8.5 Basic probability theory

Probability is the branch of mathematics that analyses random experiments. A **random experiment** is one in which we cannot predict the precise outcome. Examples of random experiments are ‘tossing a coin’ or ‘rolling a dice’ or ‘predicting the gold, silver, and bronze medalists in a 100 m sprint’.

> There are some assumptions being made:
> **1** the coin is unbiased
> **2** the dice is unbiased
> **3** all sprinters are evenly matched

It is impossible to predict the outcome of a random experiment **precisely** but it is possible to

a list the set of all possible outcomes of the experiment

b decide how likely a particular outcome may be.

When tossing a coin, there are two possible outcomes: **heads** (H) and **tails** (T).

Also, the likelihood of getting a head is the same as getting a tail, so the probability of getting a head is one chance out of two. The probability of getting a tail is the same.

In other words, the set of equally likely outcomes is $\{H, T\}$ and $P(H) = P(T) = \frac{1}{2}$.

When rolling a dice, the set of equally likely possible outcomes has six elements and is $\{1, 2, 3, 4, 5, 6\}$.

As all six outcomes are equally likely, $P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$.

Let event A be ‘rolling an even number’.

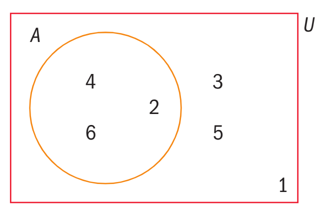
To find $P(A)$, consider the set of equally likely outcomes $\{1, 2, 3, 4, 5, 6\}$.

There are six equally likely outcomes and three of these are even numbers, so $P(A) = \frac{3}{6}$.

Let B be the event ‘rolling a prime number’.

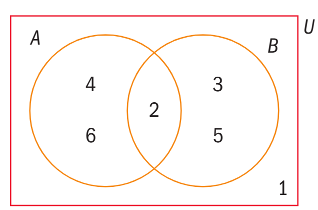
To find $P(B)$, look again at the set of outcomes. There are three prime numbers: 2, 3, and 5 so, $P(B) = \frac{3}{6}$.

We can show the equally likely possible outcomes of rolling a dice on a Venn diagram using $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{\text{even numbers}\}$.



$$P(A) = \frac{n(A)}{n(U)} = \frac{3}{6}$$

Set B can be added to the Venn diagram to represent the event B.



$$P(B) = \frac{n(B)}{n(U)} = \frac{3}{6}$$

→ If all of the equally likely possible outcomes of a random experiment can be listed as U, the universal set, and an event A is defined and represented by a set A, then:

$$P(A) = \frac{n(A)}{n(U)}$$

There are three consequences of this law:

1 $P(U) = \frac{n(U)}{n(U)} = 1$ (the probability of a **certain** event is 1)

2 $P(\varnothing) = \frac{n(\varnothing)}{n(U)} = 0$ (the probability of an **impossible** event is 0)

3 $0 \le P(A) \le 1$ (the probability of any event **always** lies between 0 and 1)

Example 11

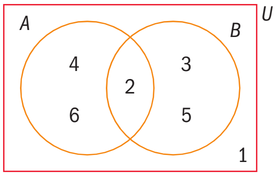
Find the probability that these events occur for the random experiment 'rolling a fair dice'.

a Rolling an odd number
b Rolling an even prime number
c Rolling an odd prime number
d Rolling a number that is either prime or even

Unless stated otherwise, we will always be talking about a cubical dice with faces numbered 1 to 6.

Answers

Use the Venn diagram drawn earlier, where A is the event 'rolling an even number' and B is the event 'rolling a prime number'.



a $P(A') = \frac{n(A')}{n(U)} = \frac{3}{6}$

A is the event 'rolling an even number', so the probability of rolling an odd number is P(A′). From the Venn diagram, A′ = {1, 3, 5}.

b $P(A \cap B) = \frac{n(A \cap B)}{n(U)} = \frac{1}{6}$

A is the event 'rolling an even number', and B is the event 'rolling a prime number', so the probability of rolling an even prime number is P(A ∩ B).

c $P(A' \cap B) = \frac{n(A' \cap B)}{n(U)} = \frac{2}{6}$

The probability of rolling an odd prime number is P(A′ ∩ B).

d $P(A \cup B) = \frac{n(A \cup B)}{n(U)} = \frac{5}{6}$

The probability of rolling a number that is either prime or even is P(A ∪ B).

