7 Sequences and series

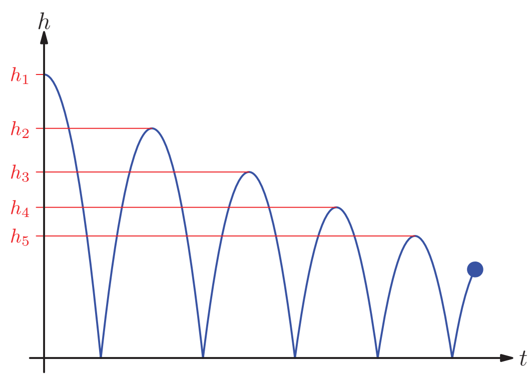
In this chapter you will learn:

- how to describe sequences mathematically
- a way to describe sums of sequences
- about sequences with a constant difference between terms
- about finite sums of sequences with a constant difference between terms
- about sequences with a constant ratio between terms
- about finite sums of sequences with a constant ratio between terms
- about infinite sums of sequences with a constant ratio between terms
- how to apply sequences to real life problems.

Introductory problem

A mortgage of $100 000 is fixed at 5% compound interest. It needs to be paid off over 25 years by annual instalments. Interest is added at the end of each year, just before the payment is made. How much should be paid each year?

If you drop a tennis ball, it will bounce a little lower each time it hits the ground. The heights to which the ball bounces form a **sequence**. Although the study of sequences may just seem to be the maths of number patterns, it also has a remarkable number of applications in the real world, from calculating mortgages to estimating the harvests on farms.



7A General sequences

A sequence is a list of numbers in a specified order. You may recognise a pattern in each of the following examples:

1, 3, 5, 7, 9, 11, ...

1, 4, 9, 16, 25, ...

100, 50, 25, 12.5, ...

To study sequences further, it is useful to have a notation to describe them.

KEY POINT 7.1

u_n is the value of the nth term of a sequence.

EXAM HINT

u_n is a conventional symbol for a sequence, but there is nothing special about the letters used. We could also have a sequence t_x or a_h. The important thing is that the letter with a subscript represents a value and the subscript represents where the term is in the sequence.

So in the sequence 1, 3, 5, 7, 9, 11, ... above, we could say that $u_1 = 1$, $u_2 = 3$, $u_5 = 9$.

The whole of a sequence u_1, u_2, u_3, … is sometimes written $\{u_n\}$.

We are mainly interested in sequences with well-defined mathematical rules. There are two types: **recursive definitions** and **deductive** rules.

Recursive definitions link new terms to previous terms in the sequence. For example, if each term is three times the previous term we would write $u_{n+1} = 3u_n$.

Worked example 7.1

A sequence is defined by $u_{n+1} = u_n + u_{n-1}$ with $u_1 = 1$ and $u_2 = 1$. What is the fifth term of this sequence?

The sequence is defined inductively, so we have to work our way up to u_5
To find u_3 we set $n = 2$

$$u_3 = u_2 + u_1 = 1 + 1 = 2$$

To find u_4 we set $n = 3$

$$u_4 = u_3 + u_2 = 2 + 1 = 3$$

To find u_5 we set $n = 4$

$$u_5 = u_4 + u_3 = 3 + 2 = 5$$

You may recognise this as the famous Fibonacci Sequence, based on a model Leonardo Fibonacci made for the breeding of rabbits. This has many applications from the arrangement of seeds in pine cones to a proof of the infinity of prime numbers.

There is also a beautiful link to the golden ratio: $\frac{1 \pm \sqrt{5}}{2}$

Deductive rules link the value of the term to where it is in the sequence. For example, if each term is the square of its position in the sequence then we would write $u_n = n^2$.

Worked example 7.2

A sequence is defined by $u_n = 2n - 1$. List the first four terms of this sequence.

With a deductive rule, we can find the first four terms by setting $n = 1, 2, 3, 4$

$u_1 = 2 \times 1 - 1 = 1$
$u_2 = 2 \times 2 - 1 = 3$
$u_3 = 2 \times 3 - 1 = 5$
$u_4 = 2 \times 4 - 1 = 7$

EXAM HINT

There are several alternative names used for deductive and recursive definitions.

