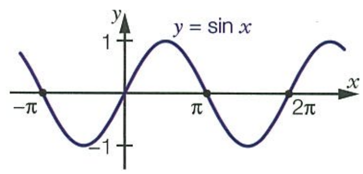
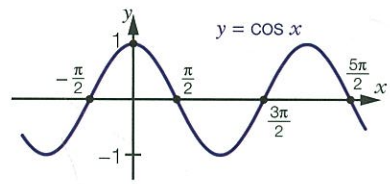
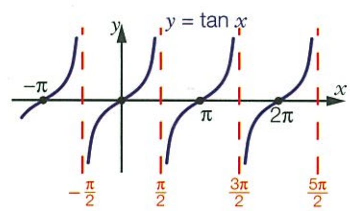
6 TRIGONOMETRY

WHAT YOU NEED TO KNOW

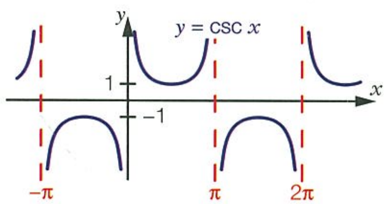
- The graphs of trigonometric functions:

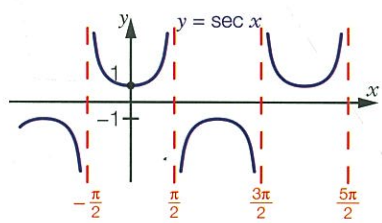


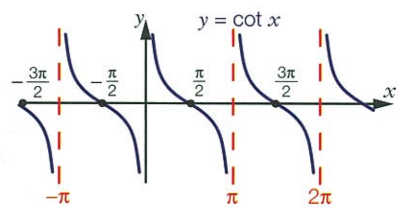




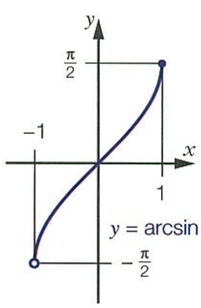
- The graphs of reciprocal trigonometric functions:

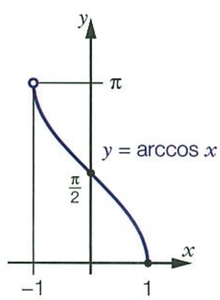


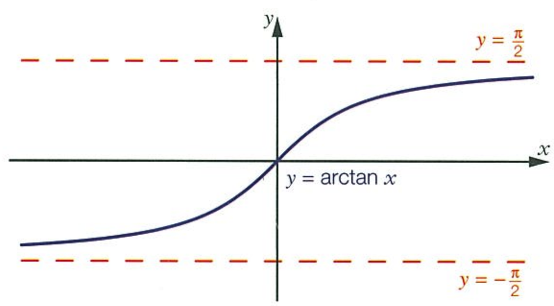




- The graphs of inverse trigonometric functions:







- The following exact values:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	

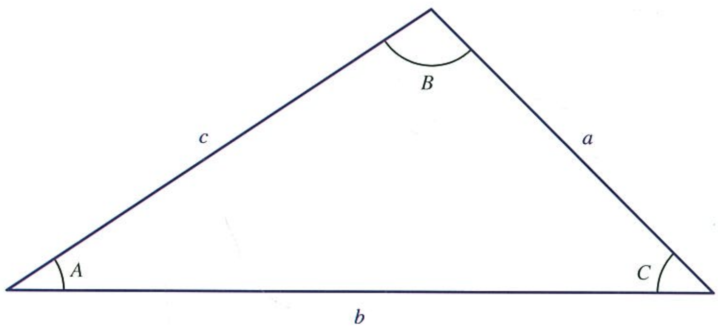
- Trigonometric functions are related through:
  - Identities:

    $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad\sec\theta = \dfrac{1}{\cos\theta}$ $\quad\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