This example illustrates the basics of probability theory: list the equally likely possible outcomes of a random experiment and count. Drawing a Venn diagram may clarify the situation.

Two further laws of probability:

- For complementary events, $P(A') = 1 - P(A)$
- For combined events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Use the Venn diagram to illustrate these laws.

Exercise 8J

1 A random experiment is: roll an unbiased six-faced dice.
Let A be the event 'roll a square number' and let B be the event 'roll a factor of 6'.

a List the elements of set A.
b List the elements of set B.
c Show sets A and B on a Venn diagram.
d Write down $P(A)$.
e Write down $P(B)$.
f Find the probability that the number rolled is not a square number.
g Find the probability that the number rolled is both a square number and a factor of 6.
h Find the probability that the number rolled is either a square number or a factor of 6 or both.
i Verify that both $P(A') = 1 - P(A)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

2 The numbers 3, 4, 5, 6, 7, 8, 9, 10 are written on identical pieces of card and placed in a bag. A random experiment is: a card is selected at random from the bag.
Let A be the event 'a prime number is chosen' and let B be the event 'an even number is chosen'.

a List the elements of set A.
b List the elements of set B.
c Show sets A and B on a Venn diagram.
d Write down $P(A)$.
e Write down $P(B)$.
f Find the probability that the number rolled is composite (not a prime).
g Find the probability that the number rolled is odd.
h Find the probability that the number rolled is both even and prime.
i Find the probability that the number rolled is either even or prime or both.
j Verify that both $P(A') = 1 - P(A)$ and $P(B') = 1 - P(B)$.
k Verify that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
l Find the probability that the number rolled is both odd and composite.
m Find the probability that the number rolled is either odd or composite or both.
n Verify that $P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$

3 The numbers 2, 3, 4, 5, 6, 7, 8, 9 are written on identical pieces of card and placed in a bag. A random experiment is: a card is selected at random from the bag.
Let A be the event 'an odd number is chosen' and let B be the event 'a square number is chosen'.

a List the elements of set A.
b List the elements of set B.
c Show sets A and B on a Venn diagram.
d Write down P(A).
e Write down P(B).
f Find the probability that an odd square number is chosen.
g Find the probability that either an odd number or a square number is chosen.
h Verify that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The first book written on probability was *The Book of Chance and Games* by Italian philosopher and mathematician Jerome Cardan (1501–75). It explained techniques on how to cheat and catch others at cheating.

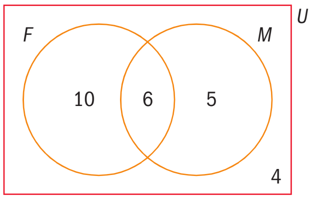
4 A random experiment is: toss two unbiased coins.
a List the set of four equally likely possible outcomes.
b Find P(two heads show), P(one head shows), P(no heads show).

5 A random experiment is: toss three unbiased coins.
a List the set of eight equally likely possible outcomes.
b Find P(no heads), P(one head), P(two heads), P(three heads).

6 A random experiment is: toss four unbiased coins.
a Find P(no heads).
b Find P(four heads).
c Find P(one head).
d Find P(three heads).
e Use the answers **a** to **d** to deduce P(two heads).
f List the equally likely possible outcomes.

8.6 Conditional probability

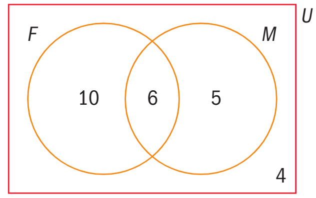
In a class of 25 students, 16 students study French, 11 students study Malay and 4 students study neither language. This information can be shown in a Venn diagram.



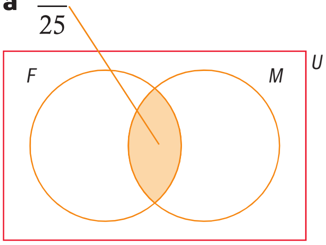
Suppose a student is chosen at random from the class. We can use the techniques we have looked at already to find the probability that

a the student studies French and Malay
b the student studies exactly one language
c the student does not study two languages
d the student does not study French.

