real roots.

Example 4

Find the value(s) of the real parameter m so that:

a $x^2 - 6x + m = 0$ has two real roots
b $x^2 - mx + m - 1 = 0$ has one repeated real root
c $mx^2 + (2m - 1)x + 1 = 0$ has no real roots

Answers

a $x^2 - 6x + m = 0 \Rightarrow$
$\Delta = (-6)^2 - 4 \cdot 1 \cdot m$ — *Find the discriminant*
$\Delta = 36 - 4m$ — *Simplify Δ and set $\Delta > 0$*
$36 - 4m > 0$ — *Solve the inequality for m*
$36 > 4m \Rightarrow m < 9$

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b $x^2 - mx + m - 1 = 0 \Rightarrow$	
$\Delta = (-m)^2 - 4 \cdot 1 \cdot (m-1)$	*Find the discriminant*
$\Delta = m^2 - 4m + 4$	
$m^2 - 4m + 4 = 0$	*Set $\Delta = 0$*
$(m-2)^2 = 0 \Rightarrow m = 2$	*Solve the equation for m*
c $mx^2 - (2m-1)x + m = 0 \Rightarrow$	
$\Delta = (2m-1)^2 - 4 \cdot m \cdot m$	*Find the discriminant*
$\Delta = 4m^2 - 4m + 1 - 4m^2 \Rightarrow$	*Simplify Δ and set $\Delta < 0$*
$1 - 4m < 0$	
$1 < 4m \Rightarrow m > \frac{1}{4}$	*Solve the inequality for m*

Exercise 3B

1 Without solving the equations, determine the nature of the roots.

a $x^2 - 2x - 3 = 0$ **b** $x^2 + 10x + 25 = 0$

c $4x^2 - 3x + 2 = 0$ **d** $5x^2 - 11x + 6 = 0$

e $\frac{3}{5}x^2 - \frac{4}{7}x + \frac{2}{3} = 0$ **f** $2x^2 + 2\sqrt{26}x + 13 = 0$

2 Find the value(s) of the real parameter *k* so that:

a $x^2 - 2x - k = 0$ has one real root

b $kx^2 + 3x - 2 = 0$ has two real roots

c $3x^2 + 5x + 2k - 1 = 0$ has no real roots

d $x^2 - (3k+2)x + k^2 = 0$ has one real root

e $kx^2 + 2kx + k - 2 = 0$ has two real roots

f $2kx^2 + (4k+3)x + k - 3 = 0$ has no real roots

Sum and product of roots of a quadratic equation

Investigation – Viète's theorem

A general quadratic equation $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$, $a \neq 0$ has two solutions, x_1 and x_2. By using the quadratic formula find expressions for the sum, $x_1 + x_2$, and product, $x_1 \cdot x_2$, of the two roots in terms of the coefficients a, b, c.

François Viète (1540–1603) discovered a relationship between the parameters a, b and c of a quadratic equation and the solutions x_1 and x_2.

The expressions you found in the investigation are known as Viète's theorem.

→ For a quadratic equation $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$ and solutions x_1 and x_2, then the sum of the roots, $x_1 + x_2 = -\frac{b}{a}$ and the product of the roots, $x_1 \cdot x_2 = \frac{c}{a}$

Example 5

The roots of a quadratic equation $3x^2 - 5x + 2 = 0$ are x_1 and x_2.
Without solving the equation, find:

a $\dfrac{1}{x_1} + \dfrac{1}{x_2}$ **b** $x_1^2 + x_2^2$ **c** $\dfrac{2}{x_1^3} + \dfrac{2}{x_2^3}$

Answers

a $\dfrac{1}{x_1} + \dfrac{1}{x_2} = \dfrac{x_2 + x_1}{x_1 \cdot x_2}$

$= \dfrac{\frac{5}{3}}{\frac{2}{3}} = \dfrac{5}{2}$

Apply the theorem:

$x_1 + x_2 = \dfrac{5}{3}$ *and* $x_1 \cdot x_2 = \dfrac{2}{3}$

b $x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2$

$= \left(\dfrac{5}{3}\right)^2 - 2 \cdot \dfrac{2}{3}$

$= \dfrac{25}{9} - \dfrac{4}{3} = \dfrac{13}{9}$

Use the binomial formula
$(A + B)^2 \equiv A^2 + 2AB + B^2$

c $\dfrac{2}{x_1^3} + \dfrac{2}{x_2^3}$

$= 2 \cdot \dfrac{x_2^3 + x_1^3}{x_1^3 x_2^3}$

$= 2 \cdot \dfrac{(x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2)}{(x_1x_2)^3}$

Use the binomial formula
$(A + B)^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3$

$= 2 \cdot \dfrac{\left(\frac{5}{3}\right)^3 - 3 \cdot \frac{2}{3} \cdot \frac{5}{3}}{\left(\frac{2}{3}\right)^3}$

$= 2 \cdot \dfrac{\frac{125}{27} - \frac{10}{3}}{\frac{8}{27}} = 2 \cdot \dfrac{\frac{35}{27}}{\frac{8}{27}} = \dfrac{35}{4}$

The binomial formula is discussed in Section 1.8

Exercise 3C

1 Given a quadratic equation whose roots are x_1 and x_2, find the indicated expression without solving the equation.

a $x^2 - 3x + 2 = 0,\ \dfrac{2}{x_1} + \dfrac{2}{x_2}$

b $3x^2 - 5x + 1 = 0,\ 3x_1^2 + 3x_2^2$

c $5x^2 + x + 3 = 0,\ \dfrac{1}{x_1^2} + \dfrac{1}{x_2^2}$

d $x^2 - 2x + 4 = 0,\ (x_1 - x_2)^2$

e $2x^2 - 4x + 3 = 0,\ x_1^3 + x_2^3$

f $x^2 + 3x + 1 = 0,\ \dfrac{1}{x_1^4} + \dfrac{1}{x_2^4}$

g $4x^2 - 7x + 1 = 0,\ x_1^3x_2^2 + x_1^2x_2^3$

h $7x^2 + 4x - 5 = 0,\ (x_1 - x_2)^4$

Algebraic vs. geometric introduction to complex numbers

Algebraic approach

Historically, complex numbers were first encountered when solving cubic equations. However, in modern mathematics, these numbers appear naturally as solutions of quadratic equations as we shall see in this section.

Since the square of a real number is always a non-negative number, a quadratic equation of the form $x^2 = c$, $c \in \mathbb{R}^-$ has no real solution. If you say that the simplest such equation $x^2 = -1$ has solutions you can develop a whole new algebra starting from $x = \pm\sqrt{-1}$.

The first person to mention the square root of a negative number was **Heron of Alexandria** (c.10–c.60. CE) when discussing the volume of frustum of a pyramid whose side lengths were impossible.

In medieval Italy, mathematical tournaments were very popular and solving cubic equations distinguished the winners. These mathematicians discovered the formula for solutions of cubic equations and basically introduced complex numbers.

Scipione dal Ferro (1465–1526) solved a cubic equation with no quadratic term which helped **Niccolò Fontana Tartaglia** (1499–1557) to discover the formula. He shared his knowledge with **Gerolamo Cardano** (1501–1576) who published it in his algebra book *Ars Magna*. Cardano introduced complex numbers of the form $a+\sqrt{-b}$, $a \in \mathbb{R}$, $b \in \mathbb{R}^+$. Mathematicians realised that the two parts could not be combined and the second part was called an imaginary or even impossible part.

René Descartes (1596–1650) was the first person to establish the term imaginary part and **John Wallis** (1616–1703) made huge progress in giving a geometric interpretation to $\sqrt{-1}$.

Leonhard Euler (1707–1783) was the first mathematician to use the symbol $i = \sqrt{-1}$ and he called it an **'imaginary unit'**.

Dose the terminology 'complex' and 'imaginary' make these numbers seem unnatural? Are they simply the inventions of mathematical minds?

Today complex numbers are used in many real world applications.

You can write all the solutions of the equation $x^2 = c$, $c \in \mathbb{R}^-$ as $x = \pm i\sqrt{-c} = \pm\, id$, $d \in \mathbb{R}^+$. Numbers like $\pm\, id$ are purely imaginary number.

$c = -1 \times -c$

$\sqrt{c} = \sqrt{-1 \times -c}$

$= \pm i\sqrt{-c}$

c is negative so $-c$ is positive and has a real square root.

$\pm\sqrt{-c} = \pm\sqrt{d}$ where $d \in \mathbb{R}^+$

Complex numbers have the form $z = a + ib$, $a, b \in \mathbb{R}$, where a is called a **real part**, $\text{Re}(z) = a$, and b is called an **imaginary part**, $\text{Im}(z) = b$, of the complex number z.

→ When $b = 0$, $z = a + i \cdot 0 = a$. Since the complex number does not have a part containing i, it reduces to a real number. Similarly, when $a = 0$, $z = 0 + ib = ib$. Since the complex number has only a part containing the imaginary unit i, it is called a purely imaginary number.

Example 6

Find the real and imaginary parts of these complex numbers.

a $z = 3 + 2i$ **b** $z = 5i - 4$

c $z = -\frac{2}{3} + \sqrt{3}i$ **d** $z = \frac{\sqrt[3]{11}i - \sqrt{11}}{\pi}$

Answers

a $z = 3 + 2i \Rightarrow \begin{cases} \text{Re}(z) = 3 \\ \text{Im}(z) = 2 \end{cases}$

b $z = 5i - 4 \Rightarrow \begin{cases} \text{Re}(z) = -4 \\ \text{Im}(z) = 5 \end{cases}$

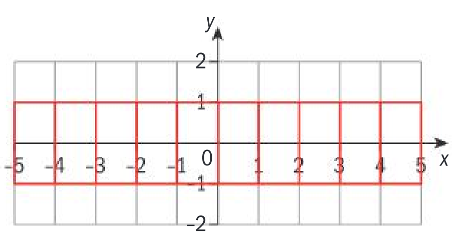
c $z = -\frac{2}{3} + \sqrt{3}i \Rightarrow \begin{cases} \text{Re}(z) = -\frac{2}{3} \\ \text{Im}(z) = \sqrt{3} \end{cases}$

d $z = \frac{\sqrt[3]{11}i + \sqrt{23}}{\pi} \Rightarrow \begin{cases} \text{Re}(z) = \frac{\sqrt{23}}{\pi} \\ \text{Im}(z) = \frac{\sqrt[3]{11}}{\pi} \end{cases}$

Geometric approach

Real numbers can be visualised on the number line that was introduced by John Wallis. Each point on the line represents one real number. In order to have numbers other than real numbers, we need to expand the line into the second dimension, which results in the complex plane.

The first person to set up the plane model of complex numbers was **Jean-Robert Argand** (1768–1822). **Carl Friedrich Gauss** (1777–1855) independently developed and refined the plane model and therefore the geometrical visualization of complex numbers in a plane is known as an **Argand diagram** or **Gaussian plane**.

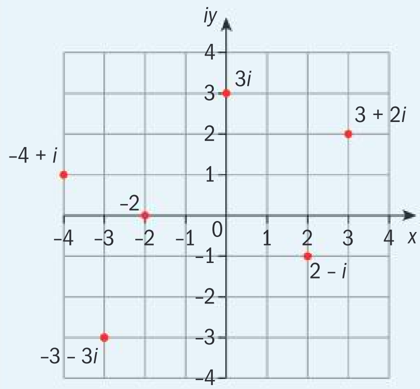


The complex plane is a two-dimensional coordinate plane where the usual coordinate axes x and y are now called the real and imaginary axes respectively. Each complex number $z = x + iy$ is represented by a point P(x, y) in the plane where the coordinates are the real and imaginary parts of the complex number itself.

Example 7

Plot these complex numbers in the Argand diagram.
$3 + 2i$, $2 - i$, $-3 - 3i$, $-4 + i$, $3i$ and -2.

Answer

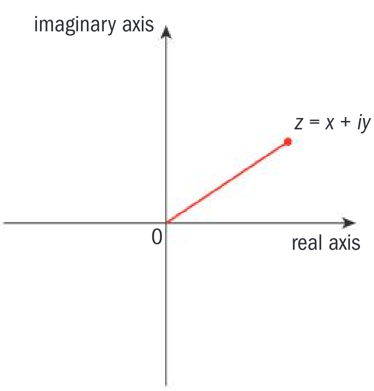


The real part is measured along the real axis (horizontal axis) and the imagniary part along the imaginary axis (vertical axis).

Modulus of a complex number

You saw in Chapter 2 that the modulus, or absolute value, of a real number was algebraically defined as $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. Geometrically it represents the distance from the number x on the number line to the origin 0. You can extend this idea to complex numbers: the modulus of a complex number $|z|$ is the distance from the point P(x, y) (which represents the complex number $z = x + iy$) to the origin (0, 0) in the complex plane.

To find the distance between two points in a coordinate plane use Pythagoras theorem.



$$|z| = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2} = \sqrt{\mathrm{Re}^2(z) + \mathrm{Im}^2(z)}$$

The geometric interpretation will be discussed further in Chapter 12.

→ $|z| = |x + iy| = \sqrt{x^2 + y^2}$

Example 8

Find the modulus of these complex numbers.

a $3 - 4i$ **b** $-7 + \sqrt{11}i$ **c** $\dfrac{-5 - 12i}{13}$

Answers

a $|3 - 4i| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

b $|-7 + \sqrt{11}i| = \sqrt{(-7)^2 + (\sqrt{11})^2} = \sqrt{49 + 11} = \sqrt{60} = 2\sqrt{15}$

c $\left|\dfrac{-5 - 12i}{13}\right| = \sqrt{\left(-\dfrac{5}{13}\right)^2 + \left(-\dfrac{12}{13}\right)^2} = \sqrt{\dfrac{25 + 144}{169}} = \sqrt{\dfrac{169}{169}} = 1$

Exercise 3D

1 Find the real and imaginary parts of these complex numbers.

a $z = 3i$ **b** $z = -7$ **c** $z = \dfrac{18 - 12i}{8}$

d $z = \dfrac{11}{4} + i\dfrac{\sqrt{7}}{5}$ **e** $z = \dfrac{4i - 2}{3\pi^2}$

2 Find the modulus of these complex numbers.

a $12 + 5i$ **b** $-24 - 7i$ **c** $2\sqrt{2} + i\sqrt{5}$

d $\dfrac{-21 + 20i}{29}$ **e** $\dfrac{-3 + 4i}{\pi}$

3.2 Operations with complex numbers

Two complex numbers are **equal** if, and only if, their **real** and **imaginary parts** are **equal**.

So given that $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ and $a_1, b_1, a_2, b_2 \in \mathbb{R}$

$(z_1 = z_2) \Leftrightarrow (a_1 = a_2 \text{ and } b_1 = b_2)$

or

$(z_1 = z_2) \Leftrightarrow (\text{Re}(z_1) = \text{Re}(z_2) \text{ and } \text{Im}(z_1) = \text{Im}(z_2))$

Why is it not possible to define inequality relations (<, >) on complex numbers? Find reasons to declare the following statements false:

- $i > 0$
- $i < 0$

Addition and subtraction of complex numbers

The addition of complex numbers is defined in a very natural way:

→ $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Likewise,

→ $z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

Multiplication of complex numbers by a real number

To multiply a complex number by a real number use the distributive property.

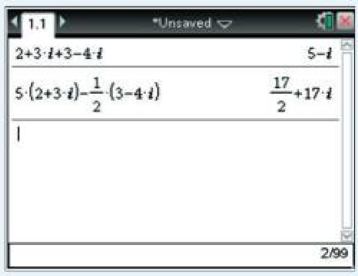
→ $\lambda z = \lambda(a + ib) = (\lambda a) + i(\lambda b),\ a, b, \lambda \in \mathbb{R}$

Example 9

If $z_1 = 2 + 3i$ and $z_2 = 3 - 4i$, calculate these and check your answers with a GDC.

a $z_1 + z_2$ **b** $5z_1 - \frac{1}{2}z_2$

Answers



a $z_1 + z_2 = 2 + 3i + 3 - 4i$

$= (2 + 3) + (3 - 4)i = 5 - i$

b $5z_1 - \frac{1}{2}z_2 = 5(2 + 3i) - \frac{1}{2}(3 - 4i)$

$= 10 + 15i - \frac{3}{2} + 2i$

$= \frac{17}{2} + 17i$

Multiplication of complex numbers

Use the distributive property and the fact that $i^2 = -1$ to multiply two complex numbers.

→ $z_1 \cdot z_2 = (a_1 + ib_1)\cdot(a_2 + ib_2) = a_1a_2 + ib_1a_2 + a_1ib_2 + \underbrace{i^2}_{-1} b_1b_2 sdf$

$= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

This formula is not simple to memorize. In practice it is easier to apply the distributive property each time when multiplying complex numbers

Example 10

Given that $z_1 = 2 + 3i$, $z_2 = 3 - 4i$ and $z_3 = 1 - i$, calculate these and check your answers with a GDC.

a $z_1 \cdot z_2$

b $z_1 \cdot z_3 - 3z_2$

Answers

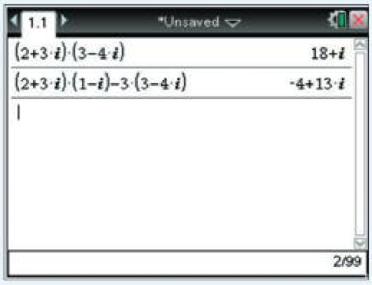
a $z_1 \cdot z_2 = (2 + 3i)\cdot(3 - 4i)$

$= 6 + 9i - 8i - 12i^2 = 6 + i - 12 \cdot -1$

$= 18 + i$

Continued on next page

b $z_1 \cdot z_3 - 3z_2 = (2 + 3i) \cdot (1 - i) - 3(3 - 4i)$
$= 2 - 2i + 3i - 3i - 9 + 12i = -7 + 13i - 3 \cdot -1 = -4 + 13i$



Exercise 3E

1 Given that $z_1 = 2 + 3i$, $z_2 = \frac{3}{2} - 4i$, $z_3 = 1 - 5i$ and $z_4 = \frac{3+4i}{5}$,

calculate these and check your answers with a GDC.

a $z_1 + z_3$ **b** $z_1 - 2z_2$ **c** $z_2 + z_4$ **d** $5z_4 - 2z_2$

e $3z_1 + 4z_2 - z_3 - 5z_4$ **f** $z_1 \cdot z_2 - z_3 \cdot z_4$ **g** $z_3^{\,2} - \frac{2}{3} z_2 \cdot z_4$

Example 11

Find a complex number z that satisfies $(4 - 2i) \cdot z = 3z + 2 - 5i$.

