16 Basic differentiation and its applications

In this chapter you will learn:

- how to find the gradients of curves from first principles, a process called differentiation
- how to differentiate x^n
- how to differentiate $\sin x$, $\cos x$ and $\tan x$
- how to differentiate e^x and $\ln x$
- to find the equations of tangents and normals to curves at given points
- to find maximum and minimum points on curves.

Introductory problem

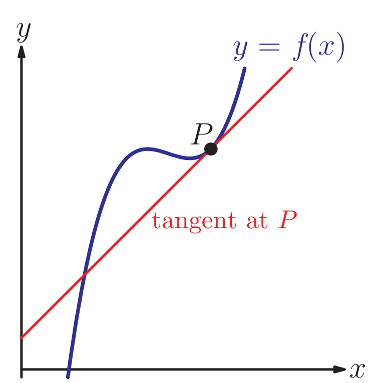
The cost of petrol used in a car, in £ per hour, is $\frac{12+v^2}{100}$ where v is measured in miles per hour and $v > 0$.

If Daniel wants to travel 50 miles as cheaply as possible, at what speed should he travel?

In real life, things change. Planets move, babies grow and prices rise. Calculus is the study of things that change, and one of its important tools is differentiation; the ability to find the rate at which the y-coordinate of a curve is changing when the x-coordinate changes. For a straight-line graph this is determined by the **gradient**, but it requires more work to apply the same idea to curves, where the gradient is different at different points.

16A Sketching derivatives

Our first task is to establish exactly what is meant by the gradient of a curve. We are clear on what is meant by the gradient of a straight line and we can use this idea to make a more general definition: the gradient of a curve at a point P is the gradient of the tangent to the curve at that point.



A **tangent** is a straight line which touches the curve without crossing it.

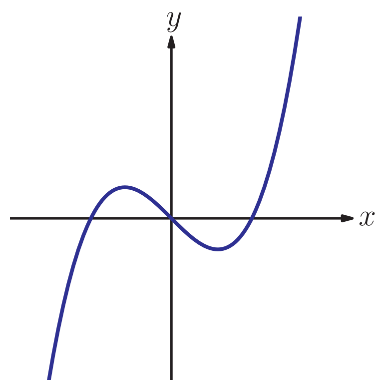
We have already met tangents in chapter 3.

Note that when we say that the tangent at P does not cross the curve we mean that this is only the case locally (close to the point P). The tangent might also intersect a different part of the curve.

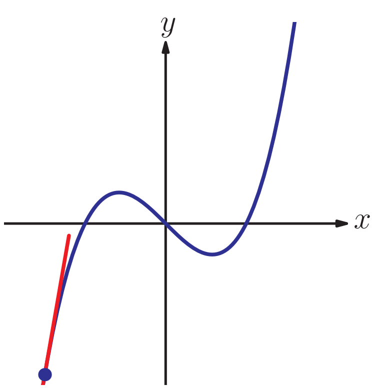
The **derivative** of a function, $f(x)$, is another function that gives the gradient of $y = f(x)$ at any point in the x domain.
It is often useful to be able to roughly sketch the derivative.

Worked example 16.1

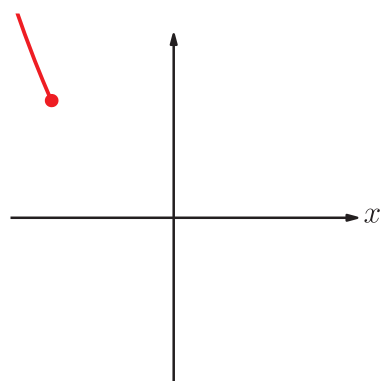
Sketch the derivative of this function.



Imagine we are tracking a point moving along the curve from left to right; we will track the tangent to the curve at the moving point and form the graph of its gradient



The curve is increasing from left to right, but more and more slowly...



... so the gradient is positive and falling

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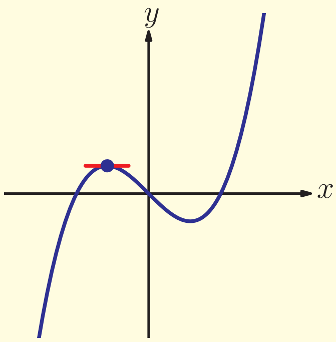
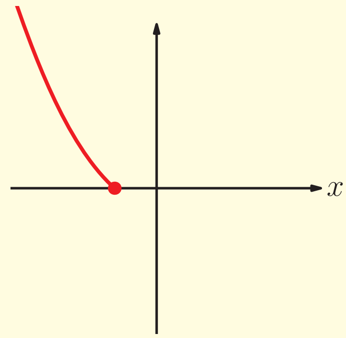
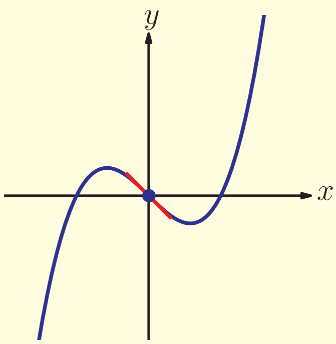


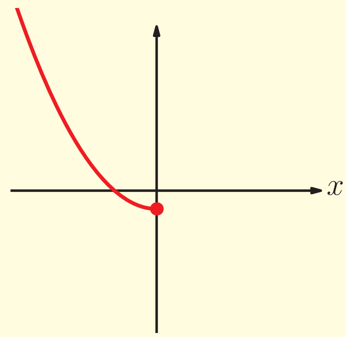
The tangent is horizontal...



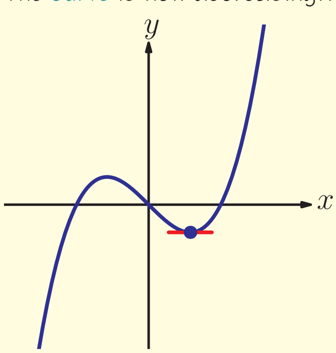
... so the gradient is zero



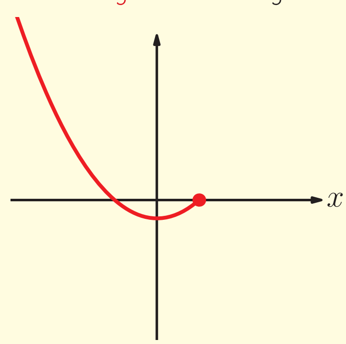
The curve is now decreasing. . . .



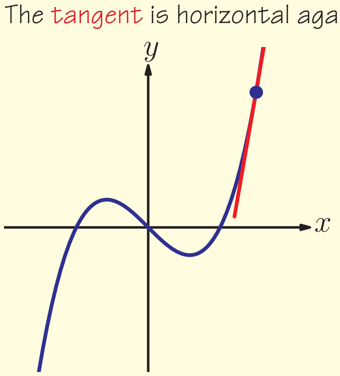
... so the gradient is negative



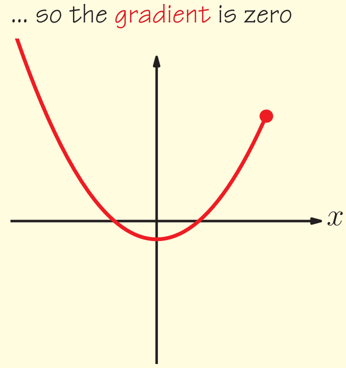
The tangent is horizontal again...



... so the gradient is zero



The curve is increasing, and does so faster and faster...

