55 $P(n) = 1000 + ae^{kn}$

Now $P(0) = 2000$

$\therefore\ 1000 + a = 2000$ and so $a = 1000$

Also, $P(12) = 4000$

$\therefore\ 1000 + 1000e^{12k} = 4000$

$\therefore\ 1000e^{12k} = 3000$

$\therefore\ e^{12k} = 3$

$\therefore\ e^{k} = 3^{\frac{1}{12}}$

If $P(n) = 10\,000$, $1000 + ae^{kn} = 10\,000$

$\therefore\ 1000e^{kn} = 9000$

$\therefore\ e^{kn} = 9$

$\therefore\ 3^{\frac{n}{12}} = 3^2$

$\therefore\ n = 24$ months

It will take 2 years for the population to reach 10 000.

56 **a** Since a is a solution of the equation,

$3a^3 - 11a^2 + 8a = 12a$

$\therefore\ 3a^3 - 11a^2 - 4a = 0$

$\therefore\ a(3a^2 - 11a - 4) = 0$

$\therefore\ a(3a+1)(a-4) = 0$

$\therefore\ a = 0, -\frac{1}{3}$, or 4

b If $a = 0$, $3x^3 - 11x^2 + 8x = 0$

$\therefore\ x(3x^2 - 11x + 8) = 0$

$\therefore\ x(3x-8)(x-1) = 0$

$\therefore\ x = 0, \frac{8}{3}$, or 1

If $a = -\frac{1}{3}$, $3x^3 - 11x^2 + 8x = 12(-\frac{1}{3})$

$\therefore\ 3x^3 - 11x^2 + 8x + 4 = 0$

$x = a$ is a solution, and so $(3x+1)$ must be a factor.

$\therefore\ 3x^3 - 11x^2 + 8x + 4 = (3x+1)(x^2 + ax + 4)$ for some a

Equating coefficients of x^2 gives $-11 = 1 + 3a$

$\therefore\ a = -4$

$\therefore\ (3x+1)(x^2 - 4x + 4) = 0$

$\therefore\ (3x+1)(x-2)^2 = 0$

$\therefore\ x = -\frac{1}{3}$ or 2

If $a = 4$, $3x^3 - 11x^2 + 8x = 12(4)$

$\therefore\ 3x^3 - 11x^2 + 8x - 48 = 0$

$x = a$ is a solution, so $(x-4)$ is a factor.

$\therefore\ 3x^3 - 11x^2 + 8x - 48 = (x-4)(3x^2 + ax + 12)$ for some a

Equating coefficients of x^2 gives $-11 = a - 12$

$\therefore\ a = 1$

$\therefore\ (x-4)(3x^2 + x + 12) = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 3 \times 12}}{2 \times 3}$$

$$x = 4 \text{ or } \frac{-1 \pm i\sqrt{143}}{6}$$

57 $P(x)$ is a real polynomial, so $3 + 2i$ must also be a zero of $P(x)$.

$(3+2i) + (3-2i) = 6$ and

$(3+2i)(3-2i) = 9 + 4$

$= 13$

So, $x^2 - 6x + 13$ is a factor of $P(x)$.

$\therefore\ 2x^3 + mx^2 - (m+1)x + (3 - 4m)$

$= (x^2 - 6x + 13)(2x + b)$ for some b

$= 2x^3 + (b-12)x^2 + (26 - 6b)x + 13b$

Equating coefficients of x^2: $m = b - 12$ .... (1)

Equating constants: $3 - 4m = 13b$ .... (2)

Substituting (1) into (2): $13b = 3 - 4(b - 12)$

$\therefore\ 13b = 3 - 4b + 48$

$\therefore\ 17b = 51$

$\therefore\ b = 3$

$\therefore\ m = 3 - 12 = -9$

$\therefore\ P(x) = (x^2 - 6x + 13)(2x + 3)$

$\therefore$ the zeros of $P(x)$ are $3 \pm 2i$ and $-\frac{3}{2}$

58 $a^2 \times a^3 + a^2 - a^4 \times a - 2 = 0$

$\therefore\ a^2 - 2 = 0$

$\therefore\ a = \pm\sqrt{2}$

$\therefore\ P(z) = 2z^3 + z^2 - 4z - 2$

Since $P(z)$ is the same whether $a = \pm\sqrt{2}$, both $z = \sqrt{2}$ and $z = -\sqrt{2}$ must be zeros of $P(z)$.

Hence $(z - \sqrt{2})(z + \sqrt{2}) = (z^2 - 2)$ is a factor of $P(z)$.

$\therefore\ P(z) = (z^2 - 2)(2z + 1)$

$\therefore$ the zeros of $P(x)$ are $\pm\sqrt{2}$ and $-\frac{1}{2}$

59 $(f \circ g)(x) = f(g(x))$

$= f(2x^3)$

$= 2(2x^3) - 1$

$= 4x^3 - 1$

So the function $(f \circ g)^{-1}$ is $x = 4y^3 - 1$

$\therefore\ 4y^3 = x + 1$

$$\therefore\ y^3 = \frac{x+1}{4}$$

$$\therefore\ y = \left(\frac{x+1}{4}\right)^{\frac{1}{3}}$$

So, $(f \circ g)^{-1} : x \mapsto \left(\frac{x+1}{4}\right)^{\frac{1}{3}}$.