Answer

Let $z = a + ib$

$\Rightarrow (4 - 2i) \cdot (a + ib) = 3(a + ib) + 2 - 5i$ *Expand.*

$\Rightarrow 4a - 2ai + 4bi + 2b = 3a + 3bi + 2 - 5i$ *Collect the real and imaginary parts.*

$\Rightarrow (4a + 2b) + (-2a + 4b)\,i = (3a + 2) + (3b - 5)i$ *The real and imaginary parts are equal so set up a pair of simultaneous equations.*

$\Rightarrow \begin{cases} 4a + 2b = 3a + 2 \\ -2a + 4b = 3b - 5 \end{cases} \Rightarrow \begin{cases} a + 2b = 2 \\ -2a + b = -5 \end{cases}$ *Solve the simultaneous equations.*

$\Rightarrow \begin{cases} a = 2 - 2b \\ -2(2 - 2b) + b = -5 \end{cases}$ *Apply the method of substitution.*

$\Rightarrow \begin{cases} a = 2 - 2b \\ -4 + 4b + b = -5 \end{cases} \Rightarrow \begin{cases} a = 2 - 2b \\ 5b = -1 \end{cases}$

$\Rightarrow \begin{cases} a = 2 - 2 \cdot \left(-\frac{1}{5}\right) \\ b = -\frac{1}{5} \end{cases} \Rightarrow \begin{cases} a = \frac{12}{5} \\ b = -\frac{1}{5} \end{cases} \Rightarrow z = \frac{12}{5} - \frac{1}{5}i$

$a = 2 - 2b$
$= 2 + \frac{2}{5} = \frac{12}{5}$

Solve this problem by using the equality of two complex numbers.

Remember to write down the final answer in the form asked for in the question, especially when solving long questions involving many different parts.

Conjugate complex numbers

Two complex numbers are said to be a conjugate pair if they have equal real parts and opposite sign imaginary parts.
If $z = a + ib$ then its conjugate is $z^* = a - ib$

The conjugate of the number z is denoted by z^*.

Example 12

Given the complex number $z = a + ib$, find:

a $z + z^*$ **b** $z - z^*$ **c** $z \cdot z^*$

Answers

a $z + z^* = a + ib + a - ib = 2a$

b $z - z^* = (a + ib) - (a - ib)$
$= a + ib - a + ib = 2ib$

c $z \cdot z^* = (a + ib) \cdot (a - ib)$
$= (a)^2 - (ib)^2$
$= a^2 - \underbrace{i^2}_{-1} b^2 = a^2 + b^2$

Apply the formula
$(A + B) \cdot (A - B) = A^2 - B^2$

$z + z^* = 2a \in \mathbb{R}$ and $z \cdot z^* \in \mathbb{R}$, $a^2 + b^2 \geq 0$.

Conjugate complex numbers have these properties:

i $(z^*)^* = z$

ii $(z_1 + z_2)^* = z_1^* + z_2^*$

iii $(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$

iv $z \cdot z^* = |z|^2$

v $(z^n)^* = (z^*)^n$, $n \in \mathbb{Z}$

The first four properties can be easily proved. You are asked to do this in Exercise 3F. The fifth property can be proved using repeated application of property **iii**. In Chapter 12 you will see a simpler way of finding powers of complex numbers.

Division of complex numbers

You can divide complex numbers using several of the properties that you have learnt so far.

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2}$$

Multiply the numerator and denominator by the conjugate of the denominator.

$$= \frac{a_1a_2 + ib_1a_2 - a_1ib_2 - \overbrace{i^2}^{-1} b_1b_2}{a_2^2 + b_2^2}$$

Multiply the numerators and notice that the denominator becomes a positive real number.

$$= \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}$$

Separate the real and imaginary parts.

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}$$

Collect like parts in the numerator.

Again notice that this formula is not very simple. In practise it is easier to apply this method each time when dividing complex numbers.

→ The division formula can be written in the form $\frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{|z_2|^2}$

Example 13

Given that $z_1 = 5 + 5i$, $z_2 = 1 + 2i$ and $z_3 = 3 - 2i$, calculate these and check your answers with a GDC.

a $\dfrac{z_1}{z_2}$ **b** $\dfrac{z_1^{\,2}}{z_2 \cdot z_3^*}$

Answers

a $\dfrac{z_1}{z_2} = \dfrac{5+5i}{1+2i} \cdot \dfrac{1-2i}{1-2i}$

Multiply the numerator and denominator by the conjugate of the denominator.

$= \dfrac{5+5i-10i-10i^2}{1^2+2^2}$

Expand the numerator. Expand the denominator by using the difference of two squares.

$= \dfrac{15-5i}{5} = 3 - i$

Simplify.

b $\dfrac{z_1^2}{z_2 \cdot z_2^*} = \dfrac{(5+5i)^2}{(1+2i)\cdot(3+2i)}$

Expand the numerator and denominator.

$= \dfrac{25+50i+25i^2}{3+2i+6i+4i^2}$

$= \dfrac{50i}{-1+8i} \cdot \dfrac{-1-8i}{-1-8i}$

Multiply the numerator and denominator by the conjugate of the denominator.

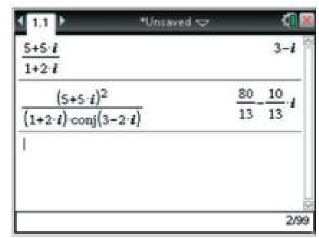
$= \dfrac{50(-i-8i^2)}{(-1)^2+8^2}$

Expand the numerator and denominator.

$= \dfrac{50(8-i)}{65} = \dfrac{10(8-i)}{13}$

Simplify.

$= \dfrac{80}{13} - \dfrac{10}{13}i$



Once you know how to divide two complex numbers you can solve linear equations in complex numbers.

Example 14

Find the complex number z that satisfies $\dfrac{z+1}{3+i} = \dfrac{z-5i}{2i-1}$

Answer

$\dfrac{z+1}{3+i} = \dfrac{z-5\text{i}}{2i-1} \Rightarrow (z+1)(2i-1) = (z-5i)(3+i)$

$\Rightarrow z(2i - 1) + (2i - 1) = z(3 + i) - 5i(3 + i)$

$\Rightarrow z(2i - 1) - z(3 + i) = -2i + 1 - 15i + 5$

$\Rightarrow z(2i - 1 - 3 - i) = -17i + 6$

$\Rightarrow z(-4 + i) = 6 - 17i$

$z = \dfrac{6-17i}{-4+i} \cdot \dfrac{-4-i}{-4-i} \quad \Rightarrow z = \dfrac{-24+68i-6i-17}{16+1} = \dfrac{-41+62i}{17}$

Exercise 3F

1 Given that $z_1 = 1 + 4i$, $z_2 = 2 - i$, $z_3 = \frac{1}{2} - \frac{5}{2}i$ and $z_4 = \frac{2i-1}{3}$,

Calculate these quotients and check your answers with a GDC.

a $\frac{z_1}{z_2}$ **b** $\frac{z_1^*}{z_1}$ **c** $\frac{z_2 \cdot z_4}{z_3}$ **d** $\frac{3z_1 - 2z_3}{z_2 + 3z_4}$ **e** $\frac{z_1^2}{\left(z_2^*\right)^2}$

2 Find the real numbers a and b that satisfy these equations.

a $(2 + i)(a + ib) = 11 - 2i$ **b** $\frac{a+ib}{2-5i} = -3+2i$

c $(3i - 2)(a + ib) = 3 + 28i$ **d** $\left(\frac{1}{2}+\frac{3}{4}i\right)(a+ib) = -3+2i$

3 Find the real and imaginary parts of these numbers.

a $\frac{3-2i}{4}$ **b** $\frac{5i-2}{3i}$ **c** $\frac{1}{3i}+\frac{2}{1+i}$ **d** $\frac{2-3i}{2+3i}-\frac{2+3i}{2-3i}$

4 Given the numbers $z_1 = 1 + 3i$ and $z_2 = 3 - i$, find:

a $z_1 \cdot z_2 + z_1 \cdot z_2^*$ **b** $z_1 \cdot z_2 - z_1^* \cdot z_2$ **c** $z_1 \cdot z_2 + (z_1 \cdot z_2)^*$

5 Find the complex number z that satisfies these equations.

a $(z + 1)i = (z + 2i)(3 + 2i)$ **b** $(2z - 1)(1 + i) = (z - 1)(2 + 3i)$

c $\frac{z-3i+2}{4+3i} = \frac{z-1}{1+i}$ **d** $\frac{3z-2i}{2+i} = \frac{2z+5}{10+15i}$

6 What conditions must the real and imaginary parts of a complex number z satisfy so that $\frac{z}{2-7i} \in \mathbb{R}$?

7 What conditions must the real and imaginary parts of a complex number z satisfy so that $\frac{3-5i}{z^*}$ is purely imaginary?

8 Solve for $z \in \mathbb{C}$:

a $|z| - z = 4 + 3i$ **b** $|z| + iz = 2 - i$ **c** $z^2 - z^* = 0$

9 Prove these properties of the modulus of a complex number.

a $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ **b** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

c $|z^n| = |z|^n$ **d** $|z_1 + z_2| \le |z_1| + |z_2|$

10 Prove these properties of conjugate complex numbers.

a $(z^*)^* = z$ **b** $(z_1 + z_2)^* = z_1^* + z_2^*$ **c** $(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$

d $z \cdot z^* = |z|^2$ **e** $|z| = |z^*|$

This table lists the fundamental properties or **axioms** of the operations on complex numbers. Other properties can be derived from these properties. The first four axioms refer to addition and the next four to multiplication, while the final axiom refers to both operations. 0 and 1 are real numbers but can be seen as complex, that is; $0 = 0 + 0i$ and $1 = 1 + 0i$.

→ **Axioms of complex numbers**

A1 For every complex numbers z_1 and z_2 then $z_1 + z_2$ is a complex number (Closure)

A2 For every complex numbers z_1 and z_2 then $z_1 + z_2 = z_2 + z_1$ (Commutativity)

A3 For every complex numbers z_1, z_2 and z_3 then $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associativity)

A4 There exists a complex number $0 = 0 + 0i$ such that for every complex number z, $0 + z = z + 0 = z$ (Additive identity)

A5 For every complex number z there exists a complex number $-z$ such that $z + -z = -z + z = 0$ (Additive inverse)

A6 For every complex numbers z_1 and z_2 then $z_1 \cdot z_2$ is a complex number (Closure)

A7 For every complex numbers z_1 and z_2 then $z_1 \cdot z_2 = z_2 \cdot z_1$ (Commutativity)

A8 For every complex numbers z_1, z_2 and z_3 then $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$ (Associativity)

A9 There exists a complex numbers $1 = 1 + 0i$ such that for every complex numbers z, $1 \cdot z = z \cdot 1 = z$ (Multiplicative identity)

A10 For every complex numbers z, $z \neq 0$, there exists a complex numbers z^{-1} such that $z \cdot z^{-1} = z^{-1} \cdot z = 0$ (Multiplicative inverse)

A11 For every complex numbers z_1, z_2 and z_3 then $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$ (Distributivity of multiplication over addition)

A structure in which addition and multiplication are defined and satisfy certain rules (shown left) is called the **field of complex numbers**. Since all real numbers can also be seen as complex and they satisfy the axioms, there is also a structure called the **field of real numbers**.

Investigation – axioms of a field

Decide if these sets of numbers satisfy the axioms of a field **A1-A11** given above.

a The integers, $\mathbb{Z}$

b The rational fractions, $\mathbb{Q}$

c The reals, $\mathbb{R}$

d Numbers of the form $p + q\sqrt{2}$ where p and q are rational fractions.

Investigation – further properties of complex numbers

Starting from the axioms of a field show these results.

a The additive and multiplicative identities 0 and 1 are unique.

b $-(-z) = z$ and $(z^{-1})^{-1} = z$ for any complex number z

c $0 \cdot z = z \cdot 0 = 0$ for any complex number z (Hint: consider $z \cdot (1 + 0)$)

d $(-z_1) \cdot z_2 = -(z_1 \cdot z_2)$ for any complex numbers z_1 and z_2

e $-z_1 \cdot -z_2 = z_1 \cdot z_2$ for any complex numbers z_1 and z_2

suppose they are not

consider $(z_1 + -z_1) \cdot z_2$

Powers and roots of complex numbers

To find powers and roots of complex numbers, you use the binomial theorem and powers of the imaginary unit, i.

Investigation – sum of powers of complex numbers

Calculate i^n, $n = 0, 1, 2, 3, \ldots$
Find a general rule for i^n, $n \in \mathbb{N}$. Use your general rule to find i^{2012}.
Use the properties of negative powers to find a general rule for i^n, $n \in \mathbb{Z}$.
Use the results you found to investigate the these:

a $\sum_{k=1}^{n} i^k, n > k$ **b** $\sum_{k=1}^{n} i^{-k}, n > k$

c $\sum_{k=1}^{n} i^k, n > k$ **d** $\sum_{k=1}^{n} i^{-k}, n > k$

Use the Σ-notation for a sum. Similarly, there is a product notation.

$\prod_{k=1}^{n} i^k = i^1 \cdot i^2 \cdot \ldots \cdot i^n$

For verification of the general rule for i^n, refer to the summary at the end of the chapter.

Example 15

Given the complex number $z = 1 - 2i$, find: **a** z^3 **b** $(z^5)^*$ **c** $(z^*)^5$
Check your answers using a GDC.

Answers

a $z^3 = (1 - 2i)^3$
$= 1^3 - 3 \cdot 1^2 \cdot 2i + 3 \cdot 1 \cdot (2i)^2 - (2i)^3$
$= 1 - 6i - 12 + 8i$
$= -11 + 2i$

Use the binomial theorem.
Use $i^2 = -1$ and $i^3 = -1$

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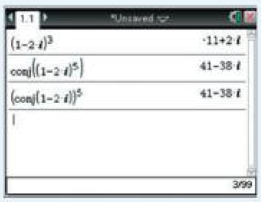
The binomial theorem states that $(a + x)^n \equiv \sum_{r=0}^{n} \binom{n}{r} a^{n-r} x^r$

b $z^5 = z^2 \cdot z^3$
$= (1 - 2i)^2 \cdot (-11 + 2i)$
$= (1 - 4i - 4) \cdot (-11 + 2i)$
$= (-3 - 4i) \cdot (-11 + 2i)$
$= 33 - 6i + 44i + 8 = 41 + 38i$
$\Rightarrow (z^5)^* = 41 - 38i$

Simplify the calculation by finding z^5 using the answer to part **a** *and then finding its conjugate. Use the square of a difference.*

c $(z^*)^5 = (1 + 2i)^5$
$= 1^5 + 5 \cdot 1^4 \cdot 2i + 10 \cdot 1^3 \cdot (2i)^2 + 10 \cdot 1^3 \cdot (2i)^3 + 5 \cdot 1 \cdot (2i)^4 + (2i)^5$
$= 1 + 10i - 40 - 80i + 80 + 32i$
$= 41 - 38i$

Use the binomial theorem.
Use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i$



Notice that the results in **b** and **c** are equal, that is, $(z^*)^n = (z^n)$, $n \in \mathbb{Z}$ as stated in the properties of conjugate complex numbers, that is, $(z^*)^n = (z^n)^*$, $n \in \mathbb{Z}$.

You can find the square roots of a complex number, z, by first squaring z so that you can work with the real and complex parts of $\sqrt{z}$ separately.

Example 16

Evaluate $\sqrt{8 - 6i}$.

Answer

Let $z = x + yi$, $x, y \in \mathbb{R}$ such that

$z = \sqrt{8-6i} \Rightarrow z^2 = 8 - 6i$
$(x + yi)^2 = 8 - 6i$
$x^2 + 2xyi - y^2 = 8 - 6i$

Expand z and use $i^2 = -1$
Equate the real and imaginary parts.

$$\begin{cases} x^2 - y^2 = 8 \\ 2xy = -6 \end{cases} \Rightarrow \begin{cases} x^2 - \left(-\dfrac{3}{x}\right)^2 = 8 \\ y = -\dfrac{3}{x} \end{cases}$$

Solve the simultaneous equations by using substitution.

$$\Rightarrow \begin{cases} x^2 - \dfrac{9}{x^2} = 8 \\ y = -\dfrac{3}{x} \end{cases} \Rightarrow \begin{cases} x^4 - 9 = 8x^2 \\ y = -\dfrac{3}{x} \end{cases}$$

$$\Rightarrow \begin{cases} x^4 - 8x^2 - 9 = 0 \\ y = -\dfrac{3}{x} \end{cases} \Rightarrow \begin{cases} (x^2 - 9)(x^2 + 1) = 0 \\ y = -\dfrac{3}{x} \end{cases}$$

Factorize the equation and apply the zero product property.

$$\Rightarrow \begin{cases} x = \pm 3 \\ y = -\dfrac{3}{\pm 3} \end{cases} \Rightarrow \begin{cases} x = \pm 3 \\ y = \mp 1 \end{cases}$$

Notice that ± 3 are the only real solutions for x.

$\Rightarrow z_1 = 3 - i$ and $z_2 = -3 + i$

A GDC will always give just one solution, but you need to be aware is that there another solution which is the negative of the number on the GDC.

The method shown in Example 16 for square roots is not an easy one and, for higher roots, algebraic skills are needed. In Chapter 12 you will learn a different method for finding roots of complex numbers.