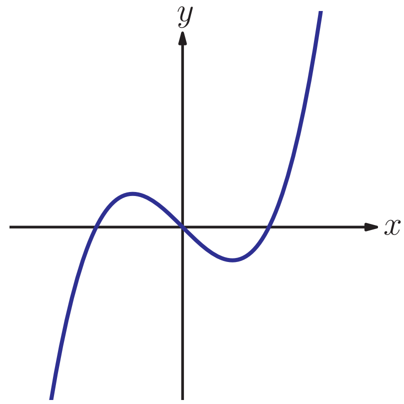


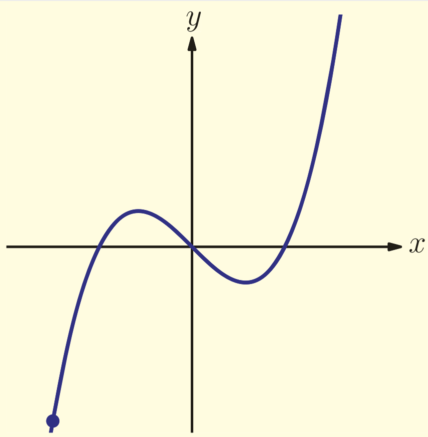
... so the gradient is positive and getting larger

We can also apply the same reasoning backwards.

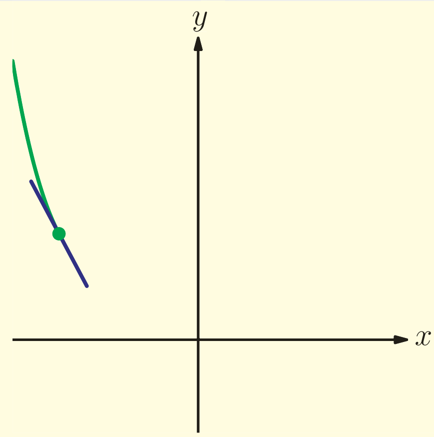
Worked example 16.2

You are given the derivative of a function. Sketch a possible graph of the original function.

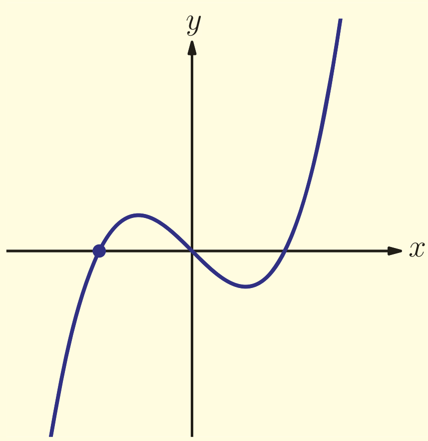




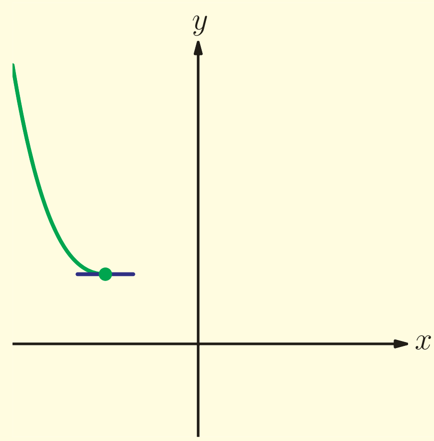
The gradient is negative…



… so the curve is decreasing.



The gradient is zero…



… so the tangent is horizontal.

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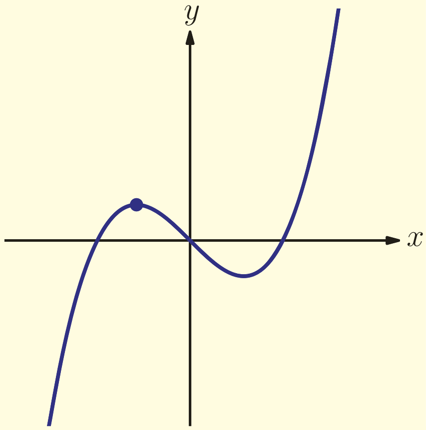
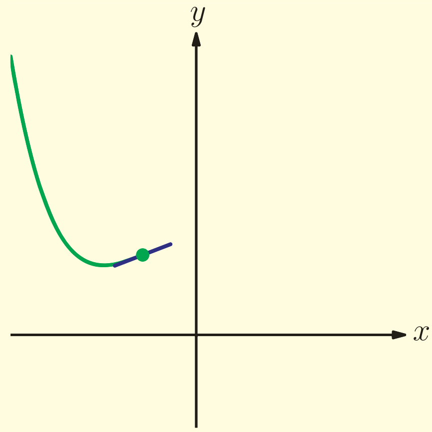
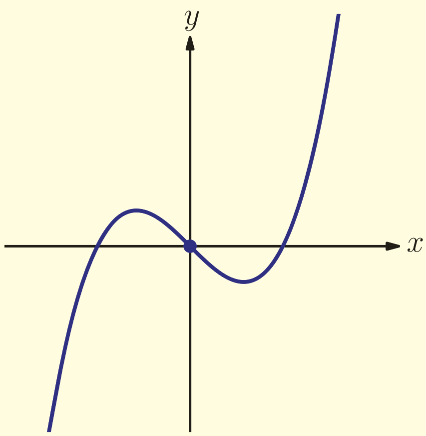


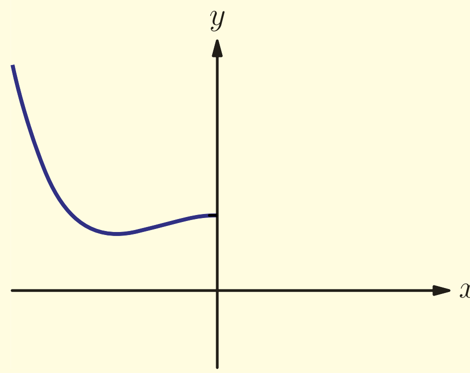
The gradient is positive...



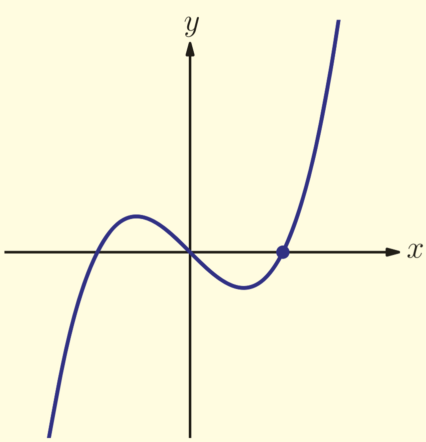
... so the curve is increasing.



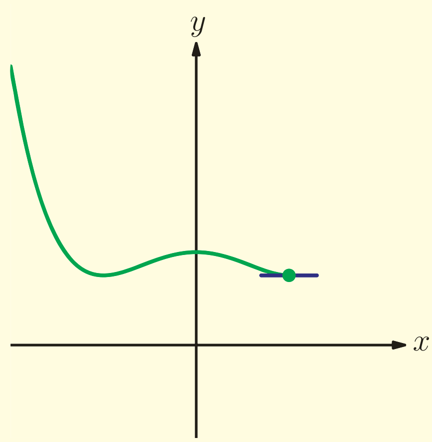
The gradient is zero...



.... so the tangent is horizontal.

