59 Given the functions $f : x \mapsto 2x - 1$ and $g : x \mapsto 2x^3$, find the function $(f \circ g)^{-1}$.

60 One zero of $x^4 + 2x^3 + 8x^2 + 6x + 15$ has form bi where $b \neq 0$, $b \in \mathbb{R}$. Find b and all zeros of the polynomial.

TOPIC 3: CIRCULAR FUNCTIONS AND TRIGONOMETRY

RADIAN MEASURE

There are $360° \equiv 2\pi$ radians in a circle.

To convert from degrees to radians, multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

APPLICATIONS OF RADIANS

For θ in radians:

- the length of an arc of radius r and angle θ is $l = r\theta$
- the area of a sector of radius r and angle θ is $A = \frac{1}{2}\theta r^2$
- the area of a segment of radius r and angle θ is $A = \frac{1}{2}\theta r^2 - \frac{1}{2}r^2 \sin\theta$.

THE UNIT CIRCLE

The **unit circle** is the circle centred at the origin O, with radius 1 unit.

The coordinates of any point P on the unit circle, where the angle θ is made by [OP] and the positive x-axis, are $(\cos\theta, \sin\theta)$.

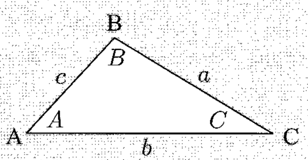
θ is **positive** when measured in an **anticlockwise** direction from the positive x-axis.

$\tan\theta$ is defined as $\dfrac{\sin\theta}{\cos\theta}$.

You should memorise or be able to quickly find the values of $\cos\theta$, $\sin\theta$, and $\tan\theta$ for θ that are multiples of $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.

NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

For the triangle alongside:



Area formula Area $= \frac{1}{2}ab\sin C$

Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

Sine rule $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

If you have the choice of rules to use, use the cosine rule to avoid the **ambiguous case**.

THE GENERAL SINE FUNCTION

If we begin with $y = \sin x$, we can perform transformations to produce the **general sine function** $f(x) = a\sin(b(x-c)) + d$.

We have a vertical stretch with factor $|a|$, a reflection in the x-axis if $a < 0$, then a horizontal stretch with factor $\frac{1}{b}$, and finally a translation with vector $\begin{pmatrix} c \\ d \end{pmatrix}$.

For the general sine function:

- the **amplitude** is $|a|$
- the **principal axis** is $y = d$
- the **period** is $\frac{2\pi}{b}$.

OTHER TRIGONOMETRIC FUNCTIONS

You should be able to graph and use:

- the cosine function $y = a\cos(b(x-c)) + d$
- the tangent function $y = a\tan(b(x-c)) + d$

$y = \tan bx$ has period $\frac{\pi}{b}$.

RECIPROCAL TRIGONOMETRIC FUNCTIONS

cosec x or $\csc x = \dfrac{1}{\sin x}$.

secant x or $\sec x = \dfrac{1}{\cos x}$.

cotangent x or $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$.

When graphing $\csc x$, $\sec x$, and $\cot x$, there will be vertical asymptotes corresponding to the zeros of $\sin x$, $\cos x$, and $\tan x$. $\cot x$ will have zeros corresponding to the vertical asymptotes of $\tan x$.

TRIGONOMETRIC IDENTITIES

$\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$ for all $k \in \mathbb{Z}$

NEGATIVE ANGLES

$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, and $\tan(-\theta) = -\tan\theta$

COMPLEMENTARY ANGLES

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

PYTHAGOREAN IDENTITIES

$\cos^2\theta + \sin^2\theta = 1$, $\tan^2 x + 1 = \sec^2 x$, and $1 + \cot^2 x = \csc^2 x$

DOUBLE ANGLE FORMULAE

$\sin 2A = 2\sin A\cos A$

$$\cos 2A = \begin{cases} \cos^2 A - \sin^2 A \\ 1 - 2\sin^2 A \\ 2\cos^2 A - 1 \end{cases}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

COMPOUND ANGLE FORMULAE

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

TRIGONOMETRIC EQUATIONS

To solve trigonometric equations we can either use graphs from technology, or algebraic methods involving the trigonometric identities. In either case we must make sure to include all solutions on the specified domain.

We need to use the inverse trigonometric functions to invert sin, cos, and tan.

Function	*Domain*	*Range*
$x \mapsto \arcsin x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$x \mapsto \arccos x$	$[-1, 1]$	$[0, \pi]$
$x \mapsto \arctan x$	$]-\infty, \infty[$	$\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

The ranges of these functions are important because our calculator will only give us the one answer in the range. Remember that other solutions may also be possible. For example, when using arcsin our calculator will always give us an acute angle answer, but the obtuse angle with the same sine may also be valid.

An equation of the form $a\sin x = b\cos x$ can always be solved as $\tan x = \frac{b}{a}$.

SKILL BUILDER QUESTIONS

1 Convert:

a $\frac{2\pi}{9}$ radians to degrees **b** $140°$ to radians.