60 Let $P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$

Since bi is a zero of the real polynomial $P(x)$, so is $-bi$.

$\therefore\ x^2 + b^2$ is a factor of $P(x)$

$\therefore\ P(x) = x^4 + 2x^3 + 8x^2 + 6x + 15$

$= (x^2 + b^2)(x^2 + cx + d)$ for some c, d

$= x^4 + cx^3 + (b^2 + d)x^2 + b^2cx + b^2d$

Equating the coefficients of x^3: $c = 2$

Equating the coefficients of x: $2b^2 = 6$

$\therefore\ b = \pm\sqrt{3}$

Equating the coefficients of x^2: $3 + d = 8$

$\therefore\ d = 5$

$\therefore\ P(x) = (x^2 + 3)(x^2 + 2x + 5)$

Now $x^2 + 2x + 5 = 0$

$$\text{when } x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$= -1 \pm 2i$

$\therefore$ the zeros are $\pm\sqrt{3}i$, $-1 \pm 2i$

SOLUTIONS TO TOPIC 3 (CIRCULAR FUNCTIONS AND TRIGONOMETRY)

1 **a** $\frac{2\pi}{9}$ radians

$= \left(\frac{2\pi}{9} \times \frac{180}{\pi}\right)^\circ$

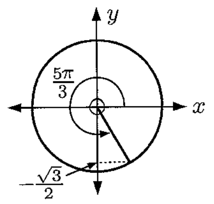
$= 40^\circ$

b 140°

$= \left(140 \times \frac{\pi}{180}\right)$ radians

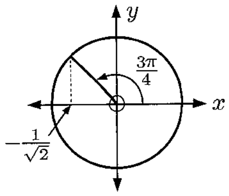
$= \frac{7\pi}{9}$

2 **a**



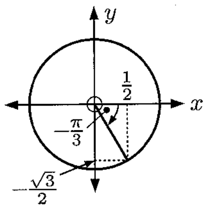
$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

b



$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

c

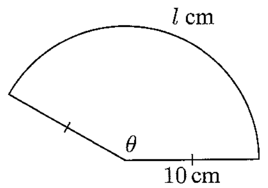


$\tan\left(-\frac{\pi}{3}\right)$

$= \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$= -\sqrt{3}$

3



perimeter $= 40$ cm

$\therefore\ 10 + 10 + l = 40$

$\therefore\ l = 20$

area $= \frac{1}{2}\theta r^2$

$= \frac{1}{2}lr \quad \{l = \theta r\}$

$= \frac{1}{2} \times 20 \times 10$

$= 100 \text{ cm}^2$

4 **a** A vertical stretch with factor 2, and a horizontal stretch with factor 3.

b A translation of $\frac{\pi}{3}$ units to the left, and a translation of 4 units downwards.

5 **a** Amplitude $= 1$

The principal axis is $y = 0$

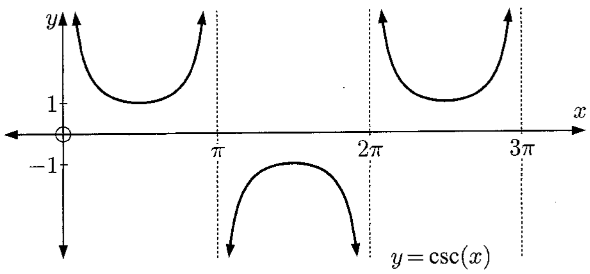
Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

b Amplitude $= 2$

The principal axis is $y = -1$

Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

6



7 $\sin\left(\frac{3\pi}{2} - \phi\right)\tan(\phi + \pi)$

$= \left(\sin\frac{3\pi}{2}\cos\phi - \cos\frac{3\pi}{2}\sin\phi\right)\tan\phi$

$= ((-1)\cos\phi - 0 \times \sin\phi)\dfrac{\sin\phi}{\cos\phi}$

$= -\sin\phi$

8 $\cos 2x = \frac{5}{8}$

$\therefore\ 1 - 2\sin^2 x = \frac{5}{8}$ {double angle formula}

$\therefore\ 2\sin^2 x = \frac{3}{8}$

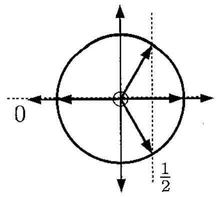
$\therefore\ \sin x = \pm\frac{\sqrt{3}}{4}$