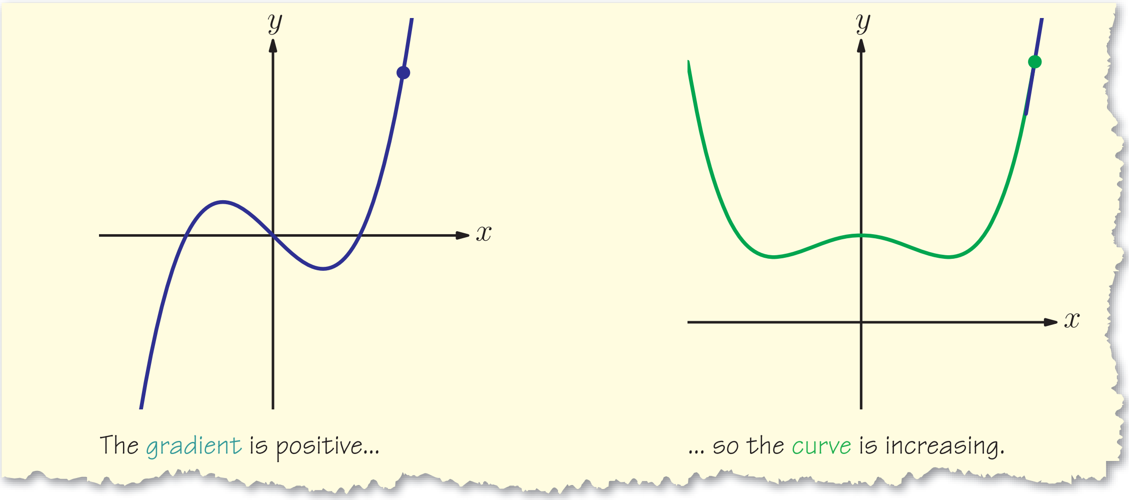


The gradient is zero...



... so the tangent is horizontal.

continued . . .



Notice in this example that there was more than one possible graph we could have drawn, depending on where we started the sketch. In chapter 17 you will learn more about this ambiguity when you 'undo' differentiation.

The relationship between a graph and its derivative can be summarised as follows:

KEY POINT 16.1

When the curve is increasing the gradient is positive.

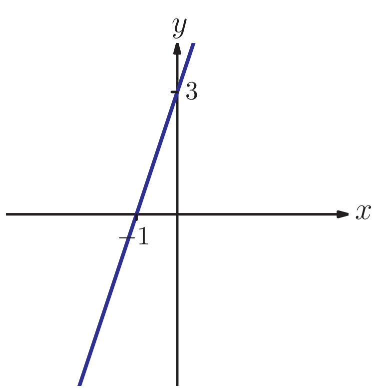
When the curve is decreasing the gradient is negative.

When the tangent is horizontal the gradient is zero; a point on the curve where this happens is called a **stationary point** or **turning point**.

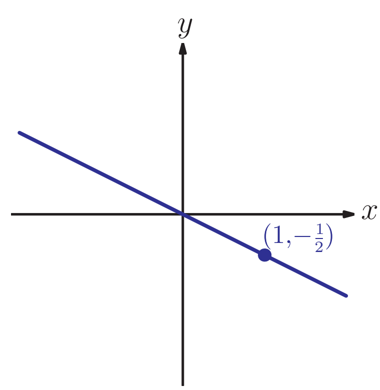
Exercise 16A

1. Sketch the derivatives of the following showing intercept with the x-axis:

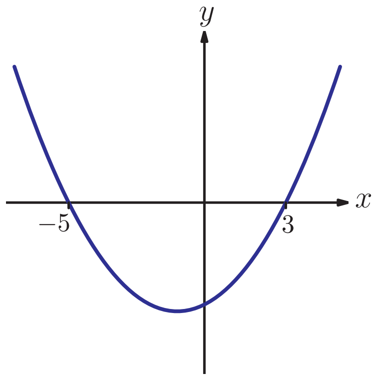
(a) (i)



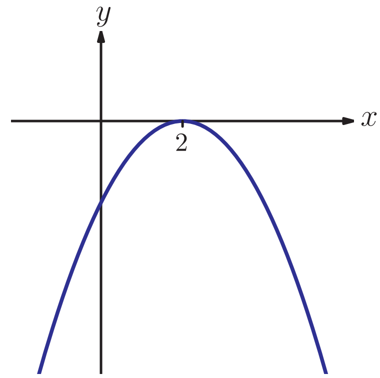
(ii)



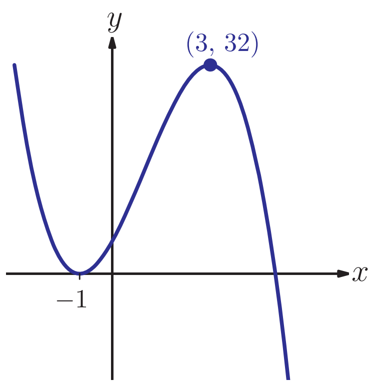
(b) (i)



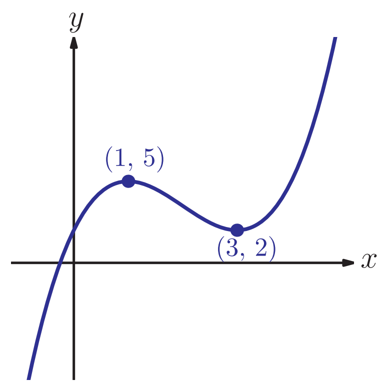
(ii)



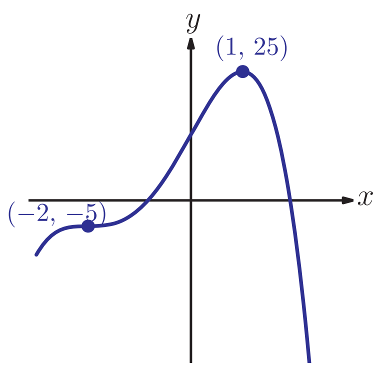
(c) (i)



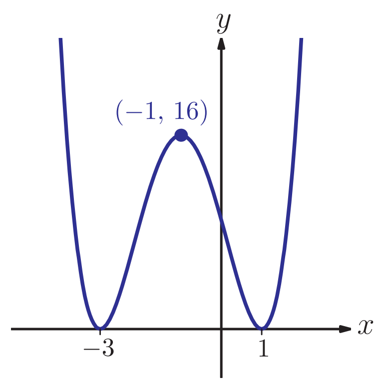
(ii)



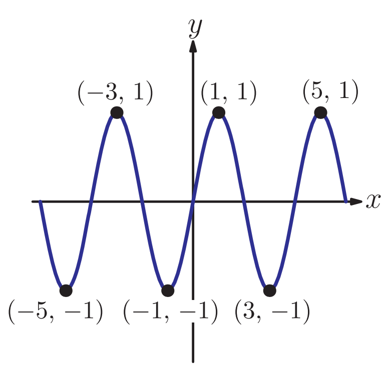
(d) (i)



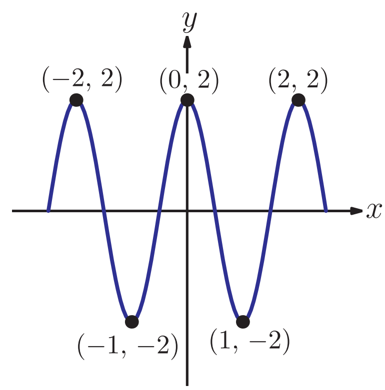
(ii)



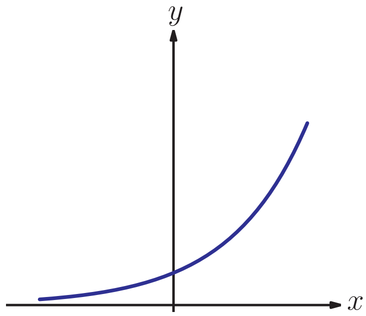
(e) (i)



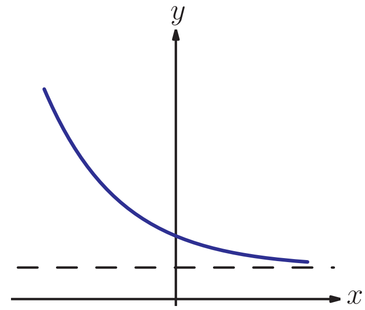
(ii)



(f) (i)