2 Find the exact value of:

a $\sin\left(\frac{5\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(-\frac{\pi}{3}\right)$

3 A sector of a circle of radius 10 cm has a perimeter of 40 cm. Find the area of the sector.

4 What consecutive transformations map the graph of $y = \sin x$ onto:

a $y = 2\sin\left(\frac{x}{3}\right)$ **b** $y = \sin\left(x + \frac{\pi}{3}\right) - 4$?

5 Find the amplitude, principal axis, and period of the following functions:

a $f(x) = \sin 4x$ **b** $f(x) = -2\sin\left(\frac{x}{2}\right) - 1$.

6 Sketch the graph of $y = \csc(x)$ for $x \in [0, 3\pi]$.

7 Simplify $\sin\left(\frac{3\pi}{2} - \phi\right)\tan(\phi + \pi)$.

8 If $\cos(2x) = \frac{5}{8}$, find the exact value of $\sin x$.

9 Solve $\sin 2x = \sin x$ for $x \in [-\pi, \pi]$, giving exact answers.

10 A sector of a circle has an arc length of 6 cm and an area of 20 cm^2. Find the angle of the sector.

11 Find the period of:

a $y = -\sin(3x)$ **b** $y = 2\sin\left(\frac{x}{2}\right) + 1$

c $y = \sin^2 x + 5$.

12 Sketch the graph of $y = \arccos x$, clearly showing the axes intercepts and endpoints.

13 Simplify $1 - \dfrac{\sin^2\theta}{1 + \cos\theta}$.

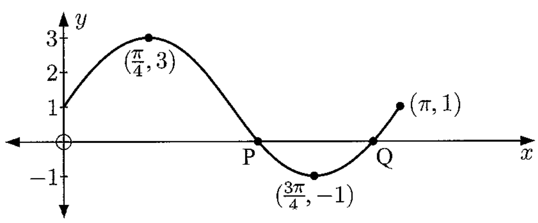
14 If $\tan\theta = 2$, find the exact values of $\tan 2\theta$ and $\tan 3\theta$.

15 Show that $\csc(2x) - \cot(2x) = \tan x$ and hence find the exact value of $\tan\left(\frac{5\pi}{12}\right)$.

16 If $\cos 2\alpha = \sin^2\alpha$, find the exact value of $\cot\alpha$.

17 A chord of a circle has length 6 cm. If the radius of the circle is 5 cm, find the area of the minor segment cut off by the chord.

18



For the illustrated sine function, find the coordinates of the points P and Q.

19 Find the period of:

a $y = \cos\left(\frac{x}{3}\right)$ **b** $y = \tan(5x)$

c $y = \sin 3x + \sin x$.

20 Find the largest angle of the triangle with sides 11 cm, 9 cm, and 7 cm.

21 Find the equations of the vertical asymptotes on $[-2\pi, 2\pi]$ for:

a $f(x) = \csc(x)$ **b** $f : x \mapsto \sec(2x)$

c $g : x \mapsto \cot\left(\frac{x}{2}\right)$.

22 Find the exact value of $\cos 79° \cos 71° - \sin 79° \sin 71°$.

23 Given that $\tan 2A = \sin A$ where $\sin A \neq 0$, find $\cos A$ in simplest radical form.

24 Suppose $\sin x - 2\cos x = A\sin(x + \alpha)$ where $A > 0$ and $0 < \alpha < 2\pi$. Find A and α.

25 $2\sin^2 x - \cos x = 1$ for $x \in [0, 2\pi]$. Find the exact value(s) of x.

26 Find x if $\arcsin(2x - 3) = -\frac{\pi}{6}$.

27 In triangle ABC, AB = 15 cm, AC = 12 cm and angle ABC measures 30°. Find the size of the angle ACB.

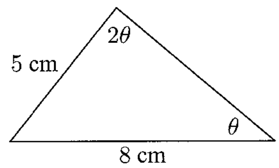
28 On the same set of axes, sketch the graphs of $f(x) = \sin x$ and $g(x) = -1 + 2f(2x + \frac{\pi}{2})$ for $-\pi \leqslant x \leqslant \pi$.

29 Find the exact value of $\arcsin(-\frac{1}{2}) + \arctan(1) + \arccos(-\frac{1}{2})$.

30 θ is obtuse and $\sin\theta = \frac{2}{3}$. Find the exact value of $\sin 2\theta$.

31 Solve for x: $\sin x + \cos x = 1$ where $0 \leqslant x \leqslant \pi$.

32



a Find $\cos\theta$. **b** Find the area of the triangle.

33 Find the exact period of $g(x) = \tan 2x + \tan 3x$.

34 Solve the equation $\cot\theta + \tan\theta = 2$ for $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

35 If $2\theta \in [\pi, \frac{3\pi}{2}]$ and $\tan(2\theta) = 2$, find the exact value of $\tan\theta$.

36 In triangle PQR, PR = 12 cm, RQ = 11 cm, and $\widehat{RPQ} = 60°$. Find the length of [PQ], giving your answer in radical form.