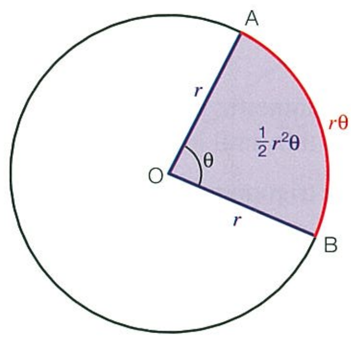
    $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$
  - Pythagorean identities:
    - $\cos^2\theta + \sin^2\theta = 1$
    - $1 + \tan^2\theta = \sec^2\theta$
    - $1 + \cot^2\theta = \csc^2\theta$
  - Double angle identities:
    - $\sin 2\theta = 2\sin\theta\cos\theta$
    - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
    - $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
  - Compound angle identities:
    - $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
    - $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
    - $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- To prove trigonometric identities, start with one side of the identity (often the left-hand side) and transform this until it equals the other side.
- When solving trigonometric equations of the form $\sin A = k$, $\cos A = k$ or $\tan A = k$:
  - Draw a graph to see how many solutions there are.
  - Use $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ to find one possible value of A: A_0.
  - Find the second solution, A_1, by using the symmetry of the graph:
    - For $\sin A = k$, $A_1 = \pi - A_0$
    - For $\cos A = k$, $A_1 = 2\pi - A_0$
    - For $\tan A = k$, $A_1 = A_0 + \pi$
  - Find all the solutions in the required interval for A by adding multiples of 2π.
- With more complicated equations, it may be necessary to first use one or more of the identities to manipulate the equation into a form that can be solved.

A common type of equation that you should look out for is a 'disguised quadratic' arising from the use of one of the Pythagorean identities or the $\cos 2x$ formulae.

- A function $a\sin x \pm b\cos x$ can be written in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$ using the relevant compound angle identity.
- The sine and cosine rules are used to find the sides and angles of any triangle:



  - Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
  - Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$, or $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$
  - The sine rule is only used when a side and its opposite angle are given.
- The area of a triangle can be found using Area $= \dfrac{1}{2}ab\sin C$.
- In a circle of radius r with an angle of θ radians subtended at the centre:
  - the length of the arc AB: $l = r\theta$
  - the area of the sector AOB: $A = \dfrac{1}{2}r^2\theta$



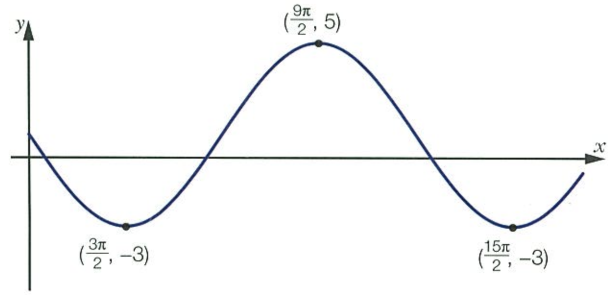
EXAM TIPS AND COMMON ERRORS

- Always check whether you should be working in degrees or radians.
- When solving equations, make sure you find **all** the solutions in the specified interval.
- Remember that $\sin 2x$ means $\sin(2x)$ – it is **not** the same as $2\sin x$.
- You need to understand that trigonometric formulae can be used in both directions. For example, knowing the double angle identity for sine, you can replace $\sin x\cos x$ by $\dfrac{1}{2}\sin 2x$.

6.1 TRANSFORMATIONS OF TRIGONOMETRIC GRAPHS

WORKED EXAMPLE 6.1

The graph shown has equation $y = c - a\sin\left(\frac{x}{b}\right)$. Find the values of a, b and c.



$$y_{max} - y_{min} = 5 - (-3) = 8$$

$$\therefore a = \frac{8}{2} = 4$$

a has the effect of stretching the sine curve vertically. The distance between the maximum and minimum values of the original sine curve is 2. The minus sign in front of a causes a vertical reflection; it does not affect the amplitude.

Transforming graphs is covered in Chapter 4.

$$\text{period} = \frac{15\pi}{2} - \frac{3\pi}{2} = 6\pi$$

$$\therefore b = \frac{6\pi}{2\pi} = 3$$

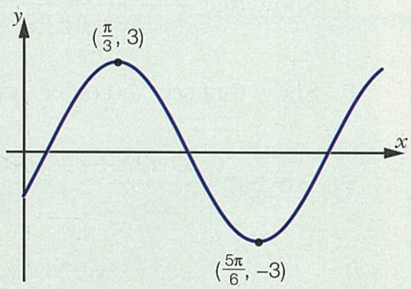
Dividing x by b stretches the sine curve horizontally by a factor of b. The period of the original sine curve is 2π.

$$c = 1$$

c causes a vertical translation. After multiplication by $-a = -4$, the minimum and maximum values would be -4 and 4 respectively, but here they are -3 and 5.