9 $\sin 2x = \sin x, \quad x \in [-\pi, \pi]$

$\therefore\ 2\sin x\cos x - \sin x = 0$

$\therefore\ \sin x(2\cos x - 1) = 0$

$\therefore\ \sin x = 0$ or $\cos x = \frac{1}{2}$



$\therefore\ x = 0, \pm\frac{\pi}{3},$ or $\pm\pi$

10 area $= 20$ cm^2

$\therefore\ \frac{1}{2}\theta r^2 = 20$

$\therefore\ \frac{1}{2}lr = 20 \quad \{l = \theta r\}$

$\therefore\ \frac{1}{2}(6)r = 20$

$\therefore\ r = \frac{20}{3}$ cm

So, $\theta = \dfrac{l}{r}$

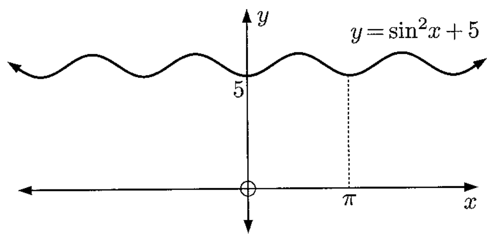
$= \dfrac{6}{\frac{20}{3}}$

$= 0.9$

11 **a** Period $= \frac{2\pi}{3}$

b Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

c



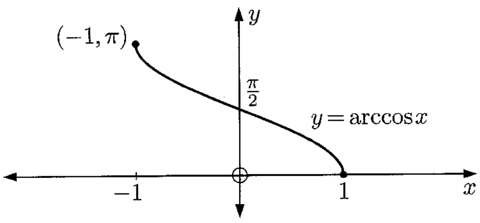
Period $= \pi$

or $\sin^2 x + 5 = \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) + 5$

$= -\frac{1}{2}\cos 2x + \frac{11}{2}$

$\therefore\ $ period $= \frac{2\pi}{2} = \pi$

12



13 $1 - \dfrac{\sin^2\theta}{1 + \cos\theta} = \dfrac{1 + \cos\theta}{1 + \cos\theta} - \dfrac{\sin^2\theta}{1 + \cos\theta}$

$= \dfrac{1 + \cos\theta - (1 - \cos^2\theta)}{1 + \cos\theta}$

$= \dfrac{\cos\theta + \cos^2\theta}{1 + \cos\theta}$

$= \dfrac{\cos\theta(1 + \cos\theta)}{1 + \cos\theta}$

$= \cos\theta$

14 $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$= \dfrac{2 \times 2}{1 - 2^2}$

$= -\frac{4}{3}$

$\tan 3\theta = \tan(2\theta + \theta)$

$= \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$

$= \dfrac{-\frac{4}{3} + 2}{1 - \left(-\frac{4}{3}\right) \times 2}$

$= \frac{2}{11}$

15 $\csc(2x) - \cot(2x) = \dfrac{1}{\sin 2x} - \dfrac{\cos 2x}{\sin 2x}$

$= \dfrac{1-\cos 2x}{\sin 2x}$

$= \dfrac{1-(1-2\sin^2 x)}{2\sin x \cos x}$ {double angle formulae}

$= \dfrac{2\sin^2 x}{2\sin x \cos x}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x$

$\therefore\ \tan\left(\frac{5\pi}{12}\right) = \csc\left(\frac{5\pi}{6}\right) - \cot\left(\frac{5\pi}{6}\right)$

$= \dfrac{1}{\sin\left(\frac{5\pi}{6}\right)} - \dfrac{1}{\tan\left(\frac{5\pi}{6}\right)}$

$= \dfrac{1}{\frac{1}{2}} - \dfrac{1}{-\frac{1}{\sqrt{3}}}$

$= 2 + \sqrt{3}$

16 $\cos 2\alpha = \sin^2\alpha$

$\therefore\ 1 - 2\sin^2\alpha = \sin^2\alpha$

$\therefore\ 1 = 3\sin^2\alpha$

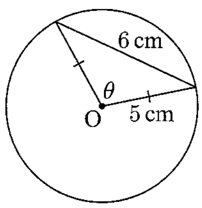
$\therefore\ \sin^2\alpha = \frac{1}{3}$

$\therefore\ \cos^2\alpha = \frac{2}{3}$

$\therefore\ \cot^2\alpha = \dfrac{\cos^2\alpha}{\sin^2\alpha} = 2$

$\therefore\ \cot\alpha = \pm\sqrt{2}$

17



$\cos\theta = \dfrac{5^2+5^2-6^2}{2\times 5\times 5}$ {cosine rule}

$\therefore\ \cos\theta = \dfrac{14}{50}$

$\therefore\ \theta \approx 1.287$

area of triangle $= \frac{1}{2}\times 5\times 5\times \sin\theta$

$= 12\ \text{cm}^2$

area of sector $= \frac{1}{2}\times\theta\times 5^2$

$\approx 16.088\ \text{cm}^2$

$\therefore$ area of minor segment = area of sector − area of triangle

$\approx 4.09\ \text{cm}^2$

18 For the sine function $y = a\sin b(x-c)+d$:

The amplitude $= 2$, so $a = 2$.

The period $= \pi$, so $\frac{2\pi}{b} = \pi \Rightarrow b = 2$.

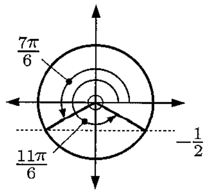
The principal axis is $y = 1$, so $d = 1$.

There is no horizontal translation, so $c = 0$.