Exercise 3G

1 Calculate:

a $i^5 + i^8 + i^{14} + i^{19}$ **b** $i^{123} + i^{172} + i^{256} + i^{375}$

c $(2 - i^{53}) \cdot (3 + 2i^{89})$ **d** $\dfrac{4i^{2010} - 3i^{2011}}{2i^{2012} + 5i^{2013}}$

e $\dfrac{i + i^2 + i^3 + \ldots + i^{2011}}{i \cdot i^2 \cdot i^3 \cdot \ldots \cdot i^{2011}}$ **f** $\dfrac{i^2 + i^4 + i^6 + \ldots + i^{2010}}{i^2 \cdot i^4 \cdot i^6 \cdot \ldots \cdot i^{2010}}$

2 Calculate these and check your answers with a GDC.

a $(2 + 3i)^2 + (1 - 4i)^2$ **b** $(3 + 2i)^2 + (3 - 2i)^2$

c $(3 + 2i)^3 + (3 - 2i)^3$ **d** $(1 + i)^4 + (1 - i)^4$

3 Evaluate these and check your answers with a GDC.

a $\sqrt{3+4i}$ **b** $\sqrt{12i-5}$ **c** $\sqrt{\frac{5}{4}+3i}$

d $\sqrt{\frac{55}{144}-\frac{1}{3}i}$ **e** $\sqrt{i}$ **f** $\sqrt{-i}$

4 Show that:

a $(1 + i)^{2n} = (2i)^n,\ n \in \mathbb{Z}$ **b** $(1 + i)^{2n+1} = (1 + i)(2i)^n,\ n \in \mathbb{Z}$

5 Given that $z = 1 - i$, find the values of $n \in \mathbb{N}$ such that:

a z^n is real **b** z^n is purely imaginary.

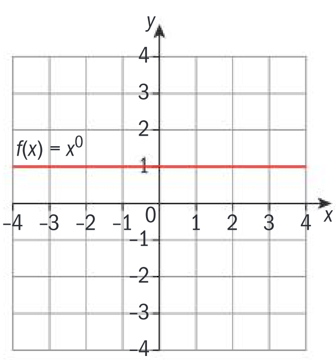
For question 5, it may help to plot $z, z^2, z^3, \ldots$ on an Argand diagram and look for a pattern.

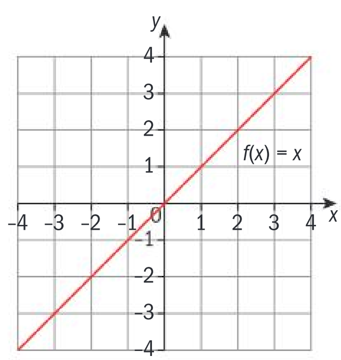
3.3 Polynomial functions: graphs and operations

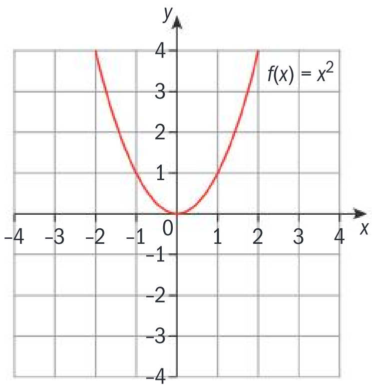
Historically, complex numbers appeared as roots of polynomial equations which means that they can be seen as zeros of polynomial functions. In this section we are going to study polynomial functions, their graphs and operations with the expressions that define them.

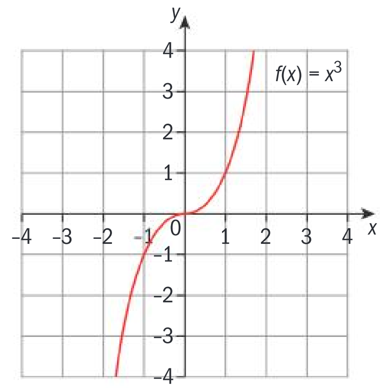
Polynomial functions and their graphs

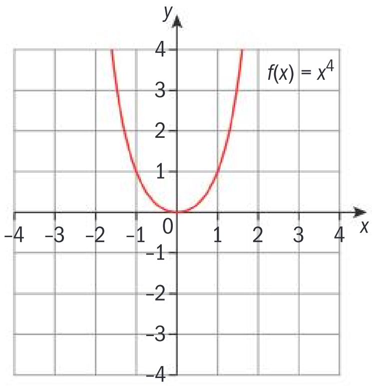
The diagrams show the graphs of $f(x) = x^n$, $n = 0, 1, 2, 3, \ldots$ where n is a natural number.

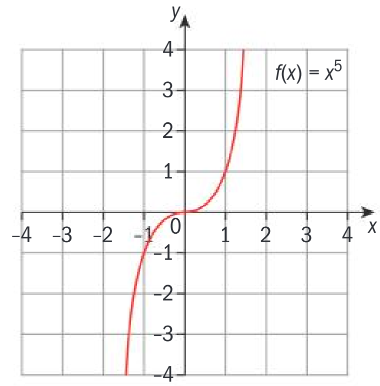


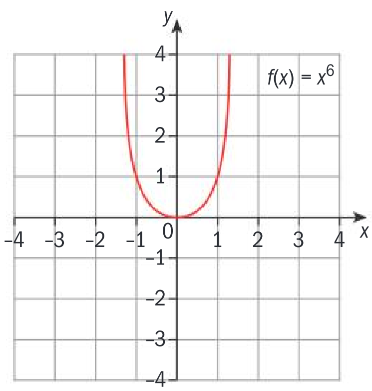


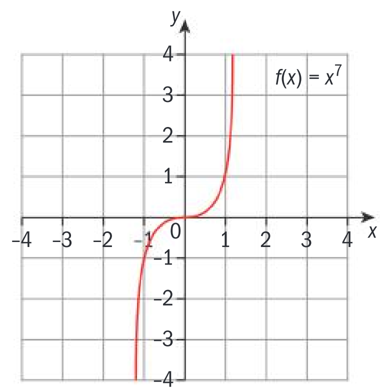












Apart from the first two powers of n, $n = 0$ and $n = 1$, notice that for:

- even powers $n = 2, 4, 6$ ... the graphs a 'U' shape
- odd powers $n = 3, 5, 7$ ... the graphs have a 'flex' shape.

The 'U' shape graph has a **local** minimum or maximum, while the 'flex' shape graph has a horizontal **inflexion**.

A linear combination of powers of x, for example $3 \cdot x^5 - 2 \cdot x^2 + 8x - 11$, is called a **polynomial**.

→ **A linear combination** of two functions f and g is an expression of the form $a \cdot f(x) + b \cdot g(x)$, where a and b are real numbers. A linear combination of n functions is an expression of the form $\sum_{k=1}^{n} a_k \cdot f_k(x)$, where f_k are functions and $a_k \in \mathbb{R}$.

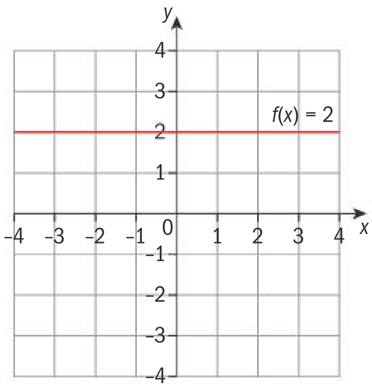
In general, polynomials can be seen as a linear combination of the power functions $\{1, x, x^2, x^3, x^4, x^5, \ldots\}$

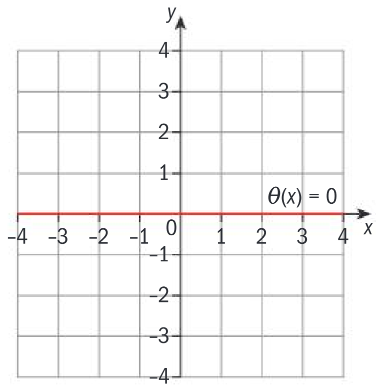
The word 'polynomial' means 'many terms'. A polynomial of one term is called a monomial, of two terms a binomial, and of three terms a trinomial.

Polynomials are real functions of the real variable $f: \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_k \in \mathbb{R}$, $k = 0, \ldots, n$ are called the coefficients. The highest power of the variable x^n is called the degree of the polynomial, $\deg(f) = n$.

Polynomials of degree 0, 1, 2 and 3

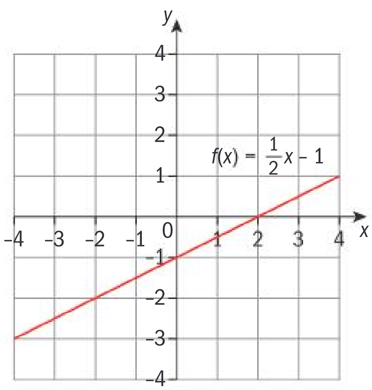
- **Constant function** $f(x) = c$, $c \in \mathbb{R}$. The graph is a horizontal line. The degree of a constant polynomial is zero.

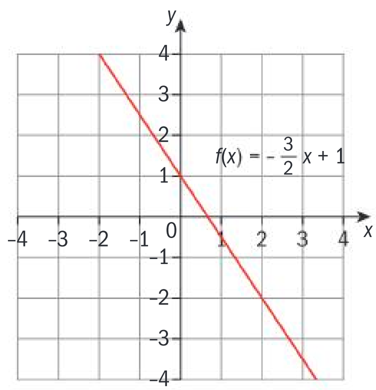




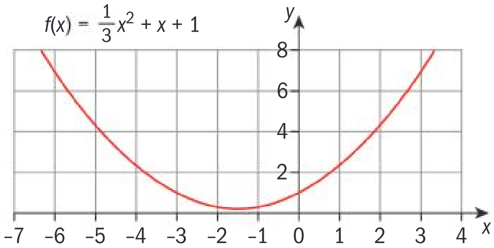
You use the notation $\theta(x) = 0$ for the zero polynomial to distinguish it from other polynomials. The zero polynomial also has an important property as an additive identity element for polynomials, that is, $f(x) + \theta(x) = \theta(x) + f(x) = f(x)$ for all polynomials f.

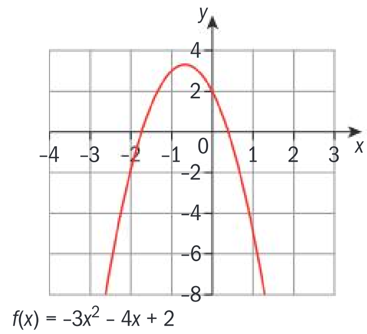
- **Zero polynomial** $\theta(x) = 0$. The graph is again a horizontal line but this time it is the x-axis itself.
- **Linear function** $f(x) = mx + c$, $m \neq 0$. This is a polynomial of the first degree. The graph is a straight line. By changing the parameters m and c, you change the steepness and the position of the line.





- **Quadratic function** $f(x) = ax^2 + bx + c$, $a \neq 0$. This is a polynomial of the second degree. The graph is a parabola, 'U' shaped, whose axis of symmetry is a vertical line. By changing the parameters a, b and c, you change the shape (wide or narrow), concavity (opens upwards or downwards) and position of the parabola.

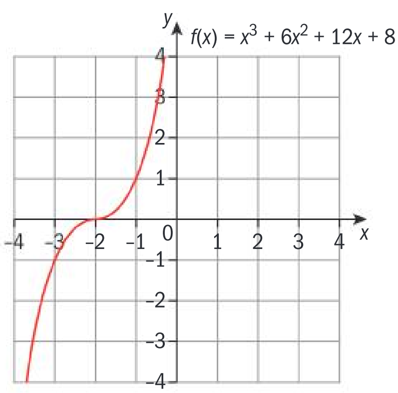


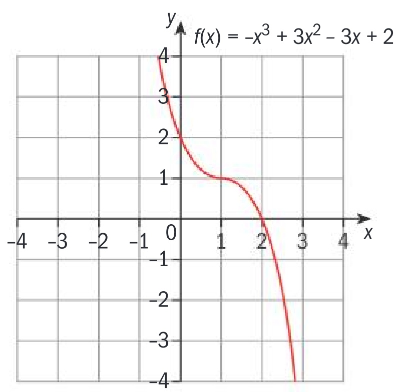


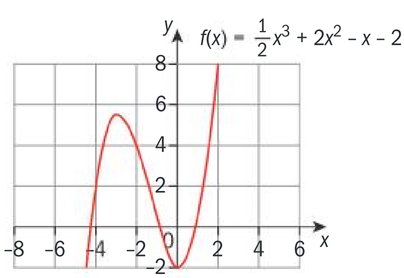
Investigation – parameters of parabolas

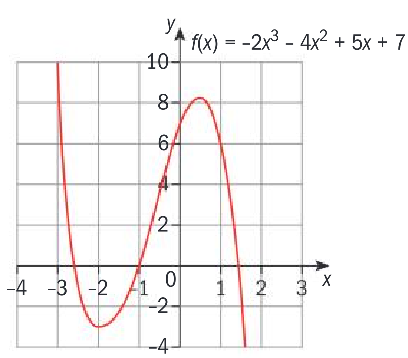
Given a quadratic function $f(x) + ax^2 + b(x) + c$, $a \neq 0$, investigate the effect of the parameters a, b and c on the shape and the position of the parabola in the coordinate system. In Chapter 2 you were investigating the form $f(x) + a(x - h)^2 + k$, $a \neq 0$, where h and k were horizontal and vertical translations, respectively. Use this to find the effect of the parameter b.

- **Cubic function**, $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. This is a polynomial of the third degree.







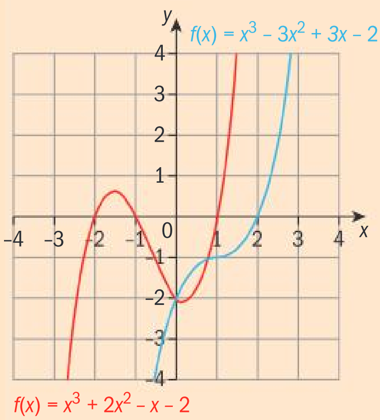


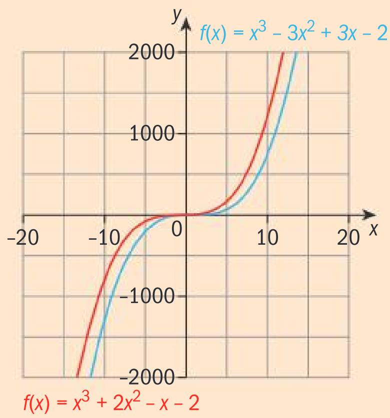
Cubic graphs have two different shapes. One shape looks like a 'flex' shape. The second shape is a combination of two 'U' shapes opening in opposite directions.

Investigation – parameters of cubics

Given a cubic function $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, investigate the effect of the parameters a, b, c and d on the shape and the position of the graphs. Start your investigation by taking two parameters at a time, for example a and b, c and c, a and d.

One interesting feature of polynomials of the same degree and with the same leading coefficient is that even though locally the graphs look very different if you change the scale on the axes they look very similar. For example, for a polynomial in x^3, $f(x)$ increases rapidly for large values of x.





The functions $f(x) = x^3 - 3x^2 + 3x - 2$ and $f(x) = x^3 + 2x^2 - x - 2$ behave like polynomials that have only the leading term, x^3, since for extremely large values of x, both positive and negative, the other terms are insignificant to the total value and can be neglected. This is the so-called 'end behavior' property of polynomials.

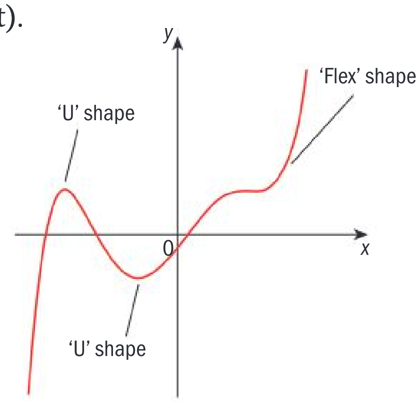
The end behavior of a polynomial function is determined by its degree and by the sign of its leading coefficients.

Polynomials are continuous functions, which means that you can draw their graphs without lifting the pen from the paper. You have to proceed in one direction (usually from left to right). Their graphs are also smooth curves with no sharp points.

Polynomials of degree 4 are called **quartic functions** and polynomials of degree 5 are called **quintic functions**. Special names are not usually used for polynomials of degree greater than 5.

In a graph of a polynomial of a higher degree you can see different types of 'U' and 'flex' shapes.

The graph shows a quintic polynomial.



Investigation – high–degree polynomials

1 Describe all the possible shapes of the graphs of polynomials of fourth degree. How many of each of the shapes of lower degree (2 and 3) can you have in those polynomials?

2 Describe all the possible shapes of the graphs of polynomials of fifth degree. How many of each of the shapes of lower degree (2, 3 and 4) can you have in those polynomials?

3 How many of each of the shapes of lower degree can you have in the polynomials of nth degree?

Operations with polynomials

Two polynomials $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_2x^2 + b_1x + b_0$ are **equal** if and only if:

i they have the **same degree**, $n = m$

ii all the **corresponding coefficients** are **equal**, $a_k = b_k$ for all $k = 0, 1, \ldots, n$.

Addition and multiplication of polynomials and multiplication by a real number follow the same rules for algebraic expressions that you met in the "Before you start" section.

→ The degree of a linear combination of two polynomials is not larger than the maximum of the degrees of either polynomial.
$\deg(\lambda \cdot f(x) + \mu \cdot g(x)) \le \max\{\deg(f(x)), \deg(g(x))\}$

Example 17

Given the polynomials $f(x) = 4x^4 + 3x^3 - 2x^2 + 6x - 2$ and $g(x) = 2x^3 - 5x^2 + x - 3$, find $f(x) \cdot g(x)$.

Answer

The standard algebraic method is difficult to follow because so many terms arise. It is usually easier to use the 'grid method'.

	$4x^4$	$3x^3$	$-2x^2$	$6x$	-2
$2x^3$	$8x^7$	$6x^6$	$-4x^5$	$12x^4$	$-4x^5$
$-5x^2$	$-20x^6$	$-15x^5$	$10x^4$	$-30x^3$	$10x^2$
x	$4x^5$	$3x^4$	$-2x^3$	$6x^2$	$-2x$
-3	$-12x^4$	$-9x^3$	$6x^2$	$-18x$	6

The 'grid method' makes it easier to simplify the like terms.

$f(x) \cdot g(x) = 8x^7 - 14x^6 - 15x^5 + 13x^4 - 45x^3 + 22x^2 - 20x + 6$

The grid method for multiplication is also knwon as the 'box method'.

→ The degree of the product of two polynomials is the sum of the degrees of the factor polynomials: $\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x))$

Example 18

Given the polynomials $f(x) = x^2 + ax - 3$ and $g(x) = x^2 - 4x + b$, find the values of the real parameters a and b such that $f(x) \cdot g(x) = x^4 - 22x^2 + 9$

Answer

$f(x) \cdot g(x) = (x^2 + ax - 3) \cdot (x^2 - 4x + b)$ — *Use distribution.*

$= x^4 + (a - 4)x^3 + (a + 12 + b)x^2 + (ab - 12)x - 3b$ — *Simplify.*

$= x^4 - 22x^2 + 9$

$$\Rightarrow \begin{cases} a - 4 = 0 \\ -4a - 3 + b = -22 \\ ab - 12 = 0 \\ -3b = 9 \end{cases}$$

Make the corresponding coefficients equal.

$$\Rightarrow \begin{cases} a = 4 \\ b = -3 \end{cases}$$

Check that the values of a and b satisfy all the equations.