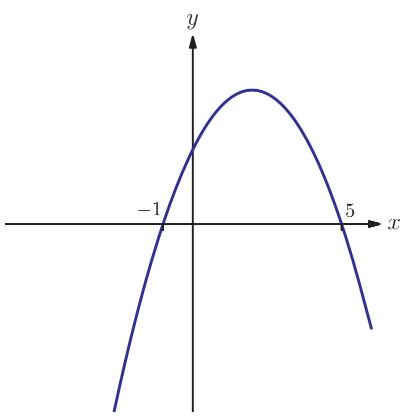


(ii)

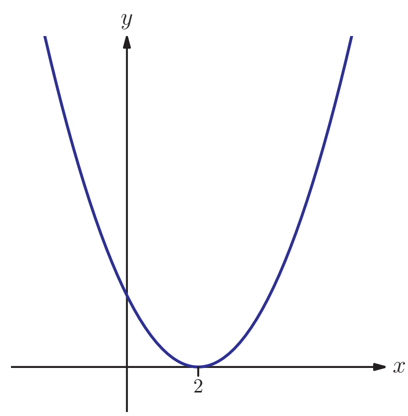


2. Each of the following represents a graph of a function's derivative. Sketch a possible graph for the original function, indicating any stationary points.

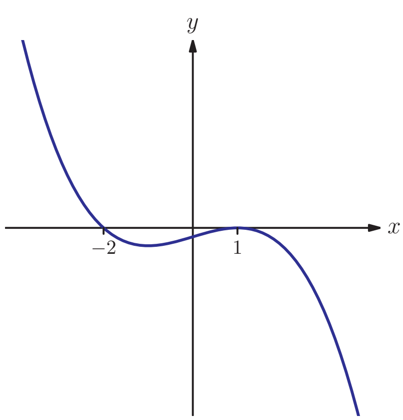
(a)



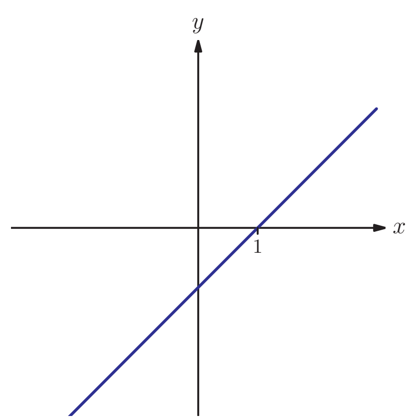
(b)



(c)



(d)



3. For each of the following statements decide if they are always true, sometimes true or always false.

(a) At a point where the derivative is positive, the original function is positive.

(b) If the original function is negative then the derivative is also negative.

(c) The derivative crossing the axis corresponds to a stationary point on the graph.

(d) When the derivative is zero, the graph is at a local maximum or minimum point.

(e) If the derivative function is always positive then part of the original function is above the x-axis.

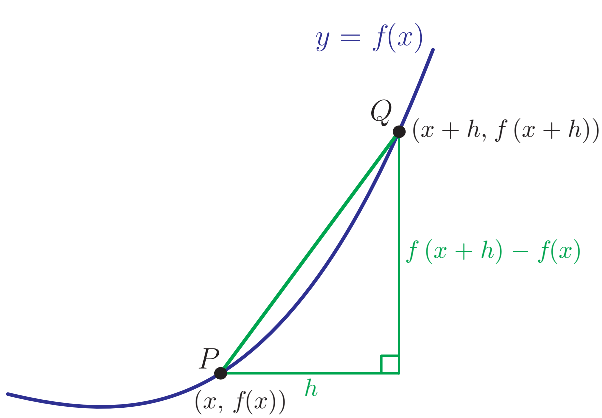
(f) At the lowest value of the original function, the derivative is zero.

16B Differentiation from first principles

You will probably find that drawing a tangent to a graph is very difficult to do accurately, and that your line actually crosses the curve at two points. The line segment between these two intersection points is called a **chord**. If the two points are close together, the gradient of the chord is very close to the gradient of the tangent. We can use this to establish a method for calculating the derivative for a given function.

Self-discovery worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM leads you through several examples of this method. Here we summarise the general procedure.

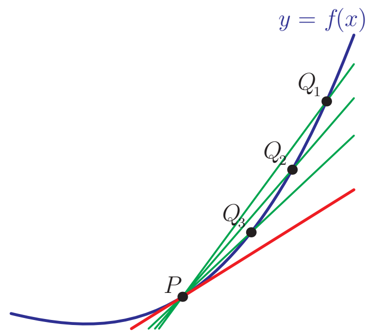
Consider a point $P(x, f(x))$ on the graph of the function $y = f(x)$ and move a distance h away from x to the point $Q(x+h, f(x+h))$.



We can find an expression for the gradient of the chord PQ:

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

As the point Q becomes closer and closer to P, the gradient of the chord PQ becomes a closer and closer approximation to the gradient of the tangent at P.



To denote this idea of the distance h approaching zero, we use $\lim\limits_{h \to 0}$, which reads as 'the limit as h tends to 0'. This idea of a limit is very much like that encountered for asymptotes on graphs in chapters 2 and 4, where the graph tends to the asymptote (the limit) as x tends to ∞.

The process of finding $\lim\limits_{h \to 0}$ of the gradient of the chord PQ is called **differentiation from first principles** and with this notation, we have the following definition:

KEY POINT 16.2

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is the **derivative of** $f(x)$. It can also be written as f', y' or $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = f(x)$. The process of finding the derivative is called **differentiation**.

EXAM HINT

Differentiation from first principles means finding the derivative using this definition, rather than any of the rules we will meet in the later sections.

We can use this definition to find the derivative of simple polynomial functions.

Worked example 16.3

For the function $y = x^2$, find $\frac{dy}{dx}$ from first principles.

Use the formula

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

We do not want to let the denominator tend to zero so first simplify the numerator and hope the h in the denominator will cancel

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

Divide top and bottom by h

$$= \lim_{h\to 0} (2x + h)$$

Finally let $h \to 0$

$$= 2x$$

We can use the same method with other functions too, but it may require more complicated algebraic manipulation.

Worked example 16.4

Differentiate $f(x) = \sqrt{x}$ from first principles.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

We do not want to let the denominator tend to zero so manipulate the numerator to get a factor of h

We can get rid of the square roots by multiplying top and bottom of the fraction by $\sqrt{x+h} + \sqrt{x}$ and using the difference of two squares

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

We can now divide top and bottom by h...

$$= \frac{1}{\sqrt{h+x} + \sqrt{x}}$$

→

… and let $h \to 0$

$$\therefore f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{h+x}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Exercise 16B

1. Find the derivatives of the following functions from first principles:
   (a) (i) $f(x) = x^3$ (ii) $f(x) = x^4$
   (b) (i) $f(x) = -4x$ (ii) $f(x) = 3x^2$
   (c) (i) $f(x) = x^2 - 6$ (ii) $f(x) = x^2 - 3x + 4$

2. Prove from first principles that the derivative of $x^2 + 1$ is $2x$.
[4 marks]

3. Prove from first principles that the derivative of 8 is zero.
[4 marks]

4. Prove from first principles that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$.
[4 marks]

5. If k is a constant prove that the derivative of $kf(x)$ is $kf'(x)$.
[4 marks]

6. Prove from first principles that the derivative of $\frac{1}{\sqrt{x}}$ is $-\frac{1}{2x\sqrt{x}}$.
[5 marks]

16C Rules of differentiation

From Exercise 16B, and the results of Self-discovery worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM, some properties of differentiation are suggested:

KEY POINT 16.3

- If $y = x^n$ then:

$$\frac{dy}{dx} = nx^{n-1}$$

- If we differentiate $kf(x)$ where k is a constant we get $kf'(x)$.
- Differentiation of the sum of various terms can proceed term by term.