$\therefore$ the function is $y = 2\sin(2x)+1$

We want to solve $2\sin(2x)+1 = 0,\ 0 \leqslant x \leqslant \pi$

$\therefore\ \sin 2x = -\frac{1}{2}$



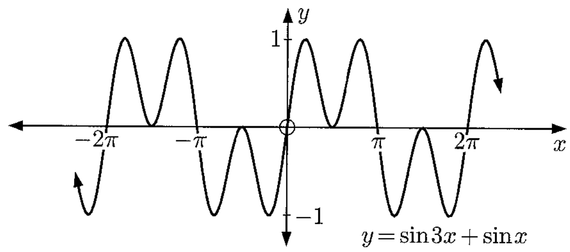
$\therefore\ 2x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

$\therefore\ x = \frac{7\pi}{12}$ or $\frac{11\pi}{12}$

So, P is $\left(\frac{7\pi}{12}, 0\right)$ and Q is $\left(\frac{11\pi}{12}, 0\right)$.

19 **a** Period $= \frac{2\pi}{\frac{1}{3}} = 6\pi$ **b** Period $= \frac{\pi}{5}$

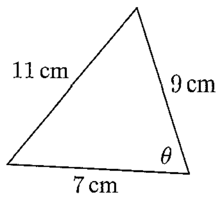
c



$y = \sin 3x$ has period $\frac{2\pi}{3}$, and $y = \sin x$ has period 2π.

So, $y = \sin 3x + \sin x$ has period 2π.

20



The largest angle is opposite the longest side.

$\cos\theta = \dfrac{9^2+7^2-11^2}{2\times 7\times 9}$ {cosine rule}

$\therefore\ \cos\theta = \frac{9}{126}$

$\therefore\ \theta \approx 85.9°$

21 **a** $\csc(x) = \dfrac{1}{\sin x}$

$\therefore$ vertical asymptotes occur when $\sin x = 0$

$\therefore$ the vertical asymptotes are $x = 0, \pm\pi$, and $\pm 2\pi$

b $\sec(2x) = \dfrac{1}{\cos 2x}$

$\therefore$ vertical asymptotes occur when $\cos 2x = 0$

$\therefore\ 2x = \pm\frac{\pi}{2} + k2\pi,\ k\in\mathbb{Z}$

$\therefore\ x = \pm\frac{\pi}{4} + k\pi,\ k\in\mathbb{Z}$

$\therefore$ the vertical asymptotes are

$x = \pm\frac{\pi}{4},\ \pm\frac{3\pi}{4},\ \pm\frac{5\pi}{4}$, and $\pm\frac{7\pi}{4}$.

c $\cot\left(\dfrac{x}{2}\right) = \dfrac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$

$\therefore$ vertical asymptotes occur when $\sin\left(\dfrac{x}{2}\right) = 0$

$\therefore\ \dfrac{x}{2} = 0 + k\pi,\ k\in\mathbb{Z}$

$\therefore\ x = 2k\pi,\ k\in\mathbb{Z}$

$\therefore\ x = 0$ and $x = \pm 2\pi$

$\therefore$ the vertical asymptotes are $x = 0$ and $x = \pm 2\pi$.

22 $\cos 79°\cos 71° - \sin 79°\sin 71° = \cos(79° + 71°)$

$= \cos(150°)$

$= -\frac{\sqrt{3}}{2}$

23 $\tan 2A = \sin A$

$\therefore\ \dfrac{\sin 2A}{\cos 2A} = \sin A$

$\therefore\ \dfrac{2\sin A\cos A}{2\cos^2 A - 1} = \sin A$ {double angle formula}

$\therefore\ \dfrac{2\cos A}{2\cos^2 A - 1} = 1$ $\{\sin A \neq 0\}$

$2\cos A = 2\cos^2 A - 1$

$\therefore\ 2\cos^2 A - 2\cos A - 1 = 0$

$\therefore\ \cos A = \dfrac{2\pm\sqrt{(-2)^2 - 4\times 2\times(-1)}}{2\times 2}$

$= \dfrac{1\pm\sqrt{3}}{2}$

But $|\cos A| \leqslant 1$, so $\cos A = \dfrac{1-\sqrt{3}}{2}$

24 $\sin x - 2\cos x = A\sin(x+\alpha)$

$$= A(\sin x\cos\alpha + \cos x\sin\alpha)$$
$$= A\sin x\cos\alpha + A\cos x\sin\alpha$$

Equating the coefficients of $\sin x$ and $\cos x$:

$$A\cos\alpha = 1 \quad \text{and} \quad A\sin\alpha = -2$$
$$\therefore \ \cos\alpha = \frac{1}{A} \quad \text{and} \quad \sin\alpha = \frac{-2}{A}$$

Now $\sin^2\alpha + \cos^2\alpha = 1$

$$\therefore \ \left(\frac{-2}{A}\right)^2 + \left(\frac{1}{A}\right)^2 = 1$$
$$\therefore \ \frac{4+1}{A^2} = 1$$
$$\therefore \ A^2 = 5$$
$$\therefore \ A = \sqrt{5} \quad \{A > 0\}$$

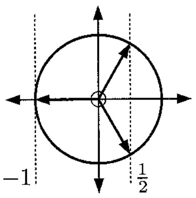
So $\cos\alpha = \frac{1}{\sqrt{5}}$, $\sin\alpha = -\frac{2}{\sqrt{5}}$

$\therefore$ α is in the 4th quadrant.