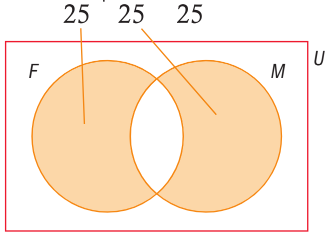
Using the Venn diagram on the right:



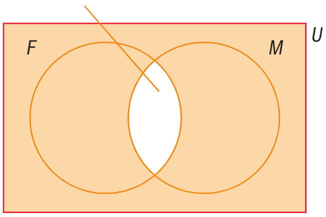
a $\frac{6}{25}$



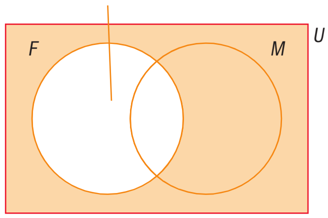
b $\frac{10}{25}+\frac{5}{25}=\frac{15}{25}$



c $1-\frac{6}{25}=\frac{19}{25}$



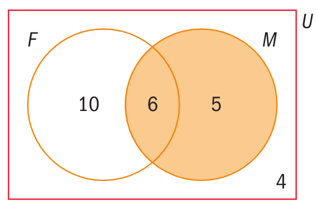
d $1-\frac{16}{25}=\frac{9}{25}$



What is the probability that a student chosen at random studies French, **given that** the student studies Malay?

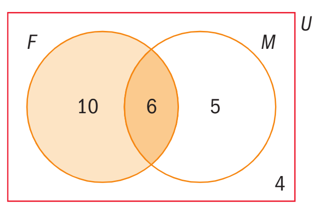
This requires a different approach because there is an extra condition: the student studies Malay.

The probability that a student studies French given that the student studies Malay is an example of a **conditional probability**. It is written $P(F|M)$.



Given that *M* has definitely occurred, then we are restricted to set M (the shaded area), rather than choosing from the universal set (the rectangle).

If we now want to determine the probability that F has also occurred, then we consider that part of F which also lies within M – the intersection of F and M (darkest shading).



The conditional probability, the probability that a student studies French given that the student studies Malay, is

$$P(F|M) = \frac{n(F \cap M)}{n(M)} = \frac{6}{11}$$

→ The conditional probability that A occurs given that B has occurred is written as $P(A|B)$ and is defined as:

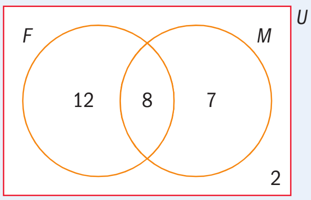
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example 12

In a class of 29 students, 20 students study French, 15 students study Malay, and 8 students study both languages. A student is chosen at random from the class.
Find the probability that the student

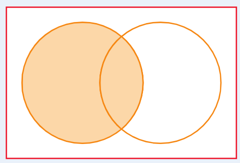
a studies French
b studies neither language
c studies at least one language
d studies both languages
e studies Malay given that they study French
f studies French given that they study Malay
g studies both languages given that they study at least one of the languages.

Answers

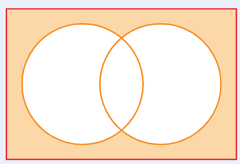


First draw a Venn diagram to show the information.

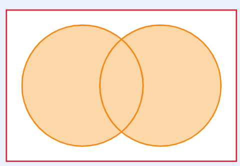
a P(studies French) = $\frac{20}{29}$



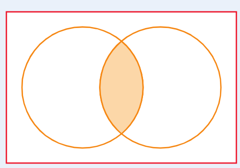
b P(studies neither language) = $\frac{2}{29}$



c P(studies at least one language) = $\frac{27}{29}$

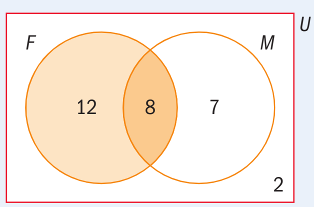


d P(studies both languages) = $\frac{8}{29}$



e P(studies Malay given that they study French)

$= \mathrm{P}(M \mid F) = \frac{n(M \cap F)}{n(F)} = \frac{8}{20}$



Probabilities **e** to **g** are conditional, and require more care

Continued on next page

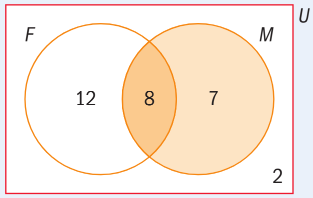
f P(studies French given that they study Malay)

$= \mathrm{P}(F \mid M) = \dfrac{n(F \cap M)}{n(M)} = \dfrac{8}{15}$

g P(studies both languages given that they study at least one language)