An recursive definition may also be referred to as 'term-to-term rule', 'recurrence relation' or 'recursive definition'.

A deductive rule may be referred to as 'position-to-term rule', 'nth term rule' or simply 'the formula' of the sequence.

Exercise 7A

1. Write out the first five terms of the following sequences, using the inductive definitions.

(a) (i) $u_{n+1} = u_n + 5,\ u_1 = 3.1$ (ii) $u_{n+1} = u_n - 3.8,\ u_1 = 10$

(b) (i) $u_{n+1} = 3u_n + 1,\ u_1 = 0$ (ii) $u_{n+1} = 9u_n - 10,\ u_1 = 1$

(c) (i) $u_{n+2} = u_{n+1} \times u_n,\ u_1 = 2, u_2 = 3$

(ii) $u_{n+2} = u_{n+1} \div u_n,\ u_1 = 2u_2 = 1$

(d) (i) $u_{n+2} = u_n + 5,\ u_1 = 3u_2 = 4$

(ii) $u_{n+2} = 2u_n + 1,\ u_1 = -3u_2 = 3$

(e) (i) $u_{n+1} = u_n + 4,\ u_4 = 12$ (ii) $u_{n+1} = u_n - 2,\ u_6 = 3$

2. Write out the first five terms of the following sequences, using the deductive definitions.

(a) (i) $u_n = 3n + 2$ (ii) $u_n = 1.5n - 6$

(b) (i) $u_n = n^3 - 1$ (ii) $u_n = 5n^2$

(c) (i) $u_n = 3^n$ (ii) $u_n = 8 \times (0.5)^n$

(d) (i) $u_n = n^n$ (ii) $u_n = \sin(90n°)$

3. Give an inductive definition for each of these sequences.

(a) (i) 7, 10, 13, 16, … (ii) 1, 0.2, −0.6, −1.4, …

(b) (i) 3, 6, 12, 24, … (ii) 12, 18, 27, 40.5, …

(c) (i) 1, 3, 6, 10, … (ii) 1, 2, 6, 24, …

4. Give a deductive definition for each of the following sequences.

(a) (i) 2, 4, 6, 8, … (ii) 1, 3, 5, 7, …

(b) (i) 2, 4, 8, 16, … (ii) 5, 25, 125, 625, …

(c) (i) 1, 4, 9, 16, … (ii) 1, 8, 27, 64, …

(d) (i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$ (ii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16} \ldots$

5. A sequence $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = 2$, $u_{n+1} = 3u_n - 2u_{n-1} - 1$ where $n \in \mathbb{Z}$.

(a) Find u_2, u_3 and u_4.

(b) (i) Based on your answer to (a), suggest a formula for u_n in terms of n.

(ii) Verify that your answer to part (b)(i) satisfies the equation $u_{n+1} = 3u_n - 2u_{n-1}$. *[6 marks]*

7B General series and sigma notation

If 10% interest is paid on money in a bank account each year, the amounts paid form a sequence. While it is good to know how much is paid in each year, you may be even more interested to know how much will be paid in altogether.

This is an example of a situation where we may want to sum a sequence. The sum of a sequence up to a certain point is called a **series**, and we often use the symbol S_n to denote the sum of the first n terms of a sequence.

Worked example 7.3

Adding up consecutive odd numbers starting at 1 forms a series.

Let S_n denote the sum of the first n terms. List the first five terms of the sequence S_n and suggest a rule for it.

Start by examining the first few terms

$S_1 = 1$

$S_2 = 1 + 3 = 4$

$S_3 = 1 + 3 + 5 = 9$

$S_4 = 1 + 3 + 5 + 7 = 16$

$S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Do we recognise these numbers?

It seems that $S_n = n^2$

Defining such sums by saying 'Add up a defined sequence from a given start point to a given end point' is too wordy and imprecise for mathematicians.