$\therefore$ $\alpha \approx 5.18$

25 $2\sin^2 x - \cos x = 1, \ x \in [0, 2\pi]$

$$\therefore \ 2(1-\cos^2 x) - \cos x = 1$$
$$\therefore \ 2 - 2\cos^2 x - \cos x = 1$$
$$\therefore \ 2\cos^2 x + \cos x - 1 = 0$$
$$\therefore \ (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore \ \cos x = \tfrac{1}{2} \quad \text{or} \quad \cos x = -1$$

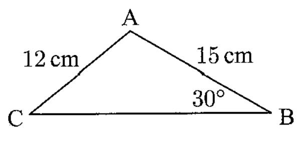


$\therefore \ x = \frac{\pi}{3}, \ \pi, \text{ or } \frac{5\pi}{3}$

26 $\arcsin(2x-3) = -\frac{\pi}{6}$

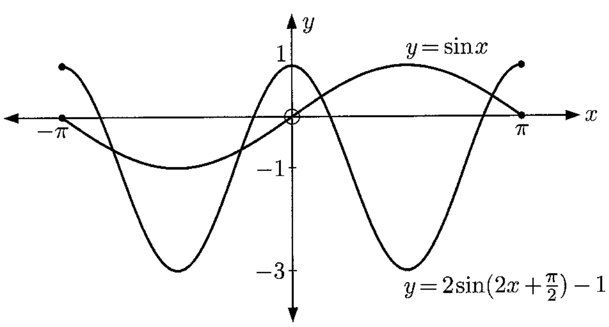
$$\therefore \ 2x - 3 = \sin\left(-\tfrac{\pi}{6}\right)$$
$$\therefore \ 2x - 3 = -\tfrac{1}{2}$$
$$\therefore \ 2x = \tfrac{5}{2}$$
$$\therefore \ x = \tfrac{5}{4}$$

27



$$\frac{\sin C}{15} = \frac{\sin 30°}{12} \quad \{\text{sine rule}\}$$
$$\therefore \ C = \arcsin\left(\frac{15\sin 30°}{12}\right)$$
$$\therefore \ C \approx 38.7° \quad \text{or} \quad 141.3°$$

28



29 $\arcsin(-\frac{1}{2}) + \arctan(1) + \arccos(-\frac{1}{2})$

$$= -\tfrac{\pi}{6} + \tfrac{\pi}{4} + \tfrac{2\pi}{3}$$
$$= \tfrac{3\pi}{4}$$

30 $\sin^2\theta + \cos^2\theta = 1$

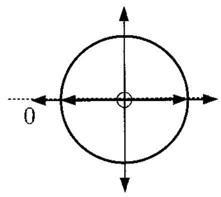
$$\therefore \ \left(\tfrac{2}{3}\right)^2 + \cos^2\theta = 1$$
$$\therefore \ \cos^2\theta = \tfrac{5}{9}$$
$$\therefore \ \cos\theta = -\tfrac{\sqrt{5}}{3} \quad \{\theta \text{ is obtuse}\}$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \tfrac{2}{3} \times -\tfrac{\sqrt{5}}{3}$$
$$= -\tfrac{4\sqrt{5}}{9}$$

31 $\sin x + \cos x = 1, \ 0 \leqslant x \leqslant \pi$

$$\therefore \ \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 \quad .... \ (*)$$

{squaring both sides}

$$\therefore \ \sin 2x + 1 = 1$$
$$\therefore \ \sin 2x = 0$$



$$\therefore \ 2x = 0 + k\pi, \ k \in \mathbb{Z}$$
$$\therefore \ x = \tfrac{k\pi}{2}, \ k \in \mathbb{Z}$$
$$\therefore \ x = 0, \ \tfrac{\pi}{2}, \ \pi \quad \{0 \leqslant x \leqslant \pi\}$$

Since we squared both sides at $(*)$, we need to check our solutions.

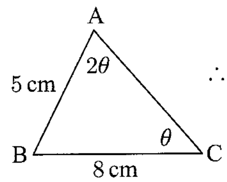
$\sin 0 + \cos 0 = 1$ ✓

$\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$ ✓

$\sin\pi + \cos\pi = -1$ ✗

So $x = 0$ or $\frac{\pi}{2}$

32 **a**



$$\frac{\sin 2\theta}{8} = \frac{\sin\theta}{5} \quad \{\text{sine rule}\}$$
$$\therefore \ \frac{2\sin\theta\cos\theta}{8} = \frac{\sin\theta}{5}$$
$$\therefore \ \cos\theta = \tfrac{4}{5}$$

b $A\widehat{B}C = \pi - 3\theta$

$$= \pi - 3\arccos(\tfrac{4}{5})$$
$$\approx 1.211$$

Area of triangle $\approx \frac{1}{2} \times 5 \times 8 \times \sin(1.211)$

$$\approx 18.7 \text{ cm}^2$$

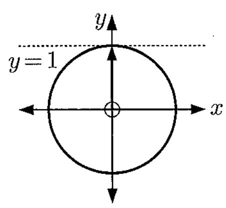
33 $\tan 2x$ has period $\frac{\pi}{2}$, and $\tan 3x$ has period $\frac{\pi}{3}$.