Exercise 3H

1 The polynomials $f(x) = 2x^2 + 3x + 1$ and $g(x) = 3x^2 - 2x - 5$ are given. Find the real parameters λ and μ such that:

a $\lambda \cdot f(x) + \mu \cdot g(x) = 13x + 13$

b $\lambda \cdot f(x) + \mu \cdot g(x) = 26x^2 + 26x$

2 Use the 'grid method' to find the product of the polynomials f and g given that:

a $f(x) = x^3 - 2x$ and $g(x) = x^2 + 2$

b $f(x) = 27x^3 - 36x^2 + 48x - 64$ and $g(x) = 3x^2 + 7x + 4$

3 Given the polynomials $f(x) = ax^2 - 3x + 5$ and $g(x) = 7x^2 + bx - 3$, find the values of the real parameters a and b such that $f(x) \cdot g(x) = 14x^4 - 17x^3 + 23x^2 + 19x - 15$

4 Given the polynomials $f(x) = x^3 + ax^2 - x + 2$ and $g(x) = 2x^2 + bx + c$, find the values of the real parameters a, b and c such that $f(x) \cdot g(x) = 2x^5 - 5x^4 + 3x^3 + 5x^2 - 8x + 4$

5 Given that a polynomial $f(x) = x^4 + 6x^3 + ax^2 + bx + 4$ can be written in the form $f(x) = (x^2 + px + q)^2$, find the values of a and b and the polynomial in the required form.

6 Find the polynomial g such that $g(x) = f(x - 2)$, where $f(x) = x^3 + 12x^2 + 6x + 3$

7 Find the polynomial f such that $f(2x - 1) = 16x^4 - 32x^3 + 12x^2$

8 All the coefficients of the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ are positive integers smaller than 10. Find the polynomial given that $f(0) = 4$ and $f(10) = 32\,584$

Division of polynomials

You divide two polynomials using long division.

Example 19

Use long division to divide
$2x^4 + 4x^3 + 3x^2 + 2x - 7$
by
$x^2 + x + 2$

Answer

$$\begin{array}{r} 2x^2 + 2x - 3 \\ x^2+x+2\overline{)\,2x^4+4x^3+3x^2+2x-7} \\ -(2x^4+2x^3+4x^2) \\ \hline 2x^3 - x^2 + 2x \\ -(2x^3-2x^2+4x) \\ \hline -3x^2-2x-7 \\ -(-3x^2-3x-6) \\ \hline x-1 \end{array}$$

Divide x^2 into $2x^4$

Multiply divisor $2x^2$

Divide x^2 into $2x^3$

Multiply divisor by $2x$

Divide x^2 into $-3x^3$

Multiply divisor by -3

Remainder is $x - 1$

$$2x^4+4x^3+3x^2+2x-7 = (x^2+x+2)\cdot(2x^2+2x-3)+(x-1)$$

Stop when the degree of the remainder is smaller than the degree of the divisor.

The same algorithm is used to divide numbers. Consider $657 \div 21$

$$\begin{array}{r} 31 \\ 21\overline{)657} \\ -63 \\ \hline 27 \\ -21 \\ \hline 6 \end{array}$$

So $657 = 21.31 + 6$

→ Theorem

For any two polynomials f and g there are unique polynomials q and r such that $f(x) = g(x) \cdot q(x) + r(x)$, for all real values of x.

dividend = divisior· quotient + remainder

The polynomial q is called the quotient and the polynomial r is called the remainder. The degree of the polynomial r is smaller than the degree of the polynomial g.

The proof of this theorem uses the Euclidian algorithm that is part of the Discrete option.

Example 20

Use long division to find the quotient and remainder when dividing $f(x) = 2x^4 - 7x^3 - 7x^2 + 14x + 5$ by $g(x) = 2x + 3$

Answer

$$\begin{array}{r} x^3 - 5x^2 + 4x + 1 \\ 2x+3\overline{)\,2x^4 - 7x^3 - 7x^2 + 14x + 5} \\ \underline{-(2x^4 + 3x^3)} \\ -10x^3 - 7x^2 + 14x + 5 \\ \underline{-(-10x^3 - 15x^2)} \\ 8x^2 + 14x + 5 \\ \underline{-(8x^2 + 12x)} \\ 2x + 5 \\ \underline{-(2x + 3)} \\ 2 \leftarrow \text{remainder} \end{array}$$

So the quotient is $q(x) = x^3 - 5x^2 + 4x + 1$ and the remainder is $r(x) = 2$

Therefore,

$2x^4 - 7x^3 - 7x^2 + 14x + 5 = (2x + 3) \cdot (x^3 - 5x^2 + 4x + 1) + 2$

Exercise 3I

1 Use long division to divide f by g if:

a $f(x) = x^4 + 5x^3 + 8x^2 + 3x - 2$ and $g(x) = x + 2$

b $f(x) = x^5 + 3x^4 + x^3 - 4x^2 - 2x + 1$ and $g(x) = x^2 - 1$

c $f(x) = 2x^5 - 3x^4 + x^3 - 2x^2 + 3x - 1$ and $g(x) = x^2 + x + 1$

2 Use long division to find the quotient and remainder when f is divided by g given that:

a $f(x) = 2x^4 + 5x^3 + 4x^2 + 4x + 3$ and $g(x) = x + 1$

b $f(x) = 3x^4 + 4x^3 + 6x^2 - 2x + 6$ and $g(x) = x^2 + 2x + 3$

c $f(x) = x^6 + x - 1$ and $g(x) = x^2 + x + 1$

Polynomial remainder theorem

→ Given a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,\ a_k \in \mathbb{R},\ k = 0, 1, \ldots, n,\ a_n \neq 0$$

and a real number p, then the remainder when $f(x)$ is divided by a linear expression $(x - p)$ is $f(p)$.

Proof:

In the unique decomposition of the polynomial $f(x) = (x - p) \cdot q(x) + r$, where the remainder r is a constant (one degree less than the divisor) we input $x = p \Rightarrow f(p) = \underbrace{(p-p)}_{0}q(p) + r \Rightarrow f(p) = r$ QED

The polynomial remainder theorem is also known as 'Bézout's little theorem'. **Étienne Bézout** (1730–1837) was inspired by the work of Euler and so decided to become a mathematician. In 1763 he was appointed examiner of the Gardes de la Marine (French Naval Academy) with the special task of composing a textbook for teaching mathematics to the students.

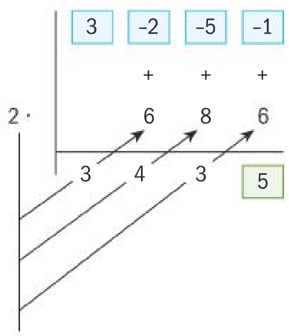
Factor theorem

> ➔ A polynomial
> $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,\ a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n,\ a_n \neq 0$
> has a factor $(x - p)$, $p \in \mathbb{R}$ if and only if $f(p) = 0$.

This theorem is a direct consequence of the remainder theorem. Its proof is left as an exercise for you.

To evaluate a polynomial for a certain value of the variable x, **William George Horner** (1786–1837) discovered an algorithm that can be used in many different cases.

If you want to find the value of $f(x) = 3x^3 - 2x^2 - 5x - 1$ when $x = 2$, select the coefficients of all terms, including missing terms, and organize them in a tabular form:



$f(x) = ((3 \cdot x - 2) \cdot x - 5) \cdot x - 1$
$f(2) = ((3 \cdot 2 - 2) \cdot 2 - 5) \cdot 2 - 1$
$= ((6 - 2) \cdot 2 - 5) \cdot 2 - 1$
$= (4 \cdot 2 - 5) \cdot 2 - 1$
$= (8 - 5) \cdot 2 - 1$
$= 3 \cdot 2 - 1$
$= 6 - 1$
$= 5$

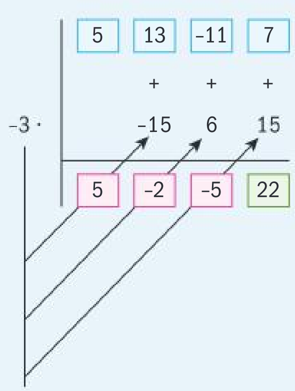
Example 21

Use Horner's algorithm to find the remainder when dividing $f(x) = 5x^3 + 13x^2 - 11x + 7$ by $g(x) = x + 3$

Answer

$x + 3 = x - (-3) \Rightarrow r = f(-3)$

Use the remainder theorem.



Use Horner's algorithm.

$f(-3) = ((5 \cdot -3 + 13) \cdot -3 - 11) \cdot -3 + 7$
$= ((-15 + 13) \cdot -3 - 11) \cdot -3 + 7$
$= (-2 \cdot -3 - 11) \cdot -3 + 7$
$= (6 - 11) \cdot -3 + 7$
$= -5 \cdot -3 + 7$
$= 15 + 7$
$= 22$

$r = f(-3) = 22$

Since the algorithm also gives the quotient you can use successive division to search for factors of a polynomial.

When you use Horner's algorithm, apart from getting the remainder (in the last row) you also obtain the coefficients of the quotient polynomial, $q(x) = 5x^2 - 2x - 5$. This is the reason why this algorithm is also known as synthetic division.

Investigate Horner's algorithm. Prove the general form of the algorithm and find in which other cases it can be used.

Example 22

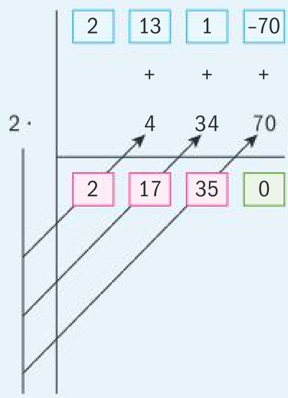
Show that $(x - 2)$ and $(x + 5)$ are factors of the polynomial
$f(x) = 2x^3 + 13x^2 + x - 70$

Answer

$x - 2 \Rightarrow r = f(2)$

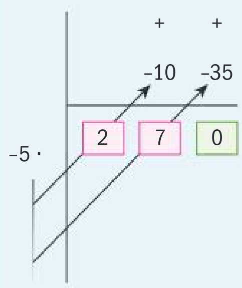
Use the remainder theorem and note that the remainders must be zeros for factors.

$x + 5 = x - (-5) \Rightarrow r = q(-5)$



Use Horner's algorithm.

$$\begin{aligned} f(2) &= ((2 \cdot 2 + 13) \cdot 2 + 1) \cdot 2 - 70 \\ &= ((4 + 13) \cdot 2 + 1) \cdot 2 - 70 \\ &= (17 \cdot 2 + 1) \cdot 2 - 70 \\ &= (34 + 1) \cdot 2 - 70 \\ &= 35 \cdot 2 - 70 \\ &= 70 - 70 \end{aligned}$$



Use Horner's algorithm.

$$\begin{aligned} q(x) &= 2x^2 + 17x + 35 \\ q(-5) &= (2 \cdot -5 + 17) \cdot -5 + 35 \\ &= (-10 + 17) \cdot -5 + 35 \\ &= 7 \cdot -5 + 35 \\ &= -35 + 35 \\ &= 0 \end{aligned}$$

Since the remainders are zeros you could proceed with the quotient polynomial as the quotient polynomial contains $x + 5$ as a factor. Notice that the last factor is $2x + 7$.

Exercise 3J

1 Use synthetic division to find the quotient and remainder when polynomial f is divided by g given that:

a $f(x) = x^3 - x^2 - 4x - 5$ and $g(x) = x - 3$

b $f(x) = 2x^3 + 5x^2 + 4x + 3$ and $g(x) = x + 1$

c $f(x) = x^5 - 3x^3 - 2x + 1$ and $g(x) = x + 2$

d $f(x) = 3x^6 - 2x^4 + 5x^2 - 2$ and $g(x) = x - 1$

2 Show that $(x - 2)$ and $(x + 3)$ are factors of $f(x) = 4x^4 - 27x^2 + 25x - 6$.

Corollary

Given a polynomial

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0,\ a_i\in\mathbb{R}, i=0,1,\ldots,n, a_n\neq 0$$

and real numbers a and b, $a \neq 0$, then the remainder when $f(x)$ is divided by a linear expression $(ax - b)$ is $f\left(\frac{b}{a}\right)$.

The proof can be conducted in a similar way to that of the theorem proof. The proof is left as an exercise for you.

A theorem easily derived from another theorem is a corollary of that theorem.

In order to use synthetic division when dividing by a linear expression $(ax - b)$ you have to modify the algorithm.

$f(x) = (ax - b) \cdot q(x) + r$

$$\Rightarrow f(x)=a\left(x-\frac{b}{a}\right)\cdot q(x)+r$$

$$\Rightarrow f(x)=\left(x-\frac{b}{a}\right)\cdot(a\cdot q(x))+r$$

Example 23

Use the synthetic division to find the quotient and remainder when dividing $f(x) = 2x^4 - 7x^3 - 7x^2 + 14x + 5$ by $g(x) = 2x + 3$

Answer

$$g(x)=2x+3=2\left(x-\left(-\frac{3}{2}\right)\right)$$

Use the remainder theorem.

Use synthetic division.

	2	-7	-7	14	5
		+	+	+	+
$-\frac{3}{2}\cdot$		-3	15	-12	-3
	2	-10	8	2	2

So the quotient is $q(x) = x^3 - 5x^2 + 4x + 1$ and the remainder is $r(x) = 2$.

The coefficients of the quotient polynomial were multiplied by 2, so you need to divide them by 2 to obtain the quotient polynomial.

Example 24

When polynomial $f(x) = x^3 - 2x^2 + ax + 11$ is divided by $(x - 2)$ the remainder is 1. Find the value of a.

Answer

Use synthetic division.

	1	-2	a	11
		+	+	+
2 ·		2	0	2a
	1	0	a	11 + 2a

$11 + 2a = 1 \Rightarrow 2a = -10 \Rightarrow a = -5$

Use the remainder r = 1

Example 25

Find the remainder when polynomial $f(x) = x^{2011} - 3x^2 + 2x - 2$ is divided by $x^2 - 1$.

Answer

$f(x) = (x^2 - 1) \cdot q(x) + \underbrace{ax + b}_{r(x)}$

Use the theorem on unique decomposition. Note that the remainder is linear.

$x = 1 \Rightarrow f(1) = 1^{2011} - 3 \cdot 1^2 + 2 \cdot 1 - 2 = -2$

$x = -1 \Rightarrow f(-1) = (-1)^{2011} - 3 \cdot (-1)^2 + 2 \cdot -1 - 2 = -8$

Calculate the value of the polynomial at the zeros of the divisor.

$$\begin{cases} f(1) = (1^2 - 1) \cdot q(1) + a \cdot 1 + b = -2 \\ f(-1) = ((-1)^2 - 1) \cdot q(-1) + a \cdot (-1) + b = -8 \end{cases}$$

Substitute f (1) = −2 and f (–1) = –8 in the unique decomposition.

$$\begin{cases} a + b = -2 \\ -a + b = -8 \end{cases} +$$

Solve the simultaneous equations by elimination.

$2b = -10 \Rightarrow b = -5$

$a - 5 = -2 \Rightarrow a = 3$

Therefore, the remainder is $r(x) = 3x - 5$

Exercise 3K

1 Use synthetic division to find the quotient and remainder when polynomial f is divided by g given that:

a $f(x) = 2x^5 - 3x^4 + 3x^3 + 3x^2 - 3$ and $g(x) = 2x - 1$

b $f(x) = 3x^4 + 4x^3 + 4x^2 - 2x + 6$ and $g(x) = 3x + 1$

2 When you divide the polynomial f by the polynomial $g(x) = x^2 + 2x - 1$ you obtain the quotient $q(x) = 3x - 4$ and the remainder $q(x) = x + 2$. Find the polynomial f.

3 Polynomial $f(x) = x^5 - 4x^4 + 3x^3 + 2x^2 - 3x + a$ is divisible by $(x - 3)$. Find the value of a.

4 Polynomial $f(x) = x^5 - 2x^4 + 2x^3 + bx - 1$ is divisible by $(x - 1)$. Find the value of b.

EXAM-STYLE QUESTIONS

5 Polynomial $f(x) = 4x^3 + 5x^2 + ax + b$ is divisible by $(x + 2)$, and when divided by $(x - 1)$ there is a remainder of 6. Find the values of a and b.

6 When polynomial f is divided by $(x - 3)$ the remainder is 2, and when divided by $(x + 1)$ the remainder is –4. Find the remainder when polynomial f is divided by $(x^2 - 2x - 3)$.

7 Find the remainder when $f(x) = x^{2011} + x^{2010} + \ldots + x + 1$ is divided by $(x + 1)$.

8 Show that the polynomial $f(x) = (x + 1)^{2n} + (x + 2)^n - 1$ is divisible by $(x^2 + 3x + 2)$ for all $n \in \mathbb{Z}^+$.

9 Given a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,\ a_i \in \mathbb{R},\ i = 1, 2, \ldots, n,\ a_n \neq 0$$

and real numbers a and b, $a \neq 0$, show that when $f(x)$ is divided by a linear expression $(ax - b)$ the remainder is $f\left(\frac{b}{a}\right)$.

3.4 Polynomial functions: zeros, sum and product

The fundamental theorem of algebra

The fundamental theorem of algebra is one of the most important theorems in mathematics. It establishes the existence of the complex zeros of a polynomial (points at which the value of the function is zero). There are many theorems and corollaries that derive from this theorem which help in algebraic manipulation of equations and polynomial functions.

→ The fundamental theorem of algebra (FTA)

A polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real or complex coefficients $(a_n \neq 0)$ has at least one zero. There is an $\omega \in \mathbb{C}$ such that $f(\omega) = 0$

This theorem was proved by Gauss, but is beyond the scope of this textbook.

→ Corollary

Each polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real or complex coefficients can be written in a factored form $f(x) = a_n (x - \omega_1)(x - \omega_2) \ldots (x - \omega_n)$ such that $\omega_k \in \mathbb{C}$, $k = 1, \ldots, n$

Extension material on CD: *See the proof of this theorem on the CD.*

These examples highlight the usefulness of the theorems above.