Practice questions 6.1

1. The graph shown has equation $y = a\sin\left(bx - \frac{\pi}{6}\right)$. Find the values of a and b.



2. Find the range of the function $f(x) = \frac{2}{5 + 2\sin x}$.

3. Find the smallest positive value of x for which $\frac{1}{3 - \sin\left(x - \frac{\pi}{4}\right)}$ takes its maximum value.

6.2 PROVING TRIGONOMETRIC IDENTITIES

WORKED EXAMPLE 6.2

Show that $\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}=2\sec x$ for $x \neq k\pi$.

$$\text{LHS}=\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}$$

We will start from the left-hand side (LHS) and transform it until we get the right-hand side (RHS).

$$=\frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)}+\frac{\cos x(1-\sin x)}{(1+\sin x)(1-\sin x)}$$

$$=\frac{\cos x+\cos x\sin x}{1-\sin^2 x}+\frac{\cos x-\cos x\sin x}{1-\sin^2 x}$$

$$=\frac{2\cos x}{1-\sin^2 x}$$

To combine these fractions, create a common denominator of $1-\sin^2 x$.

$$=\frac{2\cos x}{\cos^2 x} \quad (\text{as } \sin^2 x = 1-\cos^2 x)$$

$$=\frac{2}{\cos x}$$

$$=2\sec x=\text{RHS}$$

We can then use the identity $\sin^2 x+\cos^2 x=1$ on the denominator.

It is not always obvious how to get from one side to the other. If you're not sure, transform each side separately and meet in the middle.

Practice questions 6.2

4. Show that $\frac{\cos 4x}{\cos 2x+\sin 2x}=\cos 2x-\sin 2x$.

5. Show that $\sec^2 x(\cot^2 x-\cos^2 x)=\cot^2 x$.

6. Prove that $\frac{\cos 2x+1}{\cos^2 x}+\frac{1-\cos 2x}{\sin^2 x}=4$.

7. Show that $\frac{\csc x-\sin x}{\cos x}-\frac{\sec x-\cos x}{\sin x}=2\cot 2x$.

8. Prove that $\frac{\sin 2x+\sin x}{1+\cos 2x+\cos x}=\tan x$.

9. Show that $\csc x-\sin x=\cot x\cos x$.

6.3 USING IDENTITIES TO FIND EXACT VALUES OF TRIGONOMETRIC FUNCTIONS

WORKED EXAMPLE 6.3

Given that $\cos A = \frac{1}{3}$ and $A \in \left[0, \frac{\pi}{2}\right]$, find the exact values of:

(a) $\csc A$

(b) $\sin 2A$

(a) $\sin^2 A = 1 - \cos^2 A$

$= 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore \sin A = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}$

$\csc A = \frac{1}{\sin A}$, so we need to find $\sin A$ from the given value of $\cos A$. To do this we can use the identity $\sin^2 A + \cos^2 A = 1$.

Remember the ± when you take the square root.

For $A \in \left[0, \frac{\pi}{2}\right]$, $\sin A \geq 0$

$\therefore \sin A = \frac{2\sqrt{2}}{3}$

$\Rightarrow \csc A = \frac{1}{\sin A} = \frac{3}{2\sqrt{2}}$

We now have to choose either the positive or negative value, so consider the sine function in the given interval $0 \leq A \leq \frac{\pi}{2}$.

(b) $\sin 2A = 2\sin A \cos A$

$= 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right)$

$= \frac{4\sqrt{2}}{9}$

As we need $\sin 2A$, the only option is to use the sine double angle identity.

Substitute the values for $\sin A$ and $\cos A$.

Practice questions 6.3

10. Given that $\cos 2\theta = \frac{2}{3}$ and $\theta \in \left[\frac{\pi}{2}, \pi\right]$, find the exact value of $\cos\theta$.

11. Given that $\cos\theta = \frac{3}{4}$ and $\theta \in \left[\frac{3\pi}{2}, 2\pi\right]$, find the exact value of $\sin\theta$.

12. Given that $\tan A = 2$ and $A \in \left[0, \frac{\pi}{2}\right]$, find the exact value of $\sin A$.

6.4 SOLVING TRIGONOMETRIC EQUATIONS

WORKED EXAMPLE 6.4

Solve the equation $\csc 3x = 2$ for $x \in \left[0, \frac{\pi}{2}\right]$.