Fill-in proof sheet 15 'Differentiating polynomials' on the CD-ROM proves these results for positive integer values; however, this result holds for all rational powers.

A special case is when $n = 0$. Since $x^0 = 1$, we can say that $\frac{dy}{dx} = 0x^{-1} = 0$. This is because the gradient of the graph $y = 1$ is zero everywhere; it is a horizontal line. In fact, the derivative of any constant is zero.

You often have to simplify an expression before differentiating, using the laws of algebra, in particular the laws of exponents.

If you need to review rules of exponents, see chapter 2.

Worked example 16.5

Find the derivative of the following functions:

(a) $f(x) = x^2\sqrt{x}$ (b) $g(x) = \frac{1}{\sqrt[3]{x}}$

First rewrite the function in the form x^n using the laws of exponents

(a) $f(x) = x^2\sqrt{x} = x^2 x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{5}{2}}$

Differentiate

$f'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{\frac{3}{2}}$

Cube root can be written as a power

(b) $g(x) = \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$

$g'(x) = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}}$

EXAM HINT

Note that you cannot differentiate products by differentiating each of the factors and multiplying them together – we will see in chapter 18 that there is a more complicated rule for dealing with products.

Worked example 16.6

Find the derivative of the following functions:

(a) $f(x) = 5x^3$

(b) $g(x) = x^4 - \frac{3}{2}x^2 + 5x - 4$

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

Differentiate x^3 then multiply by 5

(a) $f'(x) = 5 \times 3x^2 = 15x^2$

Differentiate each term separately

(b) $g'(x) = 4x^3 - \frac{3}{2} \times 2x + 5 = 4x^3 - 3x + 5$

We need to write this as a sum of terms of the form x^n

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

$= \frac{4x - 14}{x^{\frac{1}{2}}}$

$= 4x^{1-\frac{1}{2}} - 14x^{-\frac{1}{2}}$

$= 4x^{\frac{1}{2}} - 14x^{-\frac{1}{2}}$

Now differentiate each term separately

$h'(x) = 4 \times \frac{1}{2}x^{\frac{1}{2}-1} - 14\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$

$= 2x^{-\frac{1}{2}} + 7x^{-\frac{3}{2}}$

Exercise 16C

1. Differentiate the following:

(a) (i) $y = x^4$ (ii) $y = x$

(b) (i) $y = 3x^7$ (ii) $y = -4x^5$

(c) (i) $y = 10$ (ii) $y = -3$

(d) (i) $y = 4x^3 - 5x^2 + 2x - 8$ (ii) $y = 2x^4 + 3x^3 - x$

(e) (i) $y = \frac{1}{3}x^6$ (ii) $y = -\frac{3}{4}x^2$

(f) (i) $y = 7x - \frac{1}{2}x^3$ (ii) $y = 2 - 5x^4 + \frac{1}{5}x^5$

(g) (i) $y = x^{\frac{3}{2}}$ (ii) $y = x^{\frac{2}{3}}$

(h) (i) $y = 6x^{\frac{4}{3}}$ (ii) $y = \frac{3}{5}x^{\frac{5}{6}}$

(i) (i) $y = 3x^4 - x^2 + 15x^{\frac{2}{5}} - 2$ (ii) $y = x^3 - \frac{3}{5}x^{\frac{5}{3}} + \frac{4}{3}x^{\frac{1}{2}}$

(j) (i) $y = x^{-1}$ (ii) $y = -x^{-3}$

(k) (i) $y = x^{-\frac{1}{2}}$ (ii) $y = -8x^{-\frac{3}{4}}$

(l) (i) $y = 5x - \frac{8}{15}x^{-\frac{5}{2}}$ (ii) $y = -\frac{7}{3}x^{-\frac{3}{7}} + \frac{4}{3}x^{-6}$

2. Find $\frac{dy}{dx}$ for the following:

(a) (i) $y = \sqrt[3]{x}$ (ii) $y = \sqrt[5]{x^4}$

(b) (i) $y = \frac{3}{x^2}$ (ii) $y = -\frac{2}{5x^{10}}$

(c) (i) $y = \frac{1}{\sqrt{x}}$ (ii) $y = \frac{8}{3\sqrt[4]{x^3}}$

(d) (i) $y = x^2(3x - 4)$ (ii) $y = \sqrt{x}(x^3 - 2x + 8)$

(e) (i) $y = (x + 2)(\sqrt[3]{x} - 1)$ (ii) $y = \left(x + \frac{2}{x}\right)^2$

(f) (i) $y = \frac{3x^5 - 2x}{x^2}$ (ii) $y = \frac{9x^2 + 3}{2\sqrt[3]{x}}$

3. Find $\frac{dy}{dx}$ if:

(a) (i) $x + y = 8$ (ii) $3x - 2y = 7$

(b) (i) $y + x + x^2 = 0$ (ii) $y - x^4 = 2x$

16D Interpreting derivatives and second derivatives

$\frac{dy}{dx}$ has two related interpretations:

- It is the gradient of the graph of y against x.
- It measures how fast y changes when x is changed – this is called the **rate of change** of y with respect to x.

Remember that $\frac{dy}{dx}$ is itself a function – its value changes with x.

For example, if $y = x^2$ then $\frac{dy}{dx}$ is equal to 6 when $x = 3$, and it is equal to –2 when $x = -1$. This corresponds to the fact that the gradient of the graph of $y = x^2$ changes with x, or that the rate of change of y varies with x.

> **EXAM HINT**
> We can also write this using function notation:
> If $f(x) = x^2$ then $f'(3) = 6$ and $f'(-1) = -2$

To calculate the gradient (or the rate of change) at any particular point, we simply substitute the value of x into the equation for the derivative.

Worked example 16.7

Find the gradient of the graph $y = 4x^3$ at the point where $x = 2$.

The gradient is given by the derivative, so find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 12x^2$$

Substitute the value for x

When $x = 2$: $\frac{dy}{dx} = 12 \times 2^2 = 48$

So the gradient is 48

EXAM HINT

Your calculator can find the gradient at a given point, but it cannot find the expression for the derivative. See Calculator sheet 8 on the CD-ROM.

If we know the gradient of a graph at a particular point, we can find the value of x at that point. This involves solving an equation.

The sign of the gradient tells us whether the function is increasing or decreasing.

Worked example 16.8

Find the values of x for which the graph of $y = x^3 - 7x + 1$ has gradient 5.

The gradient is given by the derivative

$$\frac{dy}{dx} = 3x^2 - 7$$

We know the value of $\frac{dy}{dx}$ so we can form an equation

$3x^2 - 7 = 5$

$\Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

$\Rightarrow x = 2 \text{ or } -2$

KEY POINT 16.4

If $\frac{dy}{dx}$ is positive the function is increasing – as x gets larger so does y.

If $\frac{dy}{dx}$ is negative the function is decreasing – as x gets larger y gets smaller.

In Section 16H we will discuss what happens when $\frac{dy}{dx} = 0$.

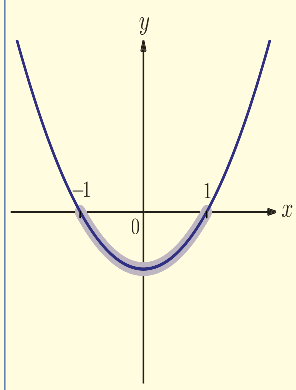
Worked example 16.9

Find the range of values of x for which the function $f(x) = 2x^3 - 6x$ is decreasing.