If a certain factor appears more than once, we say that the factor has a **multiplicity**. So, if there are fewer than n different zeros of the given polynomial, the sum of their multiplicities will add up to n.

$$f(x) = a_n (x - \omega_1)^{p_1} (x - \omega_2)^{p_2} \ldots (x - \omega_k)^{p_k},\ k < n,\ \sum_{r=1}^{k} p_r = n$$

Example 26

Factorize the polynomial $f(x) = x^4 - 6x^3 + 11x^2 - 6x$, and check your answer with a GDC.

Answer

$f(x) = x^4 - 6x^3 + 11x^2 - 6x$

$= x(x^3 - 6x^2 + 11x - 6)$

Apply Horner's algorithm for x = 1 since the sum of the coefficients is equal to zero.

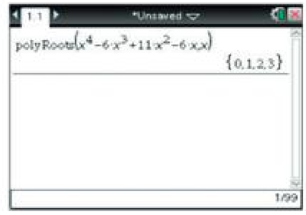
	1	-6	11	-6
		+	+	+
1 ·		1	-5	6
	1	-5	6	0

$= x(x - 1)(x^2 - 5x + 6)$

Apply the FTA to factorize the polynomial.

$= x(x - 1)(x - 2)(x - 3)$

Factorize the quadratic expression.



On a GDC you obtain zeros, but for the factor form of the polynomial you need to use the FTA.

Example 27

Given that 2 is a zero of the polynomial
$f(x) = x^5 - 4x^4 - 3x^3 + 34x^2 - 52x + 24$
and has a multiplicity of 3, factorize $f(x)$ fully and check your answer with a GDC.

Answer

Successively apply Horner's algorithm with respect to the multiplicity of the given zero.

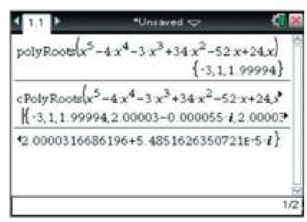
	1	-4	-3	34	-52	24
		+	+	+	+	+
2 ·		2	-4	-14	40	-24
	1	-2	-7	20	-12	0
		+	+	+	+	
2 ·		2	0	-14	12	
	1	0	-7	6	0	
		+	+	+		
2 ·		2	4	6		
	1	2	-3	0		

$f(x) = (x - 2)^3 (x^2 + 2x - 3)$

Apply the FTA to factorize the polynomial.

$= (x - 2)^3 (x + 1)(x + 3)$

Factorize the quadratic expression.



Due to the imperfection of the calculator's algorithm you obtain the approximation of the multiple zero (2), without its multiplicity. When using a complex roots finder you will find which zero has multiplicity, but an approximated value is given.

Exercise 3L

1 Given that k is a zero of multiplicity n of the polynomial f, factorize it fully and check your answers with a GDC.

a $k = -2$, $n = 2$, $f(x) = 2x^4 + 3x^3 - 10x^2 - 12x + 8$

b $k = \frac{1}{2}$, $n = 2$, $f(x) = 12x^3 - 32x^2 + 23x - 5$

2 Find a polynomial of the smallest degree, with integer coefficients, whose zeros are:

a 1, 3 and 5

b $-2, -1, 0$ and 1

c $-\frac{2}{3}$, 1, 2 and 3

3 Find a polynomial of the smallest degree, with integer coefficients, whose zeros are:

a $-\sqrt{2}$ and $\sqrt{3}$

b $-\frac{1}{2}, \frac{3}{4}$ and $\sqrt{5}$

c $-\frac{3}{5}, 1 - \sqrt{2}$ and $\sqrt[3]{3}$

4 Factorize these polynomials and check your answers with a GDC.

a $f(x) = x^3 - 2x^2 - 5x + 6$

b $f(x) = 2x^3 - x^2 - 7x + 6$

c $f(x) = 5x^4 - 12x^3 - 14x^2 + 12x + 9$

Conjugate root theorem

Given a polynomial
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$, and $a_n \neq 0$, that has a complex zero z, then its conjugate z^* is also a zero of the polynomial f.

Proof:

Using the properties of conjugate numbers, see page 110:

$$\begin{aligned} f(z) = 0 \Rightarrow f(z^*) &= a_n(z^*)^n + a_{n-1}(z^*)^{n-1} + \ldots + a_2(z^*)^2 + a_1(z^*) + a_0 \\ &= a_n(z^n)^* + a_{n-1}(z^{n-1})^* + \ldots + a_2(z^2)^* + a_1(z^*) + a_0 && \textit{Conjugate of a power.} \\ &= (a_n z^n)^* + (a_{n-1} z^{n-1})^* + \ldots + (a_2 z^2)^* + (a_1 z)^* + (a_0)^* && \textit{Conjugate of a product.} \\ &= (a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0)^* && \textit{Conjugate of a sum.} \\ &= (f(z))^* = 0^* = 0 \qquad \text{QED} \end{aligned}$$

Example 28

Given that $4 + 5i$ is a complex zero of the polynomial $f(x) = x^3 - 6x^2 + 25x + 82$, find all the remaining zeros and check your answers with a GDC.

Answer

$x_1 = 4 + 5i \Rightarrow x_2 = 4 - 5i$

Use the conjugate zero theorem.

Method 1

$(x - (4 + 5i))\,(x - (4 - 5i)) = x^2 - 8x + 41$

Find the quadratic factor.

$$\begin{array}{r} x + 2 \\ x^2 - 8x + 41 \overline{)\, x^3 - 6x^2 + 25x + 82} \\ -\underline{(x^3 - 8x^2 + 41x)} \\ 2x^2 - 16x + 82 \\ -\underline{(2x^2 - 16x + 82)} \\ 0 \end{array}$$

Use long division to find the last linear factor.

$f(x) = (x - (4 + 5i))\,(x - (4 - 5i))\,(x + 2)$

$x - 2 = 0$

$\Rightarrow x_3 = 2$

Method 2

	1	−6	25	82
		+	+	+
$(4 + 5i)\cdot$		$4 + 5i$	$-33 + 10i$	−82
	1	$-2 + 5i$	$-8 + 10i$	0
		+	+	
$(4 - 5i)\cdot$		$4 - 5i$	$8 - 10i$	
	1	2	0	

Successively apply Horner's algorithm to.

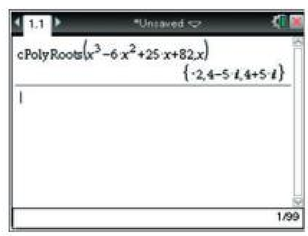
$f(x) = (x - (4 + 5i))\,(x - (4 - 5i))\,(x + 2)$

Fully factorize the polynomial.

$x + 2 = 0$

$\Rightarrow x_3 = -2$

Find the last remaining zero.



the complex numbers $4 – 5i$ and $4 + 5i$.

To check with a GDC use the feature called 'Complex Roots of Polynomials' within the polynomial tools.

Example 29

Given that i is a complex zero of the polynomial $f(x) = x^4 - 2x^3 + 6x^2 + ax + 5$, $a \in \mathbb{R}$, find the value of a. Hence, find all the remaining zeros and check your answers with a GDC.

Answer

	1	−2	6	a	5
		+	+	+	+
$i\cdot$		i	$-2i-1$	$5i+2$	$2i+ai-5$
	1	$-2+i$	$5-2i$	$2+a+5i$	$(2+a)i$

Apply Horner's algorithm for $x_1 = i$

$f(i) = 0 \Rightarrow (2 + a)\,i = 0$

$\Rightarrow 2 + a = 0 \Rightarrow a = -2$

Apply the remainder theorem.

$x_1 = i \quad \Rightarrow x_2 = -i$

Use the conjugate zero theorem.

	1	$-2+i$	$5-2i$	$5i$	0
		+	+	+	
$-i\cdot$		$-i$	$2i$	$-5i$	
	1	−2	5	0	

Use $a = -2$ and continue to apply Horner's algorithm for $x_2 = -i$ to obtain the quotient.

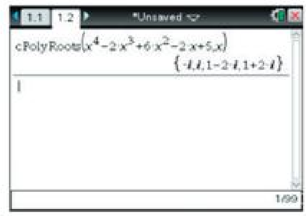
$x^2 - 2x + 5 = 0$

Find the zeros of the quotient polynomial.

$$x = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$x_3 = 1 + 2i,\ x_4 = 1 - 2i$

Apply the quadratic formula.



Exercise 3M

1 Given a polynomial f and the zero z, find all the remaining zeros.

a $f(x) = x^3 + 3x^2 + 4x + 12$, $z = 2i$

b $f(x) = x^3 - 6x^2 + 13x - 20$, $z = 1 - 2i$

c $f(x) = 5x^3 + 17x^2 + 21x + 6$, $z = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$

d $f(x) = x^4 - 4x^3 + 5x^2 - 4x + 4$, $z = -i$

e $f(x) = 2x^4 + 3x^3 + 17x^2 - 12x - 10$, $z = -1 - 3i$

f $f(x) = 2x^4 + 9x^3 + 11x^2 - 7x - 15$, $z = -2 + i$

g $f(x) = 6x^4 + 26x^3 + 35x^2 + 36x + 9$, $z = -\frac{1}{2} + \frac{\sqrt{5}}{2}i$

h $f(x) = 3x^4 - 2x^3 + 4x^2 - 2x + 1$, $z = \frac{1}{3} + \frac{\sqrt{2}}{3}i$

2 Given that z is a complex zero of the polynomial f, find the missing coefficients. Hence, find all the remaining zeros and check your answers with a GDC.

a $z = -1$, $f(x) = x^3 - 13x + a$, $a \in \mathbb{R}$

b $z = 3$, $f(x) = x^3 - 7x^2 + ax - 15$, $a \in \mathbb{R}$

c $z = -1 - i$, $f(x) = x^4 + 2x^3 - 2x^2 - 8x + a$, $a \in \mathbb{R}$

d $z = -2i$, $f(x) = x^4 - 4x^3 + 9x^2 + ax + b$, $a, b \in \mathbb{R}$

Sum and product of polynomial roots

François Viète developed formulae that connect the zeros and the coefficients of a polynomial. Viète was the first to investigate this connection for positive real zeros. Albert Girard was the first to extend that to complex zeros.

Albert Girard (1595–1632) introduced the abbreviations sin, cos and tan for trigonometric functions. He enrolled at the University of Leiden at the age of 22. Before that he was a professional musician, playing the lute.

Polynomials of the third degree

→ Theorem

Given a cubic equation $ax^3 + bx^2 + cx + d = 0$, $a, b, c, d \in \mathbb{R}$, $a \neq 0$

and solutions x_1, x_2 and x_3 then $\begin{cases} x_1 + x_2 + x_3 = -\frac{b}{a} \\ x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = \frac{c}{a} \\ x_1 \cdot x_2 \cdot x_3 = -\frac{d}{a} \end{cases}$

These are Viète's formulae for cubic equations.

Proof

Using the previous results factorize the cubic polynomial

$f(x) = ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3)$

$\Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = (x - x_1)(x - x_2)(x - x_3)$

Expand the right-hand side of the equation and equate the corresponding coefficients:

$(x - x_1)(x - x_2)(x - x_3)$

$= (x^2 - (x_1 + x_2)\,x + x_1x_2)\,(x - x_3)$

$= x^3 - (x_1 + x_2)\,x^2 + x_1x_2 \cdot x - x^2x_3 + (x_1 + x_2)\,x \cdot x_3 - x_1x_2x_3$

$= x^3 - (x_1 + x_2 + x_3)\,x^2 + (x_1x_2 + x_1x_3 + x_2x_3)\,x - x_1x_2x_3$

$$\Rightarrow \begin{cases} -(x_1 + x_2 + x_3) = \frac{b}{a} \\ x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = \frac{c}{a} \\ -(x_1 \cdot x_2 \cdot x_3) = \frac{d}{a} \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = -\frac{b}{a} \\ x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = \frac{c}{a} \\ x_1 \cdot x_2 \cdot x_3 = -\frac{d}{a} \end{cases}$$

Investigation – Coefficients of a quartic polynomial

Viéte's formulae connect the zeros and the coefficients of a cubic polynomial.

Find similar formulae that satisfy the relationship between the zeros and the coefficients of a quartic polynomial

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a, b, c, d, e \in \mathbb{R}$, $a \neq 0$

Example 30

Given that the roots of a cubic equation $2x^3 + 4x^2 - 7x + 5 = 0$ are x_1, x_2 and x_3, without solving the equation, find:

a $x_1 + x_2 + x_3$ **b** $x_1 \cdot x_2 \cdot x_3$ **c** $x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3$

d $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$ **e** $x_1^2 + x_2^2 + x_3^2$

Answers

$a = 2, b = 4, c = -7, d = 5$

Identify the coefficients of the cubic polynomial.

a $x_1 + x_2 + x_3 = -\frac{4}{2} = -2$

Use $x_1 + x_2 + x_3 = -\frac{b}{a}$.

b $x_1 \cdot x_2 \cdot x_3 = -\frac{5}{2}$

Use $x_1 \cdot x_2 \cdot x_3 = -\frac{d}{a}$.

c $x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = -\frac{7}{2}$

Use $x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = \frac{c}{a}$.

d $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{x_2x_3 + x_1x_3 + x_1x_2}{x_1x_2x_3}$

$$= \frac{-\frac{7}{2}}{-\frac{5}{2}} = \frac{7}{5}$$

Use the results found in parts **a** *and* **b**.

e $x_1^2 + x_2^2 + x_3^2$

$= (x_1 + x_2 + x_3)^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

$= (-2)^2 - 2\left(-\frac{7}{2}\right) = 4 + 7 = 11$

Use the formula $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.

Use the results found in parts **a** *and* **c**.

Theorem

Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real or complex coefficients ($a_n \neq 0$) and zeros $x_1, x_2, \ldots, x_n$ then

$$\begin{cases} x_1 + x_2 + x_3 + \ldots + x_n = -\dfrac{a_{n-1}}{a_n} \\ x_1x_2 + x_1x_3 + \ldots + x_1x_n + x_2x_3 + x_2x_4 + \ldots + x_2x_n + \ldots + x_{n-1}x_n = \dfrac{a_{n-2}}{a_n} \\ \vdots \\ x_1x_2x_3 \cdot \ldots \cdot x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

These are Viéte's formulae for an 'equation' of the nth degree.

→ As a general system:

$$\sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \left(x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k}\right) = (-1)^k \frac{a_{n-k}}{a_n}, 1 \le k \le n$$

Proof

The proof is a direct consequence of the ability to write polynomials in factorized form:

$f(x) = a_n(x - x_1)(x - x_2)\ldots(x - x_n)$.

By expanding the right hand side and comparing coeffecients you obtain the given formulae.

Example 31

Find the sum and product of the zeros of these polynomials.

a $f(x) = x^4 - 3x^3 + 11x^2 + 17x - 4$

b $f(x) = 3x^5 + 11x^4 - 4x^3 + 5x^2 - 13x + 9$

c $f(x) = 17x^{13} + 4x^{12} + 122x^2 - 14x - 17$

d $f(x) = 3x^{2012} + 7x^{370} - 4x^{25} - 15x + 2$

Answers

a $f(x) = x^4 - 3x^3 + 11x^2 + 17x - 4$

$n = 4, a_4 = 1, a_3 = -3, a_2 = 11, a_1 = 17, a_0 = -4$

Identify n and the coefficients of the polynomial.

$$x_1 + x_2 + x_3 + x_4 = -\frac{a_3}{a_4} \Rightarrow x_1 + x_2 + x_3 + x_4 = -\frac{-3}{1} = 3$$

Use $x_1 + x_2 + \ldots + x_n = -\frac{a_{n-1}}{a_n}$

$$x_1x_2x_3x_4 = (-1)^4\frac{a_0}{a_4} \Rightarrow x_1x_2x_3x_4 = \frac{-4}{1} = -4$$

Use $x_1x_2x_3 \cdot \ldots \cdot x_n = (-1)^n\frac{a_o}{a_n}$

b $f(x) = 3x^5 + 11x^4 - 4x^3 + 5x^2 - 13x + 9$

$n = 5, a_5 = 3, a_4 = 11, a_0 = 9$

For a polynomial of degree 5 you need a_5, a_4 *and* a_o

$$\sum_{r=1}^{5} x_r = -\frac{a_4}{a_5} \Rightarrow \sum_{r=1}^{5} x_r = -\frac{11}{3}$$

$$\prod_{r=1}^{5} x_r = (-1)^5\frac{a_0}{a_5} \Rightarrow \prod_{r=1}^{5} x_r = (-1)^5\frac{9}{3} = -3$$

c $f(x) = 17x^{13} + 4x^{12} + 122x^2 - 14x - 17$

$n = 13, a_{13} = 17, a_{12} = 4, a_0 = -17$

For a polynomial of degree 13 you need a_{13}, a_{12} *and* a_0

$$\sum_{r=1}^{13} x_r = -\frac{a_{12}}{a_{13}} \Rightarrow \sum_{r=1}^{5} x_r = -\frac{4}{17}$$

$$\prod_{r=1}^{13} x_r = (-1)^5\frac{a_0}{a_{13}} \Rightarrow \prod_{r=1}^{13} x_r = (-1)^{17}\frac{-17}{17} = 1$$

d $f(x) = 3x^{2012} + 7x^{370} - 4x^{25} - 15x + 2$

$n = 2012, a_{2012} = 3, a_{2011} = 0, a_0 = 2$

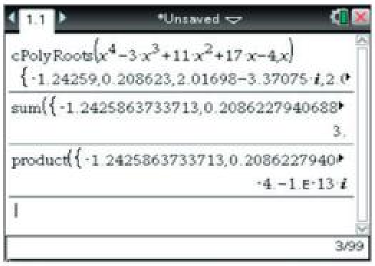
$$\sum_{r=1}^{2012} \mathrm{x}_r = -\frac{a_{2011}}{a_{2012}} \Rightarrow \sum_{r=1}^{2012} x_r = -\frac{0}{3} = 0$$

$$\prod_{r=1}^{2012} x_r = (-1)^{2012}\frac{a_0}{a_{2012}} \Rightarrow \prod_{r=1}^{2012} x_r = (-1)^{2012}\frac{2}{3} = \frac{2}{3}$$

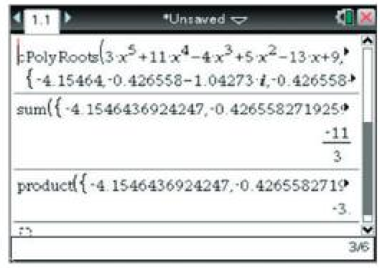
*In part **a** all the coefficients were listed but they are not all needed for the formulae.*
For a polynomial of degree n you need a_n, a_{n-1} *and* a_0

Note: You can check the results for Example 31 using a GDC.