$\csc 3x = 2 \Leftrightarrow \sin 3x = \frac{1}{2}$

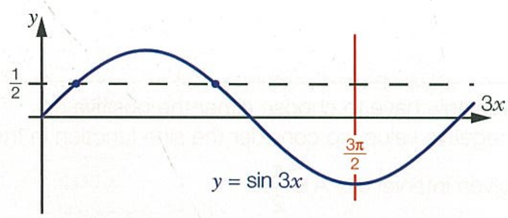
$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

First, convert the original equation into sine form so that we can use $\sin^{-1}$ to find the first value of $3x$.

Always start by converting equations involving sec, csc or cot into a form containing cos, sin or tan only.

$x \in \left[0, \frac{\pi}{2}\right] \Rightarrow 3x \in \left[0, \frac{3\pi}{2}\right]$

Find the range of possible values for $3x$.



Draw the graph to see how many solutions there are in the required interval. Here only two are needed.

$3x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore x = \frac{\pi}{18}$ or $\frac{5\pi}{18}$

For the sine function, the second value will always be π – (first value).

Finally, divide by 3 to find x.

It is a common error to divide by 3 first and then find the second value. Always find all the possible values in the interval first and then make x the subject at the end.

Practice questions 6.4

13. Solve the equation $\sec 2x = \sqrt{2}$ for $x \in [0, \pi]$.

14. Solve the equation $\tan 3x = \sqrt{3}$ for $x \in [-\pi, \pi]$.

15. Solve the equation $\sin\left(3x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$ for $x \in [-\pi, \pi]$.

16. Solve the equation $3\csc^2 2x = 4$ for $0 \le x \le 2\pi$.

6.5 USING IDENTITIES TO SOLVE TRIGONOMETRIC EQUATIONS

WORKED EXAMPLE 6.5

Solve the equation $3\sin^2 x + 1 = 4\cos x$ for $x \in [-\pi, \pi]$.

$3\sin^2 x + 1 = 4\cos x$

$\Leftrightarrow 3(1 - \cos^2 x) + 1 = 4\cos x$

$\Leftrightarrow 3 - 3\cos^2 x + 1 = 4\cos x$

Using the identity $\sin^2 x + \cos^2 x = 1$, we can replace the $\sin^2$ term and thereby form an equation that contains only one type of trigonometric function (cos).

If possible, use an identity to ensure that there is only one type of trigonometric function in the equation.

$\Leftrightarrow 3\cos^2 x + 4\cos x - 4 = 0$

$\Leftrightarrow (3\cos x - 2)(\cos x + 2) = 0$

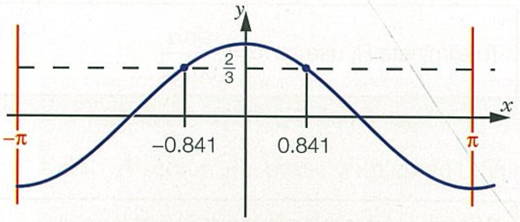
$\Leftrightarrow \cos x = \frac{2}{3}$ or $\cos x = -2$

$\therefore \cos x = \frac{2}{3}$ (as $-1 \le \cos x \le 1$)

This now becomes a standard quadratic equation, which can be factorised and solved.

Disguised quadratics are also encountered in Chapter 2 when solving exponential equations.

$\cos^{-1}\left(\frac{2}{3}\right) = 0.841$ (3 SF)



Draw the graph to see how many solutions there are in the required interval. Here only two are needed.

$2\pi - 0.841 = 5.44$ is not in the interval

$5.44 - 2\pi$ is in the interval

$\therefore x = \pm 0.841$

For the cosine function, the second value will always be 2π – (first value), but here that is not in the required interval. Therefore, subtract 2π from this value to get the second answer.

Practice questions 6.5

17. Solve the equation $3\csc^2\theta + 5\cot\theta = 5$ for $\theta \in [-\pi, \pi]$.

18. Solve the equation $\cos 2x - 2 = 5\cos x$ for $x \in [0, 2\pi]$.

19. Solve the equation $\cot x = 2\cos x$ for $-\pi < x < \pi$.

20. Find the values of $\theta \in [0, \pi]$ for which $\tan^2 2\theta - 4\sec 2\theta + 5 = 0$.

6.6 FUNCTIONS OF THE FORM $a\sin x + b\cos x$

WORKED EXAMPLE 6.6

Express $f(x) = 2\cos x - 5\sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $\alpha \in \left[0, \frac{\pi}{2}\right]$.

$$R\cos(x+\alpha) = R(\cos x\cos\alpha - \sin x\sin\alpha)$$
$$= R\cos\alpha\cos x - R\sin\alpha\sin x$$

First expand $R\cos(x+\alpha)$ using the compound angle identity.