37 Show that $\dfrac{1}{\tan\theta - \sec\theta} = -(\sec\theta + \tan\theta)$ provided $\cos\theta \neq 0$.

38 Solve for x where $x \in [-\pi, 3\pi]$, giving exact answers:

a $\sqrt{3}\tan\left(\frac{x}{2}\right) = -1$ **b** $\sqrt{3} + 2\sin(2x) = 0$.

39 If $\sin x = 2\sin\left(x - \frac{\pi}{6}\right)$, find the exact value of $\tan x$.

40 In a busy harbour, the time difference between successive high tides is about 12.3 hours. The water level varies by 2.4 metres between high and low tide. Tomorrow, the first high tide will be at 1 am, and the water level will be 4.7 metres at this time.

a Find a sine model for the height of the tide H in terms of time t tomorrow.

b Sketch a graph of the water level in the harbour tomorrow.

41 Find:

a $\sin\left(\arccos(-\frac{\sqrt{3}}{2})\right)$ **b** $\tan\left(\arcsin(\frac{1}{\sqrt{2}})\right)$

42 Find the exact solutions of $\sin x + \sqrt{3}\cos x = 0, \ x \in [0, 2\pi]$.

43 If $\dfrac{\sin\theta + 2\cos\theta}{\sin\theta - \cos\theta} = 2$, find the exact value of $\tan 2\theta$.

44 Without using technology, sketch the graph of $y = 2\sin\left(x - \frac{\pi}{3}\right) + 1$ for $x \in [-\pi, \pi]$.

45 Solve for x: $\cos 2x + \sqrt{3}\sin 2x = 1$ on the interval $[-\pi, \pi]$.

TOPIC 4: VECTORS

A **vector** is a quantity with both **magnitude** and **direction**.

Two vectors are **equal** if and only if they have the same magnitude *and* direction.

In examinations:

- scalars are written in italics a
- vectors are written in bold **a**.

On paper, you should write vector **a** as $\vec{a}$.

The basic unit vectors with magnitude 1 are:

$$\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \ \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \text{ and } \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

The **zero vector 0** is $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

The general 3-dimensional vector

$$\mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

You should understand the following for vectors in both algebraic and geometric forms:

- vector addition
- vector subtraction $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$
- multiplication by a scalar k to produce vector $k\mathbf{v}$ which is parallel to **v**
- the magnitude of vector **v**, $|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$
- the distance between two points in space is the magnitude of the vector which joins them.

The **position vector** of $A(x, y, z)$ is $\overrightarrow{OA}$ or $\mathbf{a} = \begin{pmatrix}x\\y\\z\end{pmatrix}$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

Given $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the distance $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- the midpoint of $\overrightarrow{AB}$ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$.

A, B, and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

The unit vector in the direction of **a** is $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$.

THE SCALAR OR DOT PRODUCT OF TWO VECTORS

$\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\cos\theta$ where θ is the angle between the vectors.

$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$

Properties of the scalar product:

$\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$

$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$

$(k\mathbf{v}) \bullet \mathbf{w} = k(\mathbf{v} \bullet \mathbf{w})$

$\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$

For non-zero vectors **v** and **w**:

- $\mathbf{v} \perp \mathbf{w} \Leftrightarrow \mathbf{v} \bullet \mathbf{w} = 0$
- $\mathbf{v} \parallel \mathbf{w} \Leftrightarrow |\mathbf{v} \bullet \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|$ and in this case $\mathbf{v} = k\mathbf{w}$.

The angle between vectors **v** and **w** emanating from the same point, is given by $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}$.

If $\mathbf{v} \bullet \mathbf{w} > 0$ then θ is acute.

If $\mathbf{v} \bullet \mathbf{w} < 0$ then θ is obtuse.

THE VECTOR OR CROSS PRODUCT OF TWO VECTORS

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

$\mathbf{v} \times \mathbf{w}$ is perpendicular to both **v** and **w**. Its direction is found using the right hand rule.

Properties of the vector product:

If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ then $\mathbf{v} \parallel \mathbf{w}$.

$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

$(k\mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w})$

Geometric properties:

$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin\theta$ where θ is the angle between the vectors.

$|\mathbf{v} \times \mathbf{w}|$ = area of parallelogram formed by vectors **v** and **w**.

$\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$ = area of triangle formed by vectors **v** and **w**.

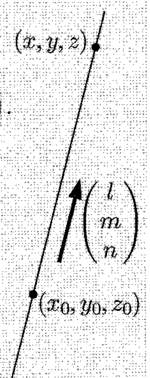
LINES

The **vector equation of a line** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where **a** is the position vector of any point on the line, **b** is a vector parallel to the line, and $\lambda \in \mathbb{R}$.

For example, if an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ for $t \geqslant 0$.

The **parametric form** for the equation of a line is $x = x_0 + \lambda l, \ y = y_0 + \lambda m, \ z = z_0 + \lambda n$

where $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{a} = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix}l\\m\\n\end{pmatrix}$.



The **Cartesian form** for the equation of a line is $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n} = \lambda$.

The **acute angle θ between two lines** is given by $\cos\theta = \dfrac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1|\,|\mathbf{b}_2|}$ where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the lines.