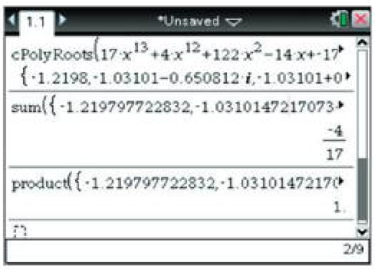
a



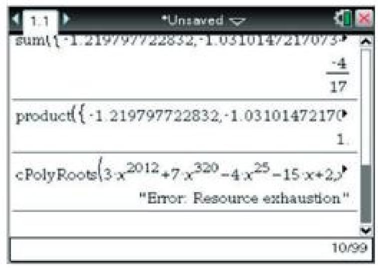
b



c



d



In part **c** you have to adjust the accuracy when converting to a decimal, while in part **d** the degree of the polynomial was too large for the algorithm for finding complex zeros. Note that for the sum and product of the solutions given in the form of a list you use the Math List menu.

Exercise 3N

1 The roots of a cubic equation $3x^3 - 2x^2 - 5x - 4 = 0$ are x_1, x_2 and x_3. Without solving the equation, find:

a $x_1 + x_2 + x_3$

b $x_1 \cdot x_2 \cdot x_3$

c $x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3$

d $\dfrac{6}{x_1} + \dfrac{6}{x_2} + \dfrac{6}{x_3}$

e $9x_1^2 + 9x_2^2 + 9x_3^2$

Check your results using a GDC.

2 The roots of a quartic equation $x^4 - 3x^3 + 2x^2 - 4x - 6 = 0$ are x_1, x_2, x_3 and x_4. Without solving the equation, find:

a $x_1 + x_2 + x_3 + x_4$

b $x_1 \cdot x_2 \cdot x_3 \cdot x_4$

c $x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_3 \cdot x_4$

d $x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4$

e $\dfrac{3}{x_1} + \dfrac{3}{x_2} + \dfrac{3}{x_3} + \dfrac{3}{x_4}$

f $\dfrac{x_1^2}{5} + \dfrac{x_2^2}{5} + \dfrac{x_3^2}{5} + \dfrac{x_4^2}{5}$

Check your results using a GDC.

3 Find the sum and product of the zeros of these polynomials.

a $f(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$

b $f(x) = 4x^6 + x^5 + 7x^4 - 3x^3 + 2x$

c $f(x) = 11x^{10} - \frac{3}{7}x^7 + \sqrt{5} \cdot x^3 - \pi x + 22$

d $f(x) = 5x^{7007} - 4x^{7006} + 2x^{231} + 10x + 8$

3.5 Polynomial equations and inequalities

Some useful theorems

Factorization is a common method used to solve polynomial equations. Descartes' rule of sign, the integer zero theorem and the rational zero theorem are valid for polynomials of all degrees and are an aid to finding factors.

Before factorizing, it is useful to know how many real zeros to expect for a given polynomial. **René Descartes** (1596–1650), in his work *La Géométrie,* noticed the following property.

Descartes' rule of signs

The number of positive real roots of a polynomial f is equal to the number of sign changes (from + to – or from – to +) of its coefficients, or an even number less. Also the number of negative real roots of a polynomial f is equal to the number of sign changes of the coefficients of $f(-x)$, or an even number less.

For example, the polynomial $f(x) = x^3 - 7x^2 - 9x + 18$ has the following sequence of signs: +, –, –, +. Here there are two sign changes so there are two or zero (an even number less) positive real roots. Now look at
$f(-x) = (-x)^3 - 7(-x)^2 - 9(-x) + 18 = -x^3 - 7x^2 + 9x + 18$,
which has the sequence of signs –, –, +, +. In this sequence there is only one sign change, so the polynomial f can have only one negative real root.

Some cases of cubic equations were solved by the Babylonians (2000–1600 BCE). They used tables with perfect squares, perfect cubes and their sums. They were able to solve equations of the form $ax^3 + bx = c$.
Later, in the 13th and 14th centuries, a group of Italian mathematicians, dal Ferro, Tartaglia and Cardano, developed a formula for solving a general cubic equation.

The following theorems are valid for polynomials with integer coefficients.

Integer zero theorem

→ Given a polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2x^2 + a_1x + a_0,\ a_k \in \mathbb{Z},\ a_n \neq 0$
and an integer p such that $f(p) = 0$, then p is a factor of a_0.

Proof

$f(p) = a_np^n + a_{n-1}p^{n-1} + \dots + a_2p^2 + a_1p + a_0 = 0$
$\Rightarrow a_np^n + a_{n-1}p^{n-1} + \dots + a_2p^2 + a_1p = -a_0$
$\Rightarrow p(a_np^{n-1} + a_{n-1}p^{n-2} + \dots + a_2p + a_1) = -a_0$

Therefore, p is a factor of a_0. QED

Sometimes, when the coefficient a_0 is not prime there are many possible factors. For example, if $a_0 = 18 \Rightarrow p \in \{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}$.

In these cases the search for all possible zeros would take a long time. To speed up the process there is a corollary that reduces the set of possible zeros.

Corollary 1

Given a polynomial
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,\ a_k \in \mathbb{Z},\ a_n \neq 0$
and an integer value p such that $f(p) = 0$, then for any integer value q, $p - q$ is a factor of $f(q)$.

Proof

$$f(p) = a_n p^n + a_{n-1} p^{n-1} + \ldots + a_2 p^2 + a_1 p + a_0 = 0 \qquad (1)$$
$$f(q) = a_n q^n + a_{n-1} q^{n-1} + \ldots + a_2 q^2 + a_1 q + a_0 \qquad (2)$$

Equation (1) – equation (2)

$$\underbrace{f(p)}_{0} - f(q) = a_n\left(p^n - q^n\right) + a_{n-1}\left(p^{n-1} - q^{n-1}\right) + \ldots + a_2\left(p^2 - q^2\right) + a_1\left(p - q\right).$$

The terms on the right-hand side of the equation are grouped in such a way that every term containing $(p^r - q^r)$, $r = 1, 2, \ldots, n$, has a factor of $p - q$, so $p - q$ is a factor of $f(q)$. QED

This corollary is useful when there are many possible factors for integer zeros as you can eliminate some and simplify the search.

$a^n - b^n$ is divisible by $a - b$ for all positive integers n. The formula $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$ was proved in Chapter 1.

Example 32

Find all the possible integer zeros of the polynomial
$f(x) = x^3 - 7x^2 - 9x + 18$

Answer

$p \in \{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}$	*List all the possible zeros, i.e. factors of 18, by using the integer zero theorem.*
$f(1) = 1^3 - 7 \times 1^2 - 9 \times 1 + 18 = 3$	*Use $q = 1$ to reduce the set of possible factors by using the corollary.*
$p - 1 \in \{\pm 1, \pm 3\}$	*$p - 1$ is a factor of 3.*
$\Rightarrow p \in \{-2, 0, 2, 4\}$	
$p \in \{-2, 2\}$	*This is the intersection of both sets.*

Note: By using the corollary 1 in Example 32, you only need to inspect two values (instead of all twelve possible values).

By using Descartes' rule of signs you can eliminate the positive solution (+2) since there must either be two or no positive roots (there cannot be only one).

Apply synthetic division for both values (2 and –2) to check this conclusion.

	1	-7	-9	18
2 ·		+	+	+
		2	-10	-38
	1	-5	-19	-20

	1	-7	-9	18
-2 ·		+	+	+
		-2	18	-18
	1	-9	9	0

Since the remainder when $f(x)$ is divided by $(x - 2)$ is –20, the polynomial is not divisible by $(x - 2)$, so Descartes' rule works well. The remainder when divided by $(x + 2)$ is zero.

$x^3 - 7x^2 - 9x + 18 = (x + 2)(x^2 - 9x + 9)$

Notice that by examining both possible integer zeros you can conclude that the only integer zero is –2, and you did not need to factorize the quadratic quotient.

The next theorem is a generalization from integers to rational zeros and the proof is similar.

Rational zero theorem

→ Given a polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,\ a_i \in \mathbb{Z},\ a_n \neq 0$ and a rational number $\frac{p}{q}$, where $\gcd(p, q) = 1$ that is $\left(\frac{p}{q}\right)$ is in its simplest form, such that $f\left(\frac{p}{q}\right) = 0$, then p is a factor of a_0 and q is a factor of a_n.

gcd(p, q) means greatest common divisor of p and q.

Proof

$$f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_2\left(\frac{p}{q}\right)^2 + a_1\frac{p}{q} + a_0 = 0 \quad (1)$$

You can also say that p and q are co-prime.

Multiply equation (1) by q^n

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_2 p^2 q^{n-2} + a_1 p\, q^{n-1} + a_0 q^n = 0 \quad (2)$$

See the proof of this theorem on the CD

Rearrange the equation (2)

$$p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \ldots + a_2 p\, q^{n-2} + a_1 q^{n-1}) = -a_0 q^n \quad (3)$$

Since the left-hand side of the equation (3) has a factor p and $\gcd(p, q) = 1$, then p must be a factor of a_0.

In a similar way, we can rearrange equation (2) to obtain

$$a_n p^n = -q(a_{n-1}p^{n-1} + \dots + a_2 p^2 q^{n-3} + a_1 p\, q^{n-2} + a_0\, q^{n-1}) \qquad (4)$$

Again, since the right-hand side of the equation (4) has a factor q and $\gcd(p, q) = 1$,
we can conclude that q is a factor of a_n. QED

Corollary 2

Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, $a_i \in Z$, $a_n \neq 0$ and a rational number $\frac{p}{q}$, where $\gcd(p, q) = 1$ such that $f\left(\frac{p}{q}\right) = 0$, then for any real value k, $(p - qk)$ is a factor of $f(k)$.

See the proof of this corollary on the CD

Example 33

Given that the polynomial $f(x) = 2x^3 - 11x^2 - 11x + 15$ has no integer zeros, find its only rational zero.

Answer

$\frac{p}{q} \in \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}\right\}$

List all the possible rational (non-integer) zeros by using the rational zero theorem.

$f(1) = 2 \times 1^3 - 11 \times 1^2 - 11 \times 1 + 15$
$= -5$

Use corollary 2 with k=1 to reduce the set of possible zeros.

$p - q \in \{\pm 1, \pm 5\}$

$p - 1 \cdot q$ is a factor of -5.

$\frac{p}{q} \in \left\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\right\}$

This is the intersection of both sets.

Sequence of signs for $f(x)$: +, –, –, +
$\Rightarrow$ 2 or 0 positive roots
Sequence of signs for $f(-x)$:
–, –, +, +
$\Rightarrow$ only 1 negative root

Apply Descartes' rule of signs.

Since there is only one rational zero it can only be the negative one $\left(-\frac{3}{2}\right)$, as complex zeros come in conjugate pairs.

$-\frac{3}{2}$	2	-11	-11	15
		+	+	+
		-3	21	-15
	2	-14	10	0

Use synthetic division.

$2x^3 - 11x^2 - 11x + 15$
$= \left(x + \frac{3}{2}\right)\underbrace{(2x^2 - 14x + 10)}_{2(x^2 - 7x + 5)}$

Factorize the quotient to simplify the divisor.

$= (2x + 3)(x^2 - 7x + 5)$
So again, the only rational zero is $-\frac{3}{2}$

The quotient is a quadratic expression $x^2 - 7x + 5$ whose discriminant is 29. Therefore, the remaining two solutions are irrational.

Using corollary 2 in Example 33, you only need to inspect three values (instead of all eight possible rational zeros).

Exercise 30

1 Solve these equations in the set of real numbers and check your answers with a GDC.

a $x^3 - 6x^2 + 11x - 6 = 0$ **b** $x^3 + 2x^2 - 7x + 4 = 0$

c $x^3 + 3x^2 - 4x - 12 = 0$ **d** $2x^3 - 5x^2 - 18x + 45 = 0$

Solving polynomial equations

In this section you will solve polynomial equations using an algebraic method and then use a GDC to verify the solutions (graphical method).

To solve a polynomial equation by a graphical method you need to find the points of intersection of the graph with the x-axis. At these points the value of the function is zero ($y = 0$), so these points are called the *zeros of the function* or the *roots*.

Example 34

Solve these equations and check your answers by using a graphical method.

a $x^3 + 2x^2 - 5x - 6 = 0$

b $6x^4 + 17x^3 + 10x^2 - 7x - 6 = 0$

Answers

a Algebraic method:

$x^3 + 2x^2 - 5x - 6 = \underbrace{x^3 + 2x^2 + x}\underbrace{- 6x - 6}$

Split the linear term (5x) for a common factor.

$= x(x^2 + 2x + 1) - 6(x + 1)$

$= x(x + 1)^2 - 6(x + 1)$

Notice the perfect square $(x + 1)^2$

$= (x + 1)(x(x + 1) - 6)$

Use distribution with the common factor $(x + 1)$.

$= (x + 1)(x^2 + x - 6)$

Factorize the quadratic factor $x^2 + x - 6$

$= (x + 1)(x - 2)(x + 3)$

$\Rightarrow (x + 1)(x - 2)(x + 3) = 0$

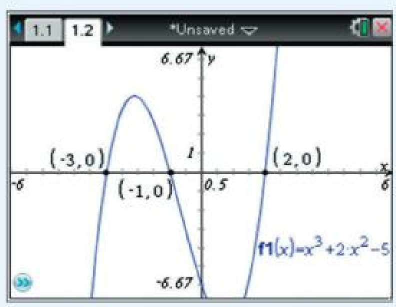
$\Rightarrow x_1 = -1, x_2 = 2, x_3 = -3$

Use the zero product theorem.

Graphical method:

$x^3 + 2x^2 - 5x - 6 = 0 \Rightarrow f(x) = x^3 + 2x^2 - 5x - 6$

The solutions are $x_1 = -3$, $x_2 = -1$ and $x_3 = 2$



b $6x^4 + 17x^3 + 10x^2 - 7x - 6 = 0$

Algebraic method:

$$\frac{p}{q} \in \left\{\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{6}\right\}$$

List all the possible rational zeros by using the rational zero theorem.

Continued on next page

	6	17	10	−7	−6
1 ·		+	+	+	+
		−6	−11	1	6
	6	11	−1	−6	0
−1 ·		+	+	+	
		−6	−5	6	
	6	5	−6	0	

Use synthetic division.

$6x^4 + 17x^3 + 10x^2 - 7x - 6$

$= (x + 1)^2 (6x^2 + 5x - 6)$

$= (x + 1)^2 (2x + 3) (3x - 2)$

$\Rightarrow (x + 1)^2 (2x + 3) (3x - 2) = 0$

Factorize the quadratic expression.

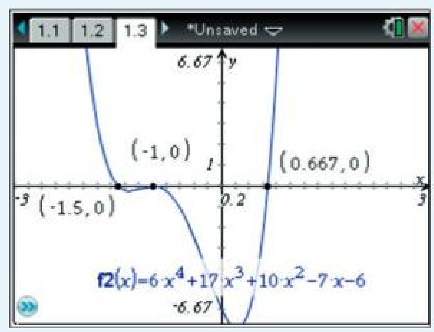
$\Rightarrow x_{1,2} = -1,\ x_3 = -\frac{3}{2},\ x_4 = \frac{2}{3}$

Use the zero product theorem.

Graphical method:

The solutions are

$x_1 = -\frac{3}{2},\ x_{2,3} = -1$ and $x_4 = \frac{2}{3}$



Note that at the point (−1, 0) the graph is just touching, that is tangent to, the x-axis. In this case (−1, 0) is a double, or repeated, zero of the function.

A polynomial of degree n can have up to n roots on the real number line. It is useful to be able to restrict any search for roots to a finite window. The next theorem provides such a search window.

Theorem

➔ All the possible zeros of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ are in the interval $\left[-\left(\frac{M}{|a_n|}+1\right), \frac{M}{|a_n|}+1\right]$ where $M = \max\{|a_n|, |a_{n-1}|, \ldots |a_1|, |a_0|\}$

See the proof of this theorem on the CD

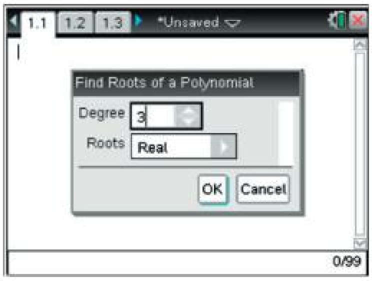
For the equations in Example 34:

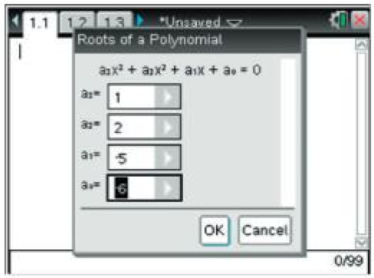
a $M = 6, |a_3| = 1 \Rightarrow \left[-\left(\frac{6}{1}+1\right), \frac{6}{1}+1\right] = [-7, 7]$

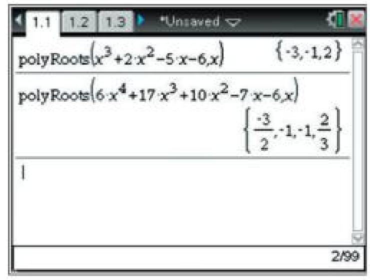
b $M = 17, |a_3| = 6 \Rightarrow \left[-\left(\frac{17}{6}+1\right), \frac{17}{6}+1\right] = \left[-\frac{23}{6}, \frac{23}{6}\right]$

Notice that the zeros satisfy the conditions of the theorem.

You could solve both equations on a GDC using the *Polynomial tools* feature.







Notice that in the solution to part **b** the GDC writes the multiplicity of the zero by repeating the value of the zero (–1) twice.

You will graph more polynomials in Chapter 4.