Comparing with $2\cos x - 5\sin x$ gives:

$R\cos\alpha = 2 \quad \cdots (1)$

$R\sin\alpha = 5 \quad \cdots (2)$

Equate the coefficients of $\cos x$ and $\sin x$ to get equations for $R\cos\alpha$ and $R\sin\alpha$.

$(1)^2 + (2)^2$:

$R^2\cos^2\alpha + R^2\sin^2\alpha = 2^2 + 5^2$

$\Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 2^2 + 5^2$

$\Rightarrow R^2 = 29$

$\therefore R = \sqrt{29}$

To solve these simultaneous equations, eliminate α by squaring and adding and then applying the identity $\cos^2\alpha + \sin^2\alpha = 1$.

$(2) \div (1)$:

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{5}{2}$

$\Leftrightarrow \tan\alpha = \frac{5}{2}$

$\therefore \alpha = \tan^{-1}\left(\frac{5}{2}\right) = 1.19$ (3 SF)

To eliminate R, use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$.

$\therefore 2\cos x - 5\sin x = \sqrt{29}\cos(x + 1.19)$

Write the expression in the required form.

Practice questions 6.6

21. Find the exact values for $R > 0$ and $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $\sqrt{3}\sin x + \cos x = R\sin(x+\theta)$.

Hence find the minimum value of $\frac{2}{3 + \sqrt{3}\sin x + \cos x}$.

⚠ These question types will always tell you which form to use: $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$.

22. Show that $5\cos x + 3\sin x$ can be written in the form $R\cos(x - \alpha)$. Hence find the coordinates of the maximum point on the graph of $y = 5\cos x + 3\sin x$ for $x \in [0, 2\pi]$.

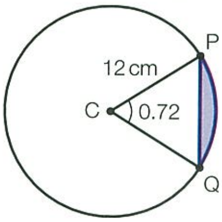
6.7 GEOMETRY OF TRIANGLES AND CIRCLES

WORKED EXAMPLE 6.7

The diagram shows a circle with centre C and radius 12 cm. Points P and Q are on the circumference on the circle and $P\hat{C}Q = 0.72$ radians.

(a) Find the length of the chord PQ.

(b) Find the area of the shaded region.



(a) Using the cosine rule in triangle PCQ:

$PQ^2 = 12^2 + 12^2 - 2\times 12\times 12\cos 0.72$

$= 71.5$

$\therefore PQ = 8.45$ cm

CP = CQ = 12 since CP and CQ are both radii. As we don't have an angle and its opposite side, we need to use the cosine rule.

Make sure your calculator is in radian mode.

(b) Area of sector PCQ:

$\frac{1}{2}r^2\theta = \frac{1}{2}\times 12^2 \times 0.72 = 51.84$

Area of triangle PCQ:

$\frac{1}{2}ab\sin C = \frac{1}{2}(12)(12)\sin 0.72 = 47.5$

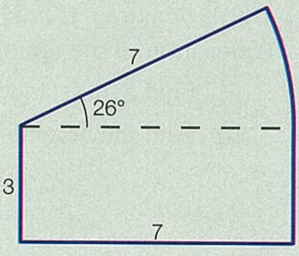
Shaded area $= 51.84 - 47.5 = 4.36\text{ cm}^2$

We can find the area of the shaded region by subtracting the area of the triangle from the area of the sector.

Practice questions 6.7

23. A sector of a circle with angle 0.65 radians has area 14.8 cm². Find the radius of the circle.

24. The diagram shows a rectangle and a sector of a circle. Find the perimeter.



25. A vertical cliff BT, of height 50 m, stands on horizontal ground. The angle of depression of the top of a lighthouse, L, from the top of the cliff is 20°. The angle of elevation of L from the bottom of the cliff is 15°. Find the height of the lighthouse.

The angle of elevation is the angle above the horizontal.
The angle of depression is the angle below the horizontal.