Exactly the same thing is written in a shorter (although not necessarily simpler) way using **sigma notation**:

KEY POINT 7.2

This is the last value taken by r, where counting ends

r is a placeholder; it shows what changes with each new term

Greek capital sigma means 'add up'

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \cdots + f(n)$$

This is the first value taken by r; where counting starts

EXAM HINT

Do not be intimidated by this complicated-looking notation.

If you struggle with it, try writing out the first few terms.

If there is only one variable in the expression being summed, it is acceptable to miss out the '$r =$' above and below the sigma.

In the example we use both the letters n and r as unknowns – but they are not the same type of unknown.

If we replaced r by any other letter (apart from f or n) then the expression on the right would be unchanged. r is called a dummy variable. If we replaced n by any other letter then the expression would change.

Worked example 7.4

$T_n = \sum_{2}^{n} r^2$ Find the value of T_4.

Put the starting value, $r = 2$ into the expression to be summed

$T_4 = 2^2 + \ldots$

We've not reached the end value, so put in $r = 3$

$T_4 = 2^2 + 3^2 + \ldots$

We've not reached the end value, so put in $r = 4$

$T_4 = 2^2 + 3^2 + 4^2$

We've reached the end value, so evaluate

$T_4 = 4 + 9 + 16 = 29$

Worked example 7.5

Write the series $T_4 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ using sigma notation.

We must write in terms of the dummy variable r what eachterm of the sequence looks like

General term $= \frac{1}{r}$

What is the first value of r?

Starts when $r = 2$

What is the final value of r?

Ends when $r = 6$

Summarise in sigma notation

$$\text{Series} = \sum_{2}^{6} \frac{1}{r}$$

Exercise 7B

1. Evaluate the following expressions.

(a) (i) $\sum_{2}^{4} 3r$ (ii) $\sum_{5}^{7} (2r+1)$

(b) (i) $\sum_{3}^{6} (2^r - 1)$ (ii) $\sum_{-1}^{4} 1.5^r$

(c) (i) $\sum_{a=1}^{a=4} b\,(a+1)$ (ii) $\sum_{q=-3}^{q=2} pq^2$

2. Write the following expressions in sigma notation. Be aware that there is more than one correct answer.

(a) (i) $2 + 3 + 4 \ldots + 43$ (ii) $6 + 8 + 10 \ldots + 60$

(b) (i) $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots + \frac{1}{128}$ (ii) $2 + \frac{2}{3} + \frac{2}{9} \ldots + \frac{2}{243}$

(c) (i) $14a + 21a + 28a \ldots + 70a$ (ii) $0 + 1 + 2^b + 3^b \ldots + 19^b, \ (b \neq 0)$

7C Arithmetic sequences

We will now focus on one particular type of sequence: one where there is a constant difference, known as the common difference, between consecutive terms.

This is called an **arithmetic sequence** (or an **arithmetic progression**). The standard notation for the difference between terms is d, so arithmetic sequences obey the recursive definition $u_{n+1} = u_n + d$.

Knowing the common difference is not enough to fully define the sequence. There are many different sequences with common difference 2, for example:

1, 3, 5, 7, 9, 11, … and 106, 108, 110, 112, 114, …

To fully define the sequence we also need the first term. Conventionally this is given the symbol u_1.

So the sequence 106, 108, 110, 112, 114, … is defined by:

$$u_1 = 106, d = 2.$$

Worked example 7.6

What is the fourth term of an arithmetic sequence with $u_1 = 300$, $d = -5$?

Use the recursive definition to find the first four terms $u_{n+1} = u_n + d$

$u_1 = 300$
$u_2 = u_1 - 5 = 295$
$u_3 = u_2 - 5 = 290$
$u_4 = u_3 - 5 = 285$

In the above example it did not take long to find the first four terms. But what if you had been asked to find the hundredth term? To do this efficiently we must move from the inductive definition of arithmetic sequences to the deductive definition.