Exercise 3P

1 Solve these equations in the set of real numbers and check your answers with a GDC.

a $12x^3 + 17x^2 + 2x - 3 = 0$ **b** $x^3 - 4x^2 - 5x + 14 = 0$

c $3x^3 - 13x^2 + 11x + 14 = 0$ **d** $x^4 - x^3 - 11x^2 + 9x + 18 = 0$

2 One of the the roots of the equation $x^3 + ax^2 - x - 3 = 0$ is equal to –3.

a Find the value of a.

b Find the other two roots.

3 The equation $ax^3 - 7x^2 + bx + 4 = 0$ has one double root which is equal to 2.

a Find the values of a and b.

b Find the remaining root.

4 Show that the polynomial $f(x) = x^3 + 5x + p$ does not have an integer zero when p is a prime number.

5 Two of the zeros of the polynomial $f(x) = x^3 + ax^2 + bx + c$, $a, b, c \in \mathbb{R}$ are opposite numbers.

a Show that $ab = c$.

b Find the third zero.

Solving polynomial inequalities

To solve polynomial inequalities by an algebraic method you factorize the polynomial and investigate the signs of the factors in a sign table. Then you find the values of x for which the inequality is true.

Quadratic in equalities are discussed in Chapter 14, section 2.12.

To solve polynomial inequalities by a graphical method use a GDC to graph the polynomial and identify the values of x for which the inequality is true.

Example 35

Solve these inequalities.

a $x^3 - x^2 - 10x - 8 \geq 0$

b $2x^3 - 5x^2 - 6x + 4 < 0$

c $1 - 4x^2 < 5x^3 + 4x$

Verify your solutions by using a graphical method.

Answers

a **Algebraic method:**

$x^3 - x^2 - 10x - 8 = 0$

List all the possible zeros.

$p \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$

Use synthetic division.

	1	-1	-10	-8
-1 ·		+	+	+
		-1	2	8
	1	-2	-8	0
-2 ·		+	+	
		-2	8	
	1	-4	0	

$x^3 - x^2 - 10x - 8 = (x + 2)(x + 1)(x - 4)$

Fully factorize the polynomial.

x	$]-\infty, -2[$	-2	$]-2, -1[$	-1	$]-1\ 4[$	4	$]4\ \infty[$
$x + 2$	–	0	+	+	+	+	+
$x + 1$	–	–	–	0	+	+	+
$x - 4$	–	–	–	–	–	0	+
$x^3 - x^2 - 10x - 8$	–	0	+	0	–	0	+

$x \in [-2, -1] \cup [4, \infty[$

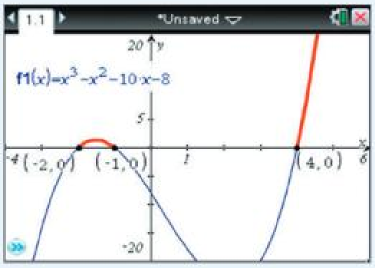
Construct the sign table.
Find the product of the signs and zeros.

Graphical method:

$f(x) = x^3 - x^2 - 10x - 8$

$x \in [-2, -1] \cup [4, \infty[$

Use a GDC to draw the graph of the polynomial and identify the parts of the graph that are above the x-axis.



Identify the values of x that satisfy the inequality.
Include zeros since it is not a strict inequality.

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b Algebraic method:

$2x^3 - 5x^2 - 6x + 4 = 0$

$\frac{p}{q} \in \left\{\pm 1, \pm 2, \pm 4, \pm \frac{1}{2}\right\}$ *List all the possible rational zeros.*

$k = -1 \Rightarrow f(-1) = 3$ *Use k = –1 to reduce the set of possible zeros.*

$p + q \in \{\pm 1, \pm 3\}$ *p – (–1) · q must be a factor of 3.*

$\frac{p}{q} \in \left\{2, -4, \pm \frac{1}{2}\right\}$ *Find the intersection of the two conditions.*

Use synthetic division.

$\frac{1}{2}\cdot$	2	-5	-6	4
		+	+	+
		1	-2	-4
	2	-4	-8	0

$$2x^3 - 5x^2 - 6x + 4 = \left(x - \frac{1}{2}\right)\underbrace{(2x^2 - 4x - 8)}_{2(x^2-2x-4)}$$

Solve the quadratic equation.

$$= (2x - 1)(x^2 - 2x - 4)$$

$$x^2 - 2x - 4 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4+16}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

$$2x^3 - 5x^2 - 6x + 4 = (2x-1)(x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$$

Construct the sign table.
Do not include the zeros since the inequality was strict.

x	$]-\infty, 1, -\sqrt{5}[$	$1-\sqrt{5}$	$]1-\sqrt{5}, \frac{1}{2}[$	$\frac{1}{2}$	$]\frac{1}{2}, 1+\sqrt{5}[$	$1+\sqrt{5}$	$]1+\sqrt{5}, \infty[$
$x-1+\sqrt{5}$	–	0	+	+	+	+	+
$2x-1$	–	–	–	0	+	+	+
$x-1-\sqrt{5}$	–	–	–	–	–	0	+
Result	–	0	+	0	–	0	+

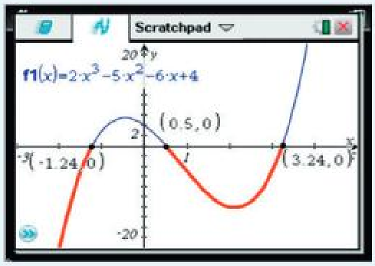
$x \in \left]-\infty, 1-\sqrt{5}\right[\cup \left]\frac{1}{2}, 1+\sqrt{5}\right[$

Continued on next page

Graphical method:

$f(x) = 2x^3 - 5x^2 - 6x + 4$

Use a GDC to draw the graph of the polynomial and identify the parts of the graph that are below the x-axis.



$x \in]-\infty, -1.24[\cup]0.5, 3.24[$

Identify the values of x that satisfy the inequality.

> $1-\sqrt{5} = -1.24$ (3 sf) and
> $1+\sqrt{5} = 3.24$ (3 sf).

c **Algebraic method:**

$1 - 4x^2 < 5x^3 + 4x$

$0 < 5x^3 + 4x^2 + 4x - 1$

Rewrite the inequality so that the leading coefficient is positive.

$5x^3 + 4x^2 + 4x - 1 = 0$

$\frac{p}{q} \in \left\{\pm 1, \pm \frac{1}{5}\right\}$

List all the possible rational zeros.

Use synthetic division.

$\frac{1}{5}$:	5	4	4	-1	0
		+	+	+	
		1	1	1	
	5	5	5	0	

$5x^3 + 4x^2 + 4x - 1 = \left(x - \frac{1}{5}\right)\underbrace{\left(5x^2 + 5x + 5\right)}_{5\left(x^2+x+1\right)}$

$= (5x - 1)(x^2 + x + 1)$

Solve the quadratic equation.

$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} \notin \mathbb{R}$

So, the only real zero is: $x = \frac{1}{5}$

The quadratic equation has no real solution so the quadratic expression is irreducible on the set of real numbers.

x	$\left]-\infty, \frac{1}{5}\right[$	$\frac{1}{5}$	$\left]\frac{1}{5}, \infty\right[$
$5x - 1$	−	0	+
$x^2 + x + 1$	+	+	+
$5x^4 + 39x^3 + 32x^2 + 27x - 7$	−	0	+

$x \in \left]\frac{1}{5}, \infty\right[$

Construct the sign table. Do not include the zeros since the inequality was strict.

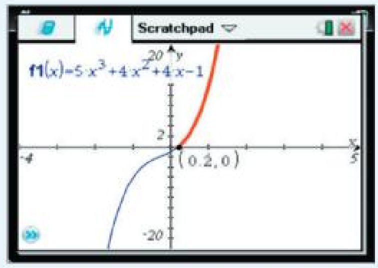
Continued on next page

Graphical method:

$f(x) = 5x^3 + 4x^2 + 4x - 1$

$x\ ]0.2, \infty[$

Use a GDC to draw the graph of the polynomial and identify the parts of the graph that are above the x-axis.



Identify the values of x that satisfy the inequality.

Example 36

Given the polynomials $f(x) = 4x^3 - 17x^2 + 30x + 5$ and $g(x) = -2x^3 + 8x^2 + 9x - 5$ find all the values of x such that $f(x) \le g(x)$.
Verify your solution by using a graphical method on a GDC.

Answer

Algebraic method:

$4x^3 - 17x^2 + 30x + 5 \le -2x^3 + 8x^2 + 9x - 5$

$6x^3 - 25x^2 + 21x + 10 \le 0$

Rewrite the inequality.

$\frac{p}{q} \in \left\{\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}\right\}$

List all the possible rational zeros.

$k = 1 \Rightarrow f(1) = 12$

Use k = 1 to reduce the set of possible zeros.

$p - q \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$

p – 1 · q must be a factor of 12.

$\frac{p}{q} \in \left\{-1, \pm 2, \pm 5, \pm\frac{1}{2}, \frac{5}{2}, \pm\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{5}{6}\right\}$

Find the intersection of both conditions.

Let $h(x) = 6x^3 - 25x^2 + 21x + 10$.
Sequence of signs for $h(x)$: +, –, +, +
Sequence of signs for $h(-x)$: –, –, –, +
There can be only one negative root and two or zero positive real roots.

Apply Descartes' rule of signs.

	6	-25	21	10
		+	+	+
2 ·		12	-26	-10
	6	-13	-5	0
		+	+	
$\frac{5}{2}$ ·		15	5	
	6	2	0	

Use synthetic division.

Continued on next page

$$6x^3 - 25x^2 + 21x + 10 = (x-2)\left(x - \frac{5}{2}\right)\underbrace{(6x+2)}_{2(3x+1)}$$

Fully factorize the polynomial.

$6x^3 - 25x^2 + 21x + 10 = (x - 2)(2x - 5)(3x + 1)$

Construct the sign table.

x	$\left]-\infty, -\frac{1}{3}\right[$	$-\frac{1}{3}$	$\left]-\frac{1}{3}, 2\right[$	2	$\left]2, \frac{5}{2}\right[$	$\frac{5}{2}$	$\left]\frac{5}{2}, \infty\right[$
$x - 2$	−	−	−	0	+	+	+
$2x - 5$	−	−	−	−	−	0	+
$3x + 1$	−	0	+	+	+	+	+
$6x^3 - 25x^2 + 21x + 10$	−	0	+	0	−	0	+

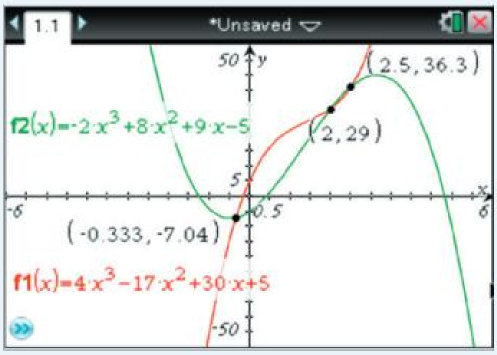
$$x \in \left]-\infty, -\frac{1}{3}\right] \cup \left[2, \frac{5}{2}\right]$$

Graphical method 1:

$f(x) = 4x^3 - 17x^2 + 30x + 5$

$g(x) = -2x^3 + 8x^2 + 9x - 5$

Use a GDC to draw the graphs of both polynomials and identify where the graph of f is below the graph of g.



$x \in]-\infty, -0.333] \cup [2, 2.5]$

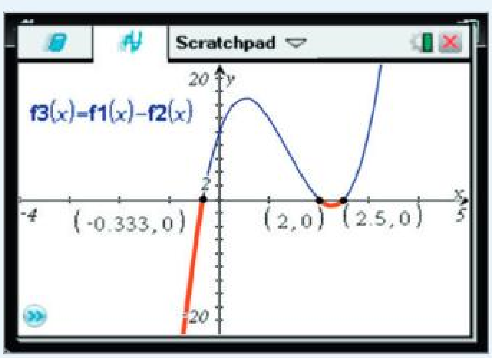
Identify the values of x that satisfy the inequality.

Graphical method 2:

Let $h(x) = f(x) - g(x) \le 0$

Rewrite the inequality and call the new function h(x).

Use a GDC to draw the graphs of the new polynomial and identify the parts where the graph is below the x-axis.



$x \in]-\infty, -0.333] \cup [2, 2.5]$

Identify the values of x that satisfy the inequality.

In Example 36 we used two different graphical methods. When using method I you may need to examine different windows to find the points of intersection between the graphs. Moreover, it can be difficult to read which function is upper and which is lower, particulary on calculators with poor resolution. Method II is more suitable because you don't need to think about the size of the window since the zeros appear along the x-axis. This saves having to explore the different windows if the intersections between the graphs are not visible in the original window.

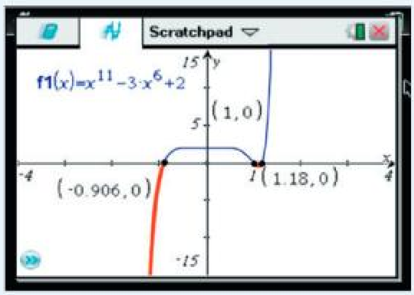
For the algebraic solution of $g(x) \geq f(x)$ the syllabus restricts polynomials to degree 3 or below. However, graphical methods on a GDC can be used for solving polynomial equations and inequalities of degree 4 or higher.

Example 37

Use a GDC to solve the inequality $x^{11} - 3x^7 + 2 \leq 0$

Answer

$f(x) = x^{11} - 3x^7 + 2$



$x \in]-\infty, -0.906] \cup [1, 1.18]$

Use a GDC to draw the graph of the polynomial and identify the parts of the graph that are below the x-axis.

Check that the window shows all the possible zeros by finding the suitable range of x-values:

$$\left[-\left(\frac{3}{|1|}+1\right), \frac{3}{|1|}+1\right] = [-4, 4].$$

Identify the values of x that satisfy the inequality.

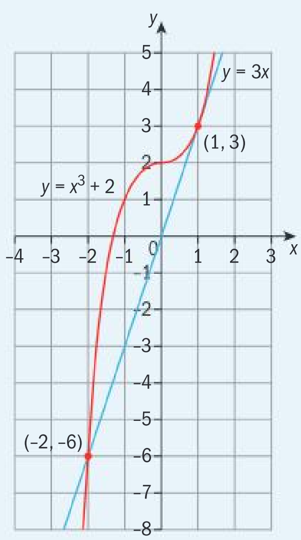
Sometimes, when equations or inequalities can be easily split into simpler polynomial curves, you can sketch these and find the solution by inspection.

Example 38

Use simple polynomial graphs to solve the inequality $x^3 - 3x + 2 \leq 0$

Answer

$$\underbrace{x^3 + 2}_{f(x)} \leq \underbrace{3x}_{g(x)}$$



$x \in]-\infty, -6] \cup \{3\}$

Split the inequality into a cubic and linear function.

Sketch the graphs of both the cubic and linear functions. Identify the values of x for which the cubic graph is below the linear graph.

Note: You could split the inequality: $\underbrace{x^3}_{f(x)} \leq \underbrace{3x - 2}_{g(x)}$

In this case the graphs would be exactly the same shape but shifted 2 units down.

Exercise 3Q

1 Solve these inequalities in the set of real numbers and check your answers with a GDC.

a $x^3 - 6x^2 + 11x - 6 \geq 0$ **b** $x^3 + 2x^2 - 7x + 4 \leq 0$

c $x^3 + 3x^2 - 4x - 12 < 0$ **d** $2x^3 - 5x^2 - 18x + 45 > 0$

e $12x^3 + 17x^2 + 2x - 3 \leq 0$ **f** $x^3 - 4x^2 - 5x + 14 > 0$

g $3x^3 - 13x^2 + 11x + 14 < 0$ **h** $x^4 - x^3 - 11x^2 + 9x + 18 \geq 0$

2 Given the polynomials $f(x) = 4x^3 - 17x^2 + 30x + 5$ and $g(x) = -2x^3 + 8x^2 + 9x - 5$, find all the values of x such that $f(x) > g(x)$.
Verify your solution by using a graphical method on a GDC.

3 Use a GDC to solve these inequalities.

a $x^7 - 2x^3 - 1 \geq 0$ **b** $x^9 - 2x^8 + 2x^5 + x \leq 0$

4 Use simple polynomial graphs to find the solutions of these inequalities.

a $x^3 + x - 2 > 0$ **b** $-2x^3 + 3x + 1 \geq 0$ **c** $x^4 + 2x + 1 \leq 0$

3.6 Solving systems of equations

Systems of two linear equations with two unknowns with complex coefficients

When solving simultaneous equations with complex coefficients the methods of elimination and substitution can be very demanding. The method shown here will lead to general formulae for the solutions.

For more on solving systems of two linear equations with two unknowns with real coefficients, see Chapter 14 section 2.5.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

Multiply first equation by d and second equation by b to obtain equal coefficients for the variable y.

$$\Rightarrow \left.\begin{cases} adx + bdy = ed \\ bcx + bdy = fb \end{cases}\right\} -$$

Subtract the equations to eliminate the variable y.

$$\Rightarrow adx - bcx = ed - fb$$

$$\Rightarrow x(ad - bc) = ed - fb$$

Factorize the left hand side.

$$\Rightarrow x = \frac{ed - fb}{ad - bc},\ ad - bc \neq 0$$

$$a \times \frac{ed - fb}{ad - bc} + by = e$$

Substitute the value of x in the first equation to find the value of y.

$$\Rightarrow by = e - a \times \frac{ed - fb}{ad - bc}$$

$$\Rightarrow by = \frac{e(ad-bc)-a(ed-fb)}{ad-bc}$$

$$= \frac{ead-ebc-aed+afb}{ad-bc}$$

$$= \frac{afb-ebc}{ad-bc} \Rightarrow y = \frac{b(af-ec)}{ad-bc} \times \frac{1}{b}$$

$$= \frac{af-ec}{ad-bc},\ ad-bc \neq 0.$$

So the general form of the solution is

$$(x, y) = \left(\frac{ed-fb}{ad-bc}, \frac{af-ec}{ad-bc}\right),\ ad-bc \neq 0$$

Notice that we could have substituted the value of x in the second equation to find the value of y.

These formulae are very efficient when the coefficients of the simultaneous liner equations are complex numbers.