The lowest common multiple of $\frac{\pi}{2}$ and $\frac{\pi}{3}$ is π.

$\therefore$ the period $= \pi$

34 $\cot\theta + \tan\theta = 2, \ \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$$\therefore \ \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = 2$$
$$\therefore \ \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = 2$$
$$\therefore \ 1 = 2\sin\theta\cos\theta$$
$$\therefore \ 1 = \sin 2\theta$$
$$\therefore \ 2\theta = \tfrac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$$
$$\therefore \ \theta = \tfrac{\pi}{4} + k\pi, \ k \in \mathbb{Z}$$
$$\therefore \ \theta = \tfrac{\pi}{4} \quad \{\text{since } \theta \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[\}$$



35

$$\tan 2\theta = 2$$

$$\therefore \frac{2\tan\theta}{1-\tan^2\theta} = 2 \qquad \{\text{double angle formula}\}$$

$$2\tan\theta = 2 - 2\tan^2\theta$$

$$\tan^2\theta + \tan\theta - 1 = 0$$

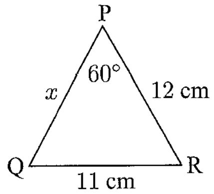
$$\therefore \tan\theta = \frac{-1 \pm \sqrt{1^2 - 4\times 1\times(-1)}}{2\times 1}$$

$$= \frac{-1\pm\sqrt{5}}{2}$$

Now $2\theta \in [\pi, \frac{3\pi}{2}]$, so $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

$$\therefore \tan\theta = \frac{-1-\sqrt{5}}{2} \qquad \{\tan\theta < 0\}$$

36



$$11^2 = x^2 + 12^2 - 2\times x\times 12\cos 60^\circ$$

$$\therefore 121 = x^2 + 144 - 12x$$

$$\therefore x^2 - 12x + 23 = 0$$

$$\therefore x = \frac{12 \pm\sqrt{(-12)^2 - 4\times 1\times 23}}{2\times 1}$$

$$\therefore x = 6\pm\sqrt{13}, \text{ so PQ} = 6\pm\sqrt{13}\text{ cm}$$

37

$$\frac{1}{\tan\theta - \sec\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}} \qquad \{\text{provided } \cos\theta \neq 0\}$$

$$= \frac{1}{\left(\frac{\sin\theta - 1}{\cos\theta}\right)}$$

$$= \frac{\cos\theta}{\sin\theta - 1}\times\left(\frac{\sin\theta+1}{\sin\theta+1}\right)$$

$$= \frac{\cos\theta\sin\theta + \cos\theta}{\sin^2\theta - 1}$$

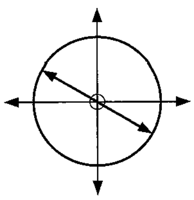
$$= \frac{\cos\theta\sin\theta+\cos\theta}{-\cos^2\theta}$$

$$= -\frac{\cos\theta\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\cos^2\theta}$$

$$= -\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}$$

$$= -(\tan\theta + \sec\theta)$$

38 **a** $\sqrt{3}\tan\left(\frac{x}{2}\right) = -1, \; x\in[-\pi, 3\pi]$



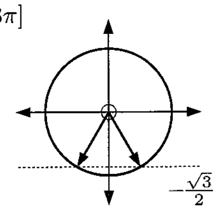
$$\therefore \tan\left(\frac{x}{2}\right) = -\tfrac{1}{\sqrt{3}}$$

$$\therefore \frac{x}{2} = \tfrac{5\pi}{6} + k\pi, \; k\in\mathbb{Z}$$

$$\therefore x = \tfrac{5\pi}{3} + 2k\pi, \; k\in\mathbb{Z}$$

$$\therefore x = \tfrac{5\pi}{3} \text{ or } -\tfrac{\pi}{3} \qquad \{x\in[-\pi, 3\pi]\}$$

b $\sqrt{3} + 2\sin(2x) = 0, \quad x\in[-\pi, 3\pi]$



$$\therefore \sin 2x = -\tfrac{\sqrt{3}}{2}$$

$$\therefore 2x = \left.\begin{matrix}\frac{4\pi}{3}\\ \frac{5\pi}{3}\end{matrix}\right\} + k2\pi, \; k\in\mathbb{Z}$$

$$\therefore x = \left.\begin{matrix}\frac{2\pi}{3}\\ \frac{5\pi}{6}\end{matrix}\right\} + k\pi, \; k\in\mathbb{Z}$$

$$\therefore x = -\tfrac{\pi}{3}, -\tfrac{\pi}{6}, \tfrac{2\pi}{3}, \tfrac{5\pi}{6}, \tfrac{5\pi}{3}, \tfrac{11\pi}{6}, \tfrac{8\pi}{3}, \text{ or } \tfrac{17\pi}{6}$$

$$\{x\in[-\pi, 3\pi]\}$$

39

$$\sin x = 2\sin(x - \tfrac{\pi}{6})$$

$$\therefore \sin x = 2(\sin x\cos\tfrac{\pi}{6} - \cos x\sin\tfrac{\pi}{6})$$

$$= 2\sin x(\tfrac{\sqrt{3}}{2}) - 2\cos x(\tfrac{1}{2})$$

$$\therefore \sin x(1-\sqrt{3}) = -\cos x$$

$$\therefore \frac{\sin x}{\cos x} = -\frac{1}{1-\sqrt{3}}$$

$$\therefore \tan x = \frac{1}{\sqrt{3}-1}$$

40 **a** Consider the sine function model $H(t) = a\sin b(t-c)+d$

The amplitude $a = \frac{2.4}{2} = 1.2$ m.