A decreasing function has negative gradient

$f'(x) < 0$
$\Rightarrow 6x^2 - 6 < 0$
$\Rightarrow x^2 - 1 < 0$

This is a quadratic inequality, so we need to look at the graph of $x^2 - 1$



$-1 < x < 1$

There is nothing special about the variables y and x. We can just as easily say that $\frac{dB}{dQ}$ is the gradient of the graph of B against Q or that $\frac{d(bananas)}{d(monkeys)}$ measures how fast bananas change when you change the variable monkeys. To emphasise which variables we are using, we call $\frac{dy}{dx}$ the **derivative of y with respect to x**.

You may wonder why it is so important to emphasise that we are differentiating with respect to x (or Q or *monkeys*). In this course we are only considering functions of one variable, but it is possible to generalise calculus to include functions which depend on several variables. This has many applications, particularly in physics and engineering.

Worked example 16.10

Given that $a = \sqrt{S}$, find the rate of change of a when $S = 9$.

The rate of change is given by the derivative

$a = S^{\frac{1}{2}}$

$\frac{da}{dS} = \frac{1}{2}S^{-\frac{1}{2}} = \frac{1}{2\sqrt{S}}$

Substitute the value for S

When $S = 9$: $\frac{da}{dS} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$\frac{d}{dx}$ is called an operator – it acts on functions to turn them into other functions. So when we differentiate $y = 3x^2$ what we are really doing is applying the $\frac{d}{dx}$ operator to both sides of the identity:

$$\frac{d}{dx}(y) = \frac{d}{dx}(3x^2)$$

$$\Rightarrow \frac{dy}{dx} = 6x$$

So $\frac{dy}{dx}$ is just $\frac{d}{dx}$ applied to y.

The $\frac{d}{dx}$ operator can also be applied to things which have already been differentiated. This is then called the **second derivative**.

KEY POINT 16.5

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is given the symbol $\frac{d^2y}{dx^2}$ or $f''(x)$ and it refers to the rate of change of the gradient.

We can differentiate again to find the third derivative $\left(\frac{d^3y}{dx^3} \text{ or } f'''(x)\right)$, fourth derivative $\left(\frac{d^4y}{dx^4} \text{ or } f^{(4)}(x)\right)$, and so on.

Worked example 16.11

Given that $f(x) = 5x^3 - 4x$:

(a) Find $f''(x)$.

(b) Find the rate of change of the gradient of the graph of $y = f(x)$ at the point where $x = -1$.

Differentiate $f(x)$ and then differentiate the result

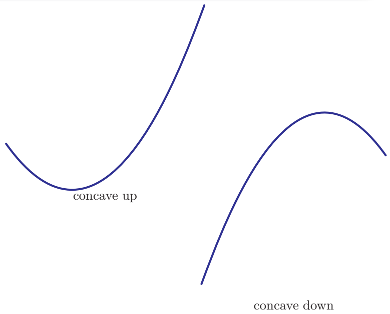
(a) $f'(x) = 15x^2 - 4$

$f''(x) = 30x$

The rate of change of the gradient means the second derivative

(b) $f''(-1) = -30$

We can use the second derivative to find out more about the shape of the graph. Remember that the second derivative is the rate of change of the gradient. So when the second derivative is positive, the gradient is increasing. This means that the graph is curving upwards; we say that it is **concave up**. When the second derivative is negative, the gradient is decreasing so the graph curves downwards; we say that it is **concave down**.



Exercise 16D

1. Write the following rates of change as derivatives:
   (a) The rate of change of z as t changes.
   (b) The rate of change of Q with respect to P.
   (c) How fast R changes when m is changed.
   (d) How quickly balloon volume (V) changes over time (t).
   (e) The rate of increase of the cost of apples (y) as the weight of the apple (x) increases.
   (f) The rate of change of the rate of change of z as y changes.
   (g) The second derivative of H with respect to m.

2. (a) (i) If $f = 5x^{\frac{1}{3}}$ what is the derivative of f with respect to x?
   (ii) If $p = 3q^5$ what is the derivative of p with respect to q?
   (b) (i) Differentiate $d = 3t + 7t^{-1}$ with respect to t.
   (ii) Differentiate $r = c + \frac{1}{c}$ with respect to c.
   (c) (i) Find the second derivative of $y = 9x^2 + x^3$ with respect to x.
   (ii) Find the second derivative of $z = \frac{3}{t}$ with respect to t.

3. (a) (i) If $y = 5x^2$, find $\frac{dy}{dx}$ when $x = 3$.

(ii) If $y = x^3 + \frac{1}{x}$, find $\frac{dy}{dx}$ when $x = 1.5$.

(b) (i) If $A = 7b + 3$, find $\frac{dA}{db}$ when $b = -1$.

(ii) If $f = \theta^2 + \theta^{-3}$, find $\frac{df}{d\theta}$ when $\theta = 0.1$.

(c) (i) Find the gradient of the graph of $A = x^3$ when $x = 2$.

(ii) Find the gradient of the tangent to the graph of $z = 2a + a^2$ when $a = -6$.

(d) (i) How quickly does $f = 4T^2$ change as T changes when $T = 3$?

(ii) How quickly does $g = y^4$ change as y changes when $y = 2$?

(e) (i) What is the rate of increase of W with respect to p when p is -3 if $W = -p^2$?

(ii) What is the rate of change of L with respect to c when $c = 6$ if $L = 7\sqrt{c} - 8$?

You may think that it is contradictory to talk about the rate of change of y as x changes if we are fixing x to have a certain value. Think about x passing through this point.

4. (a) (i) If $y = ax^2 + (1-a)x$ where a is a constant, find $\frac{dy}{dx}$.

(ii) If $y = x^3 + b^2$ where b is a constant, find $\frac{dy}{dx}$.

(b) (i) If $Q = \sqrt{ab} + \sqrt{b}$ where b is a constant, find $\frac{dQ}{da}$.

(ii) If $D = 3(av)^2$ where a is a constant, find $\frac{dD}{dv}$.

5. (a) (i) If $y = x^3 - 5x$, find $\frac{d^2y}{dx^2}$ when $x = 9$.

(ii) If $y = 8 + 2x^4$, find $\frac{d^2y}{dx^2}$ when $x = 4$.

(b) (i) If $S = 3A^2 + \frac{1}{A}$, find $\frac{d^2S}{dA^2}$ when $A = 1$.

(ii) If $J = v - \sqrt{v}$, find $\frac{d^2J}{dv^2}$ when $v = 9$.

(c) (i) Find the second derivative of B with respect to n if $B = 8n$ and $n = 2$.

(ii) Find the second derivative of g with respect to r if $g = r^7$ and $r = 1$.

6. (a) (i) If $y = 3x^3$ and $\frac{dy}{dx} = 36$, find x.

(ii) If $y = x^4 + 2x$ and $\frac{dy}{dx} = 5$, find x.

(b) (i) If $y = 2x + \frac{8}{x}$ and $\frac{dy}{dx} = -31$, find y.

(ii) If $y = \sqrt{x} + 3$ and $\frac{dy}{dx} = \frac{1}{6}$ find y.

7. (a) (i) Find the interval in which $x^3 - 4x$ is an increasing function.

(ii) Find the interval in which $x^3 - 3x^2$ is a decreasing function.

(b) (i) Find the interval in which $3x + \frac{2}{x}$ is a decreasing function.

(ii) Find the interval in which $x - \sqrt{x}$ is an increasing function.

(c) (i) Find the interval in which the graph of $y = x^3 - 4x + 3$ is concave up.

(ii) Find the interval in which the graph of $y = x^3 + 6x^2 - 1$ is concave up.

(d) (i) Find the set of values of x for which the graph of $f(x) = x^4 - 6x^3 + 12x^2$ is concave down.

(ii) Find the set of values of x for which the graph of $f(x) = x^4 - 54x^2$ is concave down.