Example 39

Solve the simultaneous equations

$$\begin{cases} 2x+(3-i)y=3 \\ ix+(1+2i)y=2i \end{cases}$$

Answers

$a=2, b=3-i, c=i, d=1+2i, e=3, f=2i$ — *Identify the coefficients.*

$2\times(1+2i)-(3-i)\times i = 2+4i-3i-1 = 1+i$ — *Find the denominator ad – bc*

$3\times(1+2i)-(3-i)\times 2i = 3+6i-6i-2 = 1$ — *Find ed – fb, the numerator for x*

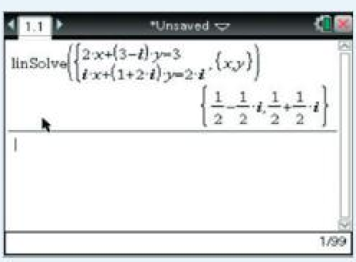
$2\times 2i - 3\times i = 4i-3i = i$ — *Find af – ec, the numerator for y*

$x = \frac{1}{1+i}\times\frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2}-\frac{1}{2}i$ — *Apply the formulae for x and y.*

$y = \frac{i}{1+i}\times\frac{1-i}{1-i} = \frac{i+1}{2} = \frac{1}{2}+\frac{1}{2}i$

$(x, y) = \left(\frac{1}{2}-\frac{1}{2}i, \frac{1}{2}+\frac{1}{2}i\right)$

The same result can be obtained on a GDC, using Solve Systems of Linear Equations.



For a method using determinants See the CD.

Systems of three linear equations with three unknowns

You can use the methods of substitution and elimination to reduce a system of three equations with three unknowns to a system of two equations with two unknowns.

Example 40

Solve the system of equations: $\begin{cases} 2x+4y+z=5 \\ 3x-5y-z=4 \\ x+y-z=6 \end{cases}$

Use:

a the method of substitution

b the method of elimination.

Answers

a $x+y-6=z \Rightarrow \begin{cases} 2x+4y+(x+y-6)=5 \\ 3x-5y-(x+y-6)=4 \end{cases}$

Use the third equation to express z in terms of x and y and substitute for z in the first two equations. Use z because the coefficients of z are simpler.

$\Rightarrow \begin{cases} 3x+5y=11 \\ 2x-6y=-2 \end{cases}$

Use $2x - 6y = -2$ to *express x in terms of y and substitute for x in* $3x + 5y = 11$

$\Rightarrow \begin{cases} 3(3y-1)+5y=11 \\ x=3y-1 \end{cases}$

$\Rightarrow \begin{cases} 9y-3+5y=11 \\ x=3y-1 \end{cases}$

$\Rightarrow \begin{cases} 14y=14 \\ x=3y-1 \end{cases}$

$\Rightarrow \begin{cases} y=1 \\ x=3\cdot 1-1=2 \end{cases}$

$\Rightarrow z = 2 + 1 - 6 = -3$

$\Rightarrow (x, y, z) = (2, 1, -3)$

> If you use substitution to obtain a system of two equations with two unknowns, you don't have to use the same method to solve for the unknowns in this new system – you can use elimination.

b $\left.\begin{array}{r} 2x+4y+z=5 \\ 3x-5y-z=4 \end{array}\right) + \quad \left.\begin{array}{r} 2x+4y+z=5 \\ x+y-z=6 \end{array}\right) + \quad \Rightarrow \begin{cases} 5x-y=9 \\ 3x+5y=11 \end{cases}$

To eliminate z, add the first and second equations and the first and third equations.
You must eliminate the same unknown from both pairs of equations. Eliminate z because z has the simplest coeffiecients.

$\Rightarrow \left.\begin{cases} 25x-5y=45 \\ 3x+5y=11 \end{cases}\right) +$

To eliminate y, multiply $5x - y = 9$ *by 5 and then add* $3x + 5y = 11$.

$\Rightarrow 28x = 56$

$\Rightarrow x = 2$

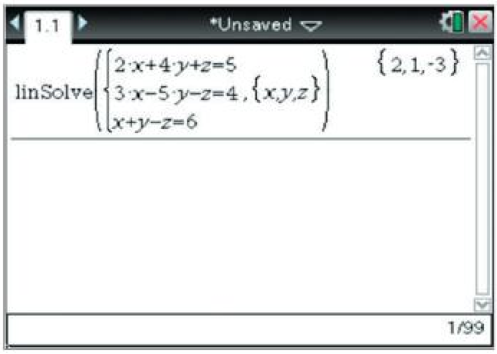
$\Rightarrow 5 \cdot 2 - y = 9 \Rightarrow 1 = y$

$\Rightarrow 2 + 1 - z = 6 \Rightarrow -3 = z$

$\Rightarrow (x, y, z) = (2, 1, -3)$

To find y substitute $x = 2$ *in* $5x - y = 9$
To find z substitute $x = 2$ *and* $y = 1$ *in* $x + y - z = 6$

Linear systems with three unknowns can also be solved using a GDC.



The geometrical interpretation of a linear equation in three variables as a plane is developed in Chapter 11.

For a method using determinants see the CD

When solving systems of three simultaneous linear equations with three unknowns there are again three possible types of solution:

i A unique triplet of numbers (the three variables) that satisfies all three equations.

ii No triplet of real numbers that satisfies all the equations.

iii Infinitely many triplets of real numbers that satisfy all the equations.

To solve three linear equations in three unknowns a special method of elimination was invented by **Johann Carl Friedrich Gauss** (1777–1855). The method, called the Gaussian method, is more suitable to use when the coefficients of the system are in matrix form. It involves eliminating variables in order until you reach the last variable. Consequently you find the variables in reverse order to the order of elimination.

Example 41

Use the Gaussian method to solve the simultaneous equations $\begin{cases} x+3y-2z=3 \\ 2x-4y+3z=5 \\ 4x+y-z=6 \end{cases}$

Answer

$$\begin{cases} x+3y-2z=3 & (1) \\ 2x-4y+3z=5 & (2) \\ 4x+y-z=6 & (3) \end{cases}$$

Eliminate x from equations (2) and (3)

$$\begin{cases} x+3y-2z=3 & (1) \\ 10y-7z=1 & (4) \\ 11y-7z=6 & (5) \end{cases}$$

To obtain equation (4) subtract (2) from 2 · (1)

To obtain equation (5) subtract (3) from 4 · (1)

$$\begin{cases} x+3y-2z=3 & (1) \\ 10y-7z=1 & (4) \\ -\frac{7}{10}z=-\frac{49}{10} & (7) \\ z=7 & \end{cases}$$

To obtain equation (7) subtract (5) from $\frac{11}{10}$ · (4)

Use equation (7) to find z

Continued on next page

$$\begin{cases} x+3y-2z=3 \\ 10y-7\cdot 7=1 \\ y=5 \end{cases}$$

Substitute z = 7 in equation (4) to find y

$$\begin{cases} x+3\cdot 5-2.7\neq 3 \\ x+1\neq 3 \\ x=2 \end{cases}$$

Substitute y = 5 and z = 7 in equation (1) to find the value of x

The solution is $(x, y, z)=(2, 5, 7)$

Example 42

Discuss all the possible types of solution of this system of equations with respect to the real parameter a

$$\begin{cases} ax+y+z=3 \\ x+y+z=1 \\ x+2y-z=2 \end{cases}$$

Answer

$$\begin{cases} ax+y+z=3 & (1) \\ x+y+z=1 & (2) \\ x+2y-z=2 & (3) \end{cases} \Rightarrow \begin{cases} (a-1)x=2 & (4) \\ 2x+3y=3 & (5) \end{cases}$$

Eliminate z
To obtain equation (4) subtract (2) from (1).
To obtain equation (5) add (2) and (3).

$$\Rightarrow \begin{cases} x=\dfrac{2}{a-1} \\ 2\cdot\dfrac{2}{a-1}+3y=3 \end{cases} \Rightarrow \begin{cases} x=\dfrac{2}{a-1} \\ y=\dfrac{3a-7}{3a-3} \end{cases}$$

To find a unique solution assume that a ≠ 1.

To find y substitute for x in (5).

$$\Rightarrow \begin{cases} x=\dfrac{2}{a-1} \\ y=\dfrac{3a-7}{3a-3} \\ \dfrac{2}{a-1}+\dfrac{3a-7}{3a-3}+z=1 \end{cases}$$

To find z substitute for x and y in (2).

$$\Rightarrow (x, y, z)=\left(\frac{2}{a-1}, \frac{3a-7}{3a-3}, \frac{-2}{3a-3}\right), a\neq 1$$

The unique solution when a ≠ 1

If $a = 1 \Rightarrow 0\cdot x = 2 \Rightarrow 0 = 2$
$\Rightarrow (x, y, z) \in \phi$

Equation (4) gives a false statement therefore there is no solution when a=1.

Exercise 3R

1 Solve the following simultaneous equations and check your answers with a GDC.

a $\begin{cases} 2ix+(2+3i)y=1 \\ (1+i)x+2y=3 \end{cases}$

b $\begin{cases} (1+i)x+3iy=2+6i \\ (2-i)x-(4+3i)y=4i-3 \end{cases}$

2 Solve these systems of equations.

a $\begin{cases} x+y=-1 \\ x+z=4 \\ y+z=1 \end{cases}$ **b** $\begin{cases} x-5y+3z=-1 \\ 3x-y+2z=4 \\ 2x+y-z=2 \end{cases}$

c $\begin{cases} 2x+y+2z=0 \\ 6x-4y-5z=-2 \\ 4x+y-3z=2 \end{cases}$ **d** $\begin{cases} 3x-4y+3z=-2 \\ x+2y+6z=6 \\ 2x-6y-3z=-8 \end{cases}$

e $\begin{cases} x+2y+z=4 \\ 2x+y+2z=5 \\ 3x+2y+3z=12 \end{cases}$ **f** $\begin{cases} 2x-3y+5z=-1 \\ 9x-7y+16z=0 \\ x-2y+3z=9 \end{cases}$

3 Find the value(s) of a real parameter k so that each system of equations has no unique solution.

a $\begin{cases} x+2y+z=0 \\ 2x+y+2z=1 \\ x+2y+kz=2 \end{cases}$ **b** $\begin{cases} x+y+z=1 \\ 2x+ky+3z=-2 \\ 3x+5y+kz=-1 \end{cases}$

4 Find the value(s) of a real parameter k so that each system of equations has infinitely many solutions. Find the solutions.

a $\begin{cases} x+2y+3z=1 \\ kx+4y+3z=2 \\ 3x+6y-2z=3 \end{cases}$

b $\begin{cases} x+y+z=1 \\ 2x+ky+3z=-2 \\ 3x+5y+kz=-1 \end{cases}$

5 Find the values of a real parameter m so that the system of equations has a unique solution.

$\begin{cases} x+y+z=m \\ x+my+z=2m \\ x+y+mz=-1 \end{cases}$

Hence, find the solution in terms of m.

For more challenging systems of simultaneous equations see the CD

Review exercise

EXAM-STYLE QUESTIONS

1 When the polynomial $f(x) = x^4 - 3x^3 + ax^2 - 4x + 7$ is divided by $(x + 2)$ the remainder is 7. Find the value of a.

2 Solve the simultaneous equations:

$$\begin{cases} 3x - 2y = i - 2 \\ 4y - (1 - i)x = 3 + 3i \end{cases}$$

3 Find the value of m in the quadratic function $f(x) = m - 2 + (2m + 1)x + mx^2$ if $f(x) \leq 0$ for all real x.

4 Given that $1 - 2i$ is a complex root of the equation $z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$, find the remaining roots.

5 Find the value of m such that this system of equations has no unique solution.

$$\begin{cases} mx + 2y = 1 \\ 4x + (m + 2)y = 4 \end{cases}$$

6 Find the value of a such that the roots α and β of the quadratic equation $x^2 + ax + a + 1 = 0$ satisfy $\alpha^3 + \beta^3 = 9$

7 Given that $z = \frac{1+i}{2}$, use mathematical induction to show that

$z^{2n} = \frac{i^n}{2^n}$, $n \in \mathbb{Z}^+$

8 Show that the imaginary part of the number $\left(\frac{1+i}{1-i}\right)^{2011}$ is -1.

9 The cubic equation $x^3 - 5x^2 + 6x - 3 = 0$ has solutions α, β and γ. Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

10 a Show that $\sqrt[3]{7 - \sqrt{50}} + \sqrt[3]{7 + \sqrt{50}}$ satisfies the equation $x^3 + 3x - 14 = 0$

b Factorize the polynomial $f(z) = z^3 + 3z - 14$, $z \in \mathbb{C}$ and find all the possible zeros.

c Hence, find the value of $\sqrt[3]{7 - \sqrt{50}} + \sqrt[3]{7 + \sqrt{50}}$.

Review exercise

EXAM-STYLE QUESTIONS

1 Solve the inequality $x^3 + 5x^2 + 2x - 22 \geq 0$

2 Find all the values of the real parameter m for which the equation $(mx)^2 + 3x + 1 - m = 0$ has no real solution.

3 Solve these simultaneous equations and write your answers as fractions.

$$\begin{cases} 2x + 14y + 9z = -7 \\ 4x - 3z = 4 + 7y \\ 10x - 28y = 5 + 6z \end{cases}$$

4 Given that α, β and γ are solutions of the equation $3x^3 + 2x = 5x^2 + 4$, find the value of $\alpha^3 + \beta^3 + \gamma^3$

5 Find the smallest zero of the polynomial $f(x) = x^7 + 35x^6 - 97x^5 + 33x^2 + 4$

CHAPTER 3 SUMMARY

Zero factor property

$a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

Quadratic formula

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant

$\Delta = b^2 - 4ac$

i If $\Delta > 0$ there are two distinct real roots.

ii If $\Delta = 0$ there is one repeated real root.

iii If $\Delta < 0$ there are no real roots (conjugate complex pair of solutions.)

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Operations with complex numbers

Given that $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $a_1, b_1, a_2, b_2 \in \mathbb{R}$

$(z_1 = z_2) \Leftrightarrow (a_1 = a_2 \text{ and } b_1 = b_2)$

$z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$

$\lambda z = \lambda(a + ib) = (\lambda a) + i(\lambda b), \lambda \in \mathbb{R}$

$z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$

$|z| = |a + ib| = a^2 + b^2$

$$\frac{z_1}{z_2} = \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2} = \frac{z_1z_2^*}{|z_2|}$$

Axioms of complex numbers

A1 For every complex numbers z_1 and z_2 then $z_1 + z_2$ is a complex number (Closure)

A2 For every complex numbers z_1 and z_2 then $z_1 + z_2 = z_2 + z_1$ (Commutativity)

A3 For every complex numbers z_1, z_2 and z_3 then $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associativity)

A4 There exists a complex number $0 = 0 + 0i$ such that for every complex number z, $0 + z = z + 0 = z$ (Additive identity)

A5 For every complex number z there exists a complex number $-z$ such that $z + -z = -z + z = 0$ (Additive inverse)

A6 For every complex numbers z_1 and z_2 then $z_1 \cdot z_2$ is a complex number (Closure)

A7 For every complex numbers z_1 and z_2 then $z_1 \cdot z_2 = z_2 \cdot z_1$ (Commutativity)

A8 For every complex numbers z_1, z_2 and z_3 then $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$ (Associativity)

A9 There exists a complex numbers $1 = 1 + 0i$ such that for every complex numbers z, $1 \cdot z = z \cdot 1 = z$ (Multiplicative identity)

A10 For every complex numbers z, $z \neq 0$, there exists a complex numbers z^{-1} such that $z \cdot z^{-1} = z^{-1} \cdot z = 0$ (Multiplicative inverse)

A11 For every complex numbers z_1, z_2 and z_3 then $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$ (Distributivity of multiplication over addition)

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Viète's formulae for quadratic equations

$$ax^2 + bx + c = 0 \Rightarrow x_1 + x_2 = -\frac{b}{a} \text{ and } x_1 \cdot x_2 = \frac{c}{a}$$

Viète's formulae for cubic equations

$$ax^3 + bx^2 + cx + d = 0 \Rightarrow \begin{cases} x_1 + x_2 + x_3 = -\frac{b}{a} \\ x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = \frac{c}{a} \\ x_1 \cdot x_2 \cdot x_3 = -\frac{d}{a} \end{cases}$$

Viète's formula for equations of the *n*th degree

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0 \Rightarrow$$

$$\sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \left(x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k} \right) = (-1)^k \frac{a_{n-k}}{a_n},\ 1 \le k \le n$$

Degree of polynomials

The degree of a polynomial, $f(x) = a_n x^n + a^{n-1} x^{n-1} + \ldots + a_1 x + a_0$, is the largest power of x appearing: $\deg(f) = n$.

For a liner combination of two polynomials, $af(x) + bg(x)$ with $a, b \in \mathbb{R}$, or the product of two polynomials, $f(x) \cdot g(x)$, the degree is given by

$\deg(af + bg) = \max\{\deg(f), \deg(g)\}$

$\deg(f \cdot g) = \deg(f) + \deg(g)$

Unique decomposition

For any two polynomials f and g there are unique polynomials q and r such that $f(x) = g(x) \cdot q(x) + r(x)$, for all real values of x.

Remainder theorem

Given a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n$, $a_k \neq 0$ and a real number p, then the remainder when $f(x)$ is divided by a linear expression $(x - p)$ is $f(p)$.

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Factor theorem

A polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real coefficients ($a_n \neq 0$) has a factor $(x - p)$, $p \in \mathbb{R}$, if and only if $f(p) = 0$.

Fundamental theorem of algebra (FTA)

A polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real or complex coefficients ($a_n \neq 0$) has at least one zero.

Each polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with real coefficients can be written in a factor form $f(x) = a_n(x - \omega_1)(x - \omega_2) \ldots (x - \omega_n)$ such that $\omega_k \in \mathbb{C}$, $k, \ldots, n$.

Given a polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a_k \in \mathbb{Z}$, $a_n \neq 0$ and an integer p such that $f(p) = 0$, then p is a factor of a_0.

Given a polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, $a \in \mathbb{Z}$ $a_n \neq 0$ and a rational number

$\frac{p}{q}$, where $\gcd(p, q) = 1$ that is $\left(\frac{p}{q}\right)$ is in its simplest form, such that

$f\left(\frac{p}{q}\right) = 0$, then p is a factor of a_0 and q is a factor of a_n.

All the possible zeros of the polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ are in the interval

$\left[-\left(\frac{M}{|a_n|} + 1\right), \frac{M}{|a_n|} + 1\right]$ where $M = \max\{|a_n|, |a_{n-1}|, \ldots |a_1|, |a_0|\}$