The period is 12.3 hours, so $\frac{2\pi}{b} = 12.3$

$\therefore b \approx 0.5108$

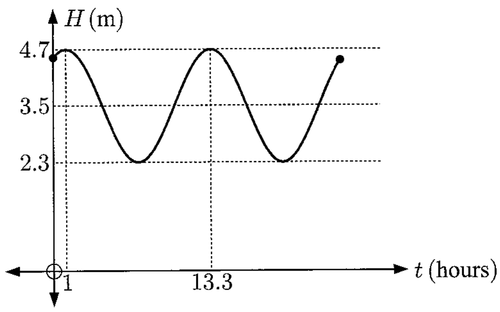
The principal axis is $H = 4.7 - 1.2 = 3.5$ m so $d = 3.5$.

The first low tide is at $t = 1 + 6.15 = 7.15$, and the next high tide is at $t = 1 + 12.3 = 13.3$

$$\therefore c = \frac{7.15 + 13.3}{2} \approx 10.2$$

$\therefore$ the model is $H(t) \approx 1.2\sin(0.5108(t - 10.2)) + 3.5$ where t is the time in hours after midnight, $0 \leqslant t \leqslant 24$.

b



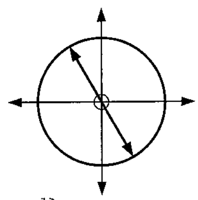
41 **a**

$$\sin\left(\arccos(-\tfrac{\sqrt{3}}{2})\right) = \sin\left(\tfrac{5\pi}{6}\right) = \tfrac{1}{2}$$

b

$$\tan\left(\arcsin\tfrac{1}{\sqrt{2}}\right) = \tan\left(\tfrac{\pi}{4}\right) = 1$$

42 $\sin x + \sqrt{3}\cos x = 0$



$$\therefore \sin x = -\sqrt{3}\cos x$$

$$\therefore \frac{\sin x}{\cos x} = -\sqrt{3}$$

$$\therefore \tan x = -\sqrt{3}$$

$$\therefore x = \tfrac{2\pi}{3} \text{ or } \tfrac{5\pi}{3} \qquad \{x\in[0, 2\pi]\}$$

43

$$\frac{\sin\theta + 2\cos\theta}{\sin\theta - \cos\theta} = 2$$

$$\therefore \sin\theta + 2\cos\theta = 2(\sin\theta - \cos\theta)$$

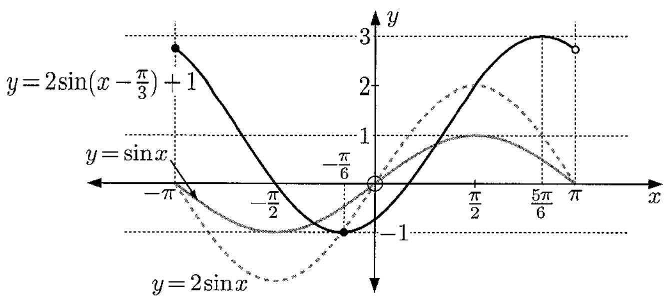
$$\therefore 4\cos\theta = \sin\theta$$

$$\therefore \tan\theta = 4$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times 4}{1-4^2} = -\tfrac{8}{15}$$

44 $y = 2\sin(x - \frac{\pi}{3}) + 1$ is a translation of $y = 2\sin x$ by $(\frac{\pi}{3}, 1)$.

So, we start with $y = \sin x$, we stretch it vertically with scale factor 2 to produce $y = 2\sin x$, then perform the translation.



45
$$\cos 2x + \sqrt{3}\sin 2x = 1, \quad x \in [-\pi, \pi]$$
$$\therefore\ 1 - 2\sin^2 x + 2\sqrt{3}\sin x\cos x = 1$$
$$\therefore\ -2\sin^2 x + 2\sqrt{3}\sin x\cos x = 0$$
$$\therefore\ 2\sin x(\sqrt{3}\cos x - \sin x) = 0$$
$$\therefore\ \sin x = 0 \text{ or } \sqrt{3}\cos x = \sin x$$
$$\therefore\ \sin x = 0 \text{ or } \tan x = \sqrt{3}$$
$$\therefore\ x = -\pi,\ -\tfrac{2\pi}{3},\ 0,\ \tfrac{\pi}{3},\ \text{or } \pi$$

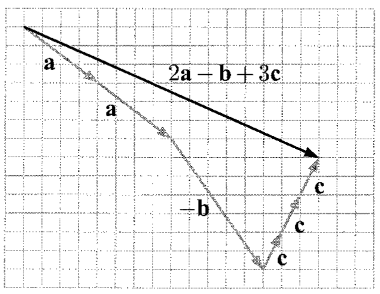
SOLUTIONS TO TOPIC 4 (VECTORS)

1 **a** $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

b magnitude $= \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$ units

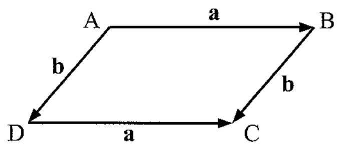
c The unit vector in the opposite direction is $-\frac{1}{\sqrt{14}}\begin{pmatrix}1\\-2\\3\end{pmatrix}$.