We need to think about how arithmetic sequences are built up. To get to the nth term we start at the first term and add on the common difference $n - 1$ times. This suggests a formula:

KEY POINT 7.3

$$u_n = u_1 + (n - 1)d$$

Worked example 7.7

The fifth term of an arithmetic sequence is 7 and the eighth term is 16. What is the 100th term?

Write down the information given and relate it to u_1 and d to give an expression for the fifth term in terms of u_1 and d

$u_5 = u_1 + 4d$

But we are told that $u_5 = 7$

$7 = u_1 + 4d$ (1)

Repeat for the eighth term

$16 = u_1 + 7d$ (2)

→

continued . . .

Solve simultaneously (2) – (1)

$$9 = 3d$$
$$\Leftrightarrow d = 3$$
$$\therefore u_1 = -5$$

Write down the general term and use it to answer the question

$$u_n = -5 + (n-1) \times 3$$
$$\therefore u_{100} = -5 + 99 \times 3 = 292$$

EXAM HINT

Many exam-style questions on sequences and series involve writing the given information in the form of simultaneous equations and then solving them.

Worked example 7.8

An arithmetic progression has first term 5 and common difference 7. What is the term number corresponding to the value 355?

The question is asking for n when $u_n = 355$. Write this as an equation

$$355 = u_1 + (n-1)d = 5 + 7(n-1)$$

Solve this equation

$$350 = 7(n-1)$$
$$\Leftrightarrow 50 = n - 1$$
$$\Leftrightarrow n = 51$$

So 355 is the 51st term.

EXAM HINT

'Arithmetic progression' is just another way of saying 'arithmetic sequence'.

Make sure you know all the alternative expressions for the same thing.

Exercise 7C

1. Using Key point 7.3, find the general formula for each arithmetic sequence given the following information.

   (a) (i) First term 9, common difference 3

   (ii) First term 57, common difference 0.2

(b) (i) First term 12, common difference −1

(ii) First term 18, common difference $\frac{1}{2}$

(c) (i) First term 1, second term 4

(ii) First term 9, second term 19

(d) (i) First term 4, second term 0

(ii) First term 27, second term 20

(e) (i) Third term 5, eighth term 60

(ii) Fifth term 8, eighth term 38

2. How many terms are there in the following sequences?

(a) (i) 1, 3, 5, …, 65

(ii) 18, 13, 8, …, −122

(b) (i) First term 8, common difference 9, last term 899

(ii) First term 0, ninth term 16, last term 450

3. An arithmetic sequence has 5 and 13 as its first two terms.

(a) Write down, in terms of n, an expression for the nth term, u_n.

(b) Find the number of terms of the sequence which are less than 400. *[8 marks]*

4. The 10th term of an arithmetic sequence is 61 and the 13th term is 79. Find the value of the 20th term. *[4 marks]*

5. The 8th term of an arithmetic sequence is 74 and the 15th term is 137. Which term has the value 227? *[4 marks]*

6. The heights of the rungs in a ladder form an arithmetic sequence. The third rung is 70 cm above the ground and the tenth rung is 210 cm above the ground. If the top rung is 350 cm above the ground, how many rungs does the ladder have? *[5 marks]*

7. The first four terms of an arithmetic sequence are 2, $a-b$, $2a+b+7$ and $a-3b$, where a and b are constants. Find a and b. *[5 marks]*

8. A book starts at page 1 and is numbered on every page.

(a) Show that the first eleven pages contain thirteen digits.

(b) If the total number of digits used is 1260, how many pages are in the book? *[8 marks]*

7D Arithmetic series

When you add up terms of an arithmetic sequence you get an arithmetic series. There is a formula for the sum of an arithmetic series, the proof of which is not required in the IB. See Fill-in proof 4 'Arithmetic series and the story of Gauss' on the CD-ROM if you are interested.

There are two different forms for the formula.

KEY POINT 7.4

If you know the first and last terms:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

If you know the first term and the common difference:

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

Worked example 7.9

Find the sum of the first 30 terms of an arithmetic progression with first term 8 and common difference 0.5.