8. Find all points of the graph of $y = x^3 - 2x^2 + 1$ where the gradient equals the y-coordinate. *[5 marks]*

9. In what interval on the graph of $y = 7x - x^2 - x^3$ is the gradient decreasing? *[5 marks]*

10. In what interval on the graph of $y = \frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 - 3x + 6$ is the gradient increasing? *[6 marks]*

11. Find an alternative expression for $\frac{d^n}{dx^n}(x^n)$.

16E Trigonometric functions

Using the techniques from Section 16A we can sketch the derivative of the graph of $y = \sin x$. The result is a graph that looks just like $y = \cos x$. On Fill-in proof sheet 17 'Differentiating trigonometric functions' on the CD-ROM you can see why this is the case. Results for $y = \cos x$ and $y = \tan x$ can be established in a similar manner giving these results:

In Section 18C we will prove the derivative of tan x using the quotient rule.

KEY POINT 16.6

Differentiating trigonometric functions gives:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

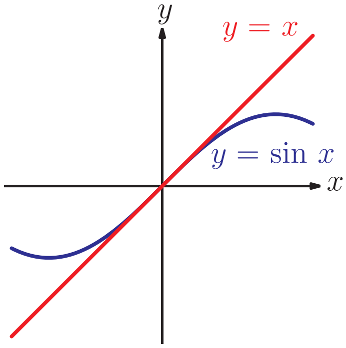
Reciprocal trigonometric functions were covered in Section 12D.

EXAM HINT

Whenever you are doing calculus you MUST work in radians.

It is possible to do calculus using degrees, or any other unit for measuring angles, but using radians gives the simplest rules, which is why they are the unit of choice for almost all mathematicians.

These rules only work if x is measured in radians since they are based upon the result that $\sin x \approx x$ for very small values of x. You can check on your calculator that $\sin x \approx x$ for radians but not for degrees. The result can also be seen on the graph and is proved on Fill-in proof sheet 16 'The small angle approximations' on the CD-ROM.



All rules of differentiation from Section 16C still apply.

Worked example 16.12

Differentiate $y = 3\tan x - 2\cos x$.

Differentiate using the rules in key point 16.6. Note that $\sec^2 x$ can also be written as $\frac{1}{\cos^2 x}$

$$\frac{dy}{dx} = 3(\sec^2 x) - 2(-\sin x)$$
$$= 3\sec^2 x + 2\sin x$$

Exercise 16E

1. Differentiate the following:
   (a) (i) $y = 3\sin x$ (ii) $y = 2\cos x$
   (b) (i) $y = 2x - 5\cos x$ (ii) $y = \tan x + 5$
   (c) (i) $y = \frac{\sin x + 2\cos x}{5}$ (ii) $y = \frac{1}{2}\tan x - \frac{1}{3}\sin x$

2. Find the gradient of $f(x) = \sin x + x^2$ at the point $x = \frac{\pi}{2}$. *[5 marks]*

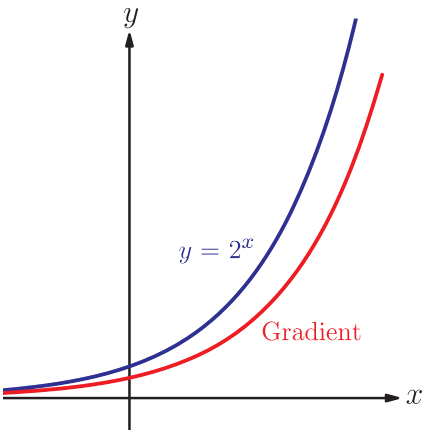
3. Find the gradient of $g(x) = \frac{1}{4}\tan x - 3\cos x - x^3$ at the point $x = \frac{\pi}{6}$. *[5 marks]*

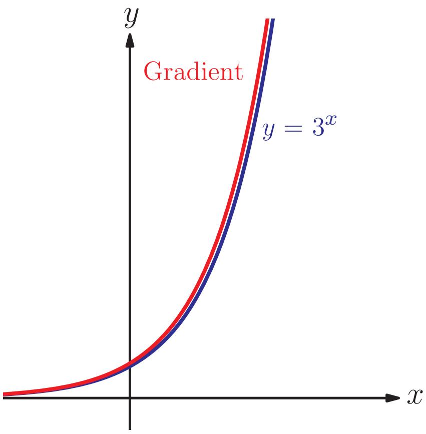
4. Given $h(x) = \sin x + \cos x \quad 0 \le x < 2\pi$, find the values of x for which $h'(x) = 0$. *[6 marks]*

5. Given $y = \frac{1}{4}\tan x + \frac{1}{x^2} \quad 0 < x \le 2\pi$ solve the equation $\frac{dy}{dx} = 1 - \frac{2}{x^3}$. *[6 marks]*

16F The exponential and natural logarithm functions

Use your calculator to plot the graphs of $y = 2^x$ and $y = 3^x$ and their derivatives. The results looks like another exponential function.





It appears that there is a number somewhere between two and three where the derivative of the graph would be exactly the same as the original exponential. It turns out that this is the graph of $y = e^x$ where $e = 2.718\ldots$ It is the same as the base of the natural logarithm defined in Section 2E.

We will see how to differentiate exponential functions with bases other than e in Section 20D.

KEY POINT 16.7

$$\frac{d}{dx}(e^x) = e^x$$

The natural logarithm function $y = \ln x$ behaves in a surprising way, having a derivative of a completely different form.

KEY POINT 16.8

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

This result is proved on Fill-in proof sheet 18 'Differentiating logarithmic functions graphically' on the CD-ROM.

Exercise 16F

1. Differentiate the following:

   (a) (i) $y = 3e^x$ (ii) $y = \frac{2e^x}{5}$

   (b) (i) $y = -2\ln x$ (ii) $y = \frac{1}{3}\ln x$

   (c) (i) $y = \frac{\ln x}{5} - 3x + 4e^x$ (ii) $y = 4 - \frac{e^x}{2} + 3\ln x$

2. (i) Find the exact value of the gradient of the graph of $f(x) = \frac{1}{2}e^x - 7\ln x$ at the point $x = \ln 4$.

   (ii) Find the exact value of the gradient of the graph $f(x) = e^x - \frac{\ln x}{2}$ when $x = \ln 3$. *[4 marks]*

3. Find the value of x where the gradient of $f(x) = 5 - 2e^x$ is −6. *[4 marks]*

4. Find the value of x where the gradient of $g(x) = x^2 - 12\ln x$ is 2. *[4 marks]*

There is an easier way to do some parts of Question 5 using a method from Section 18A. For now, you will have to use your algebra skills!

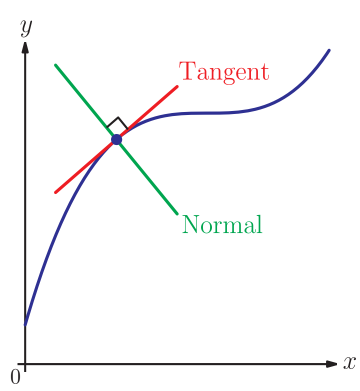
5. Differentiate:

   (a) (i) $y = \ln x^3$ (ii) $y = \ln 5x$

   (b) (i) $y = e^{x+3}$ (ii) $y = e^{x-3}$

   (c) (i) $y = e^{2\ln x}$ (ii) $y = e^{3\ln x + 2}$

   (d) (i) $y = \log_3 x$ (ii) $y = 4\log_6 x$



16G Tangents and normals

The tangent to a curve at a given point is a straight line which touches the curve and has the same gradient at that point. Finding the equation of the tangent at a point relies on knowing the gradient of the function at that point. This can be found by differentiating the function. We then have both the gradient of the line and a point on it and we can use the standard procedure for finding the equation of a straight line.

See Prior learning section W on the CD-ROM.