2



$$2\mathbf{a} - \mathbf{b} + 3\mathbf{c} = 2\begin{pmatrix}4\\-3\end{pmatrix} - \begin{pmatrix}-5\\7\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix}$$
$$= \begin{pmatrix}8+5+3\\-6-7+6\end{pmatrix}$$
$$= \begin{pmatrix}16\\-7\end{pmatrix} \quad \text{which checks with the diagram.}$$

3 **a**



b $\overrightarrow{BC} = \overrightarrow{AD} = \mathbf{b}$ and $\overrightarrow{CD} = -\overrightarrow{AB} = -\mathbf{a}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \mathbf{b} - \mathbf{a}$

c $\overrightarrow{AC} \bullet \overrightarrow{BD} = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{b} - \mathbf{a})$

$= \mathbf{a}\bullet\mathbf{b} - \mathbf{a}\bullet\mathbf{a} + \mathbf{b}\bullet\mathbf{b} - \mathbf{b}\bullet\mathbf{a}$

$= \mathbf{b}\bullet\mathbf{b} - \mathbf{a}\bullet\mathbf{a}$ {as $\mathbf{a}\bullet\mathbf{b} = \mathbf{b}\bullet\mathbf{a}$}

$= |\mathbf{b}|^2 - |\mathbf{a}|^2$

and, if $|\mathbf{b}| = |\mathbf{a}|$, $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$

d Since $\overrightarrow{AC} \bullet \overrightarrow{BD} = 0$, $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular.

4 **a** $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ where θ is the angle between **a** and **b**.
If $\mathbf{a} \bullet \mathbf{b} < 0$, then $\cos\theta < 0$ and so $90° < \theta < 180°$.

b $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix}-2\\1\\3\end{pmatrix} \bullet \begin{pmatrix}3\\-1\\1\end{pmatrix} = -6 - 1 + 3 = -4$

$|\mathbf{a}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}$ and

$|\mathbf{b}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$

and $\cos\theta = \dfrac{-4}{\sqrt{14}\sqrt{11}} \approx -0.3223$ and so $\theta \approx 108.8°$.

5 **a** $\begin{pmatrix}k\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\k\\3k\end{pmatrix}$ are parallel if $\begin{pmatrix}4\\k\\3k\end{pmatrix} = a\begin{pmatrix}k\\1\\3\end{pmatrix}$
for some a.

Thus, $4 = ak$ .... (1)
$k = a$ .... (2)
$3k = 3a$ .... (3)

From (1) and (2), $k^2 = 4$ and so $k = \pm 2$.
Hence the vectors are parallel if $k = \pm 2$.

- if $k = 2$, the vectors are $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\2\\6\end{pmatrix}$
- if $k = -2$, the vectors are $\begin{pmatrix}-2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\-2\\-6\end{pmatrix}$

b The vectors are perpendicular if $\begin{pmatrix}k\\1\\3\end{pmatrix} \bullet \begin{pmatrix}4\\k\\3k\end{pmatrix} = 0$

$\therefore\ 4k + k + 9k = 0$ and so $k = 0$.

So, the vectors $\begin{pmatrix}0\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\0\\0\end{pmatrix}$ are perpendicular.

6 **a** **i** The equation can be written as
$$\frac{x-1}{2} = \frac{3-y}{3} = z = t$$
$\therefore\ x = 2t + 1,\ y = -3t + 3,\ z = t$

or $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix} + t\begin{pmatrix}2\\-3\\1\end{pmatrix}$.

$\therefore$ a vector parallel to the line is $\begin{pmatrix}2\\-3\\1\end{pmatrix}$.

ii Letting $t = 0$, a point on the line is $(1, 3, 0)$.

iii The point $(7, -3, 2)$ lies on the line if
$7 = 2t + 1$ .... (1)
$-3 = -3t + 3$ .... (2)
$2 = t$ .... (3)

So, from (3), $t = 2$ and from (1), $t = 3$ which is not possible. Thus the point $(7, -3, 2)$ does not lie on the line.

b There are many possible answers.

Since $\begin{pmatrix}0\\1\\3\end{pmatrix}$ is perpendicular to $\begin{pmatrix}2\\-3\\1\end{pmatrix}$, a possible

line is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\-3\\2\end{pmatrix} + s\begin{pmatrix}0\\1\\3\end{pmatrix}$, which is

$x = 5,\ y = -3 + s,\ z = 2 + 3s,\ s \in \mathbb{R}$.