We have all the information we need to use the second formula

$$S_{30} = \frac{30}{2}(2\times 8 + (30-1)\times 0.5) = 457.5$$

Sometimes you have to interpret the question carefully to be sure that it is about an arithmetic sequence.

Worked example 7.10

Find the sum of all the multiples of 3 between 100 and 1000.

Write out the first few terms to see what is happening

Sum $= 102 + 105 + 108 + \ldots + 999$

This is an arithmetic series with $u_1 = 102$ and $d = 3$

To use either sum formula, we also need to know how many terms are in this sequence. We do this by setting $u_n = 999$

$$999 = 102 + 3(n-1)$$
$$\Leftrightarrow 897 = 3(n-1)$$
$$\Leftrightarrow n = 300$$

Use the first sum formula

$$S_{300} = \frac{300}{2}(102 + 999) = 165\,150$$

You must be able to work backwards too; given information which includes the sum of the series, you may be asked to find out how many terms are in the series. Remember that the number of terms can only be a positive integer.

Worked example 7.11

An arithmetic sequence has first term 5 and common difference 10.

If the sum of all the terms is 720, how many terms are in the sequence?

We need to find n and it is the only unknown in the second sum formula

$$720 = \frac{n}{2}(2\times 5 + (n-1)\times 10)$$
$$= \frac{n}{2}(10 + 10n - 10)$$
$$= 5n^2$$

Solve this equation

$$n^2 = 144$$
$$n = \pm 12$$

But n must be a positive integer, so $n = 12$

Exercise 7D

1. Find the sum of the following arithmetic sequences:
   (a) (i) 12, 33, 54, … (17 terms)
       (ii) −100, −85, −70, … (23 terms)
   (b) (i) 3, 15, …, 459
       (ii) 2, 11, …, 650
   (c) (i) 28, 23, …, −52
       (ii) 100, 97, …,40
   (d) (i) 15, 15.5, …, 29.5
       (ii) $\frac{1}{12}, \frac{1}{6}, \ldots, 1.5$
2. An arithmetic sequence has first term 4 and common difference 8.

   How many terms are required to get a sum of:
   (a) (i) 676 (ii) 4096 (iii) 11236
   (b) $x^2, x > 0$
3. The second term of an arithmetic sequence is 7. The sum of the first four terms of the sequence is 12. Find the first term, a, and the common difference, d, of the sequence. *[5 marks]*

4. Consider the arithmetic series $2 + 5 + 8 + \ldots$:

(a) Find an expression for S_n, the sum of the first n terms.

(b) Find the value of n for which $S_n = 1365$. *[5 marks]*

5. Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, ... *[6 marks]*

6. The second term of an arithmetic sequence is 6. The sum of the first four terms of the arithmetic sequence is 8. Find the first term, a, and the common difference, d, of the sequence. *[6 marks]*

7. Consider the arithmetic series $-6 + 1 + 8 + 15 + \ldots$

Find the least number of terms so that the sum of the series is greater than 10 000. *[6 marks]*

8. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term u_n. *[6 marks]*

9. A circular disc is cut into twelve sectors whose angles are in an arithmetic sequence.

The angle of the largest sector is twice the angle of the smallest sector. Find the size of the angle of the smallest sector. *[6 marks]*

10. The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$.

If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of this sequence. *[7 marks]*

11. What is the sum of all three-digit numbers which are multiples of 14 but not 21? *[8 marks]*

7E Geometric sequences

Geometric sequences have a constant ratio, called the *common ratio, r*, between terms:

$$u_{n+1} = r \times u_n$$

So examples of geometric sequences might be:

1, 2, 4, 8, 16, … $(r = 2)$

100, 50, 25, 12.5, 6.25, … $(r = \frac{1}{2})$

1, −3, 9, −27, 81, … $(r = -3)$

As with arithmetic sequences, we also need to know the first term to fully define a geometric sequence. Again this is normally given the symbol u_1.

To get immediately to the deductive rule, we can see that to get to the nth term you start at the first term and multiply by the common ratio $n - 1$ times.