Normals are lines which pass through the graph and are perpendicular to the tangent. They have many uses, such as finding centres of curvature of graphs and working out how light is reflected from curved mirrors. However, in the International Baccalaureate® you are only likely to be asked to calculate their equations. To do this you use the fact that if two lines with gradients m_1 and m_2 are perpendicular, $m_1 m_2 = -1$.

Worked example 16.13

(a) Find the equation of the tangent to the graph of the function $f(x) = \cos x + e^x$ at the point $x = 0$.

(b) Find the equation of the normal to the graph of the function $g(x) = x^3 - 5x^2 - x^{\frac{3}{2}} + 22$ at $(4, -2)$.

In each case give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

We need the gradient, which is $f'(0)$

(a) $f'(x) = -\sin x + e^x$

$\therefore f'(0) = -\sin 0 + e^0 = 1$

To find the equation of a straight line we also need coordinates of one point
The tangent passes through the point on the graph where $x = 0$. Its y-coordinate is $f(0)$

When $x = 0$,
$y = f(0) = \cos 0 + e^0$
$= 1 + 1$
$= 2$

Put all the information into the equation of a line

$y - y_1 = m(x - x_1)$
$y - 2 = 1(x - 0)$
$\Rightarrow y = x + 2$
$\Rightarrow y - x - 2 = 0$

The normal is perpendicular to the tangent, so we need the gradient of the tangent first

(b) $f'(x) = 3x^2 - 10x - \frac{3}{2}x^{\frac{1}{2}}$

Find the gradient at $x = 4$

$\therefore f'(4) = 3(4)^2 - 10(4) - \frac{3}{2}(4)^{\frac{1}{2}}$
$= 48 - 40 - 3 = 5$

For perpendicular lines, $mm_1 = -1$

Therefore gradient of normal,
$m = \frac{-1}{5}$

We are given both x and y-coordinates of the point, so put all the information into the equation of a line

$y - y_1 = m(x - x_1)$

$y - (-2) = \frac{-1}{5}(x - 4)$

$\Rightarrow \quad 5y + 10 = -x + 4$
$\Rightarrow x + 5y + 6 = 0$

The procedure for finding the equations of tangents and normals can be summarised as follows:

EXAM HINT

Your calculator may be able to find the equation of a tangent at a given point.

KEY POINT 16.9

For the point on the curve $y = f(x)$ with $x = a$:

- the gradient of the tangent is $f'(a)$
- the gradient of the normal is $-\frac{1}{f'(a)}$
- the coordinates of the point are $x_1 = a$, $y_1 = f(a)$.

To find the equation of the tangent or the normal use $y - y_1 = m(x - x_1)$ with the appropriate gradient.

You may not be given the coordinates of the point where the tangent touches the curve, but asked to find them given another point.

Worked example 16.14

The tangent at point P on the curve $y = x^2 + 1$ passes through the origin. Find the possible coordinates of P.

We want to find the equation of the tangent at P, so use unknowns for its coordinates

Let P have coordinates (p, q)

As P lies on the curve, (p, q) satisfies $y = x^2 + 1$

Then $q = p^2 + 1$

The gradient of the tangent is given by $\frac{dy}{dx}$ when $x = p$

$\frac{dy}{dx} = 2x$

When $x = p$: $\frac{dy}{dx} = 2p$

$\therefore m = 2p$

Write the equation of the tangent, remembering it passes through (p, q)

Equation of the tangent:

$y - q = 2p(x - p)$

$\Rightarrow y - (p^2 + 1) = 2p(x - p)$

→

continued . . .

Tangent passes through the origin, so set $x=0, y=0$

Passes through $(0,0)$:
$0-(p^2+1)=2p(0-p)$
$\Rightarrow -p^2-1=-2p^2$
$\Rightarrow p^2=1$
$\Rightarrow p=1$ or -1

We can now find q

When $p=1, q=2$
When $p=-1, q=2$
So the coordinates of P are (1, 2) or (–1, 2)

Exercise 16G

1. Find the equations of the tangent and normal to the following:
   (a) $y=\dfrac{x^2+4}{\sqrt{x}}$ at $x=4$
   (b) $y=3\tan x-2\sqrt{2}\sin x$ at $x=\dfrac{\pi}{4}$
   (c) $y=3-\dfrac{1}{5}e^x$ at $x=2\ln 5$

2. Find the coordinates of the point on the curve $y=\sqrt{x}+3x$ where the gradient is 5. *[4 marks]*

3. Find the equation of the tangent to the curve $y=e^x+x$ which is parallel to $y=3x$. *[4 marks]*

4. Find the x-coordinates of the points on the curve $y=x^3-3x^2$ where the tangent is parallel to the normal of the point at (1, –1). *[6 marks]*

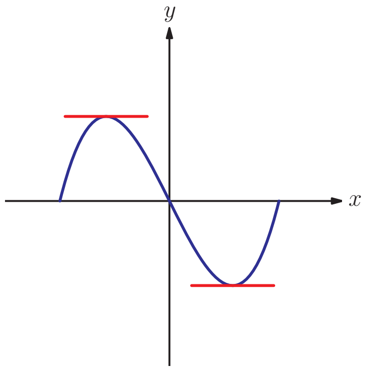
5. Find the coordinates of the point where the tangent to the curve $y=x^3-3x^2$ at $x=2$ meets the curve again. *[6 marks]*

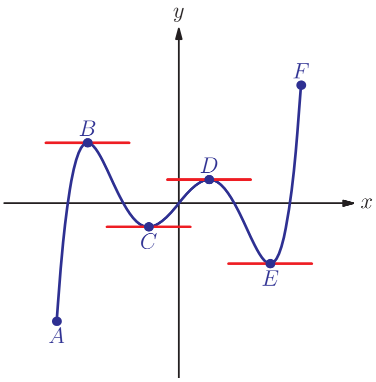
6. Find the coordinates of the point on the curve $y=(x-1)^2$ where the normal passes through the origin. *[5 marks]*

7. Points P and Q lie on the graph of $f(x)=2\sin x$ and have x-coordinates $\dfrac{\pi}{6}$ and $\dfrac{\pi}{4}$.
   (a) Evaluate $f'\left(\dfrac{\pi}{6}\right)$.
   (b) Find the angle between the tangent at P and the chord PQ, giving your answer to the nearest tenth of a degree. *[11 marks]*

8. A tangent is drawn on the graph $y=\dfrac{k}{x}$ at the point where $x=a$, $(a>0)$. The tangent intersects the y-axis at P and the x-axis at Q. If O is the origin show that the area of the triangle OPQ is independent of a. *[8 marks]*

9. Show that the tangent to the curve $y = x^3 - x$ at the point with x-coordinate a meets the curve again at a point with x-coordinate $-2a$. *[6 marks]*





16H Stationary points

In real life people are interested in maximising their profits, or minimising the drag on a car. We can use calculus to describe such things mathematically as points on a graph.

The gradient at both the maximum and minimum point on the above graph is zero and therefore:

KEY POINT 16.10

To find local maximum and local minimum points, we solve the equation $\frac{dy}{dx} = 0$.

We use the phrase **local maximum** and **local minimum** because it is possible that the largest or smallest value of the whole function occurs at the endpoint of the graph, or that there are other points which also have gradient of zero. The points that we have found are just the largest or smallest values of y in that part of the graph.

Points which have a gradient of zero are called **stationary points**.

Worked example 16.15

Find the coordinates of the stationary points of $y = 2x^3 - 15x^2 + 24x + 8$.

Stationary points have $\frac{dy}{dx} = 0$ so we need to differentiate

$$\frac{dy}{dx} = 6x^2 - 30x + 24$$

Then form an equation

For stationary points $\frac{dy}{dx} = 0$:

$6x^2 - 30x + 24 = 0$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0$

$\Rightarrow x = 1$ or $