$= \mathrm{P}(F \cap M \mid F \cup M)$

$= \dfrac{n([F \cap M] \cap [F \cup M])}{n(F \cup M)} = \dfrac{8}{27}$

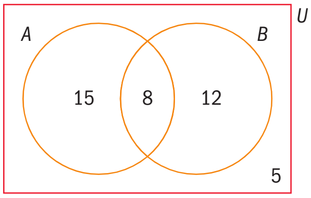


Looking at the Venn diagram you can see that $(F \cap M) \cap (F \cup M) = (F \cap M)$

Exercise 8K

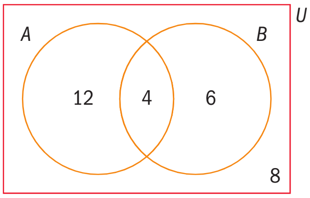
The numbers in each set are shown on the Venn diagrams.

1 Find the probability that a person chosen at random:



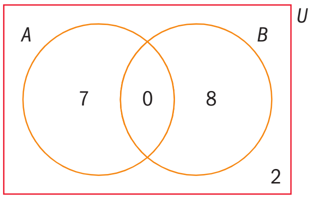
- **a** is in A
- **b** is not in either A or B
- **c** is not in A and not in B
- **d** is in A, given that they are not in B
- **e** is in B, given that they are in A
- **f** is in both A and B, given that they are in A.

2 Find the probability that a person chosen at random:



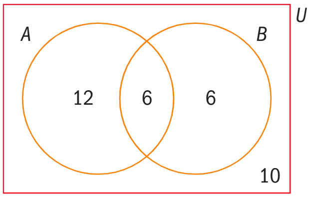
- **a** is not in A
- **b** is neither in A nor in B
- **c** is not in both A and B given that they are in B
- **d** is not in A given that they are not in B
- **e** is in B given that they are in A
- **f** is in both A and B, given that they are not in A.

3 Find the probability that a person chosen at random:



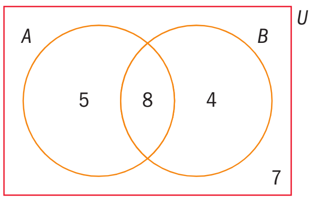
- **a** is in B but not in A
- **b** is not in A or B
- **c** is in B and not in A
- **d** is in A given that they are not in B
- **e** is in B given that they are in A
- **f** is not in both A and B, given that they are in A.

4 Find the probability that a person chosen at random:



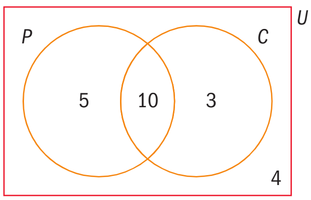
- **a** is in A but not in both A and B
- **b** is not in A and not in both
- **c** is not in both A and B
- **d** is in A given that they are not in B
- **e** is in B given that they are in A
- **f** is not in A given that they are not in B.

5 The Venn diagram shows the number of students who take Art and/or Biology in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



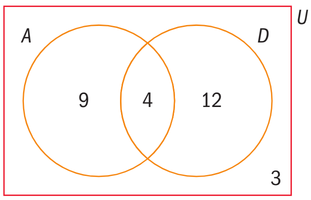
a takes Art
b takes Biology but not Art
c takes both Art and Biology
d takes at least one of the two subjects
e takes neither subject
f takes Biology
g takes exactly one of the two subjects.

6 The Venn diagram shows the number of students who take Physics and/or Chemistry in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



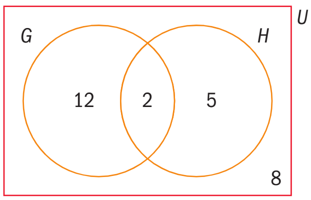
a takes Physics but not Chemistry
b takes at least one of the two subjects
c takes Chemistry given that the student takes Physics
d is a Chemist given that the student takes exactly one of the two subjects.

7 The Venn diagram shows the number of students who take Art and/or Drama in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



a takes Drama but not Art
b takes Drama given that they take Art
c takes both subjects given that they take Drama
d takes neither subject
e takes Drama given that they take exactly one of the two subjects.

8 The Venn diagram shows the number of students who take Geography and/or History in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



a takes Geography but not History
b takes Geography given that they do not take History
c takes History given that they take at least one of the two subjects
d takes Geography given they take History
e takes Geography given that they take exactly one of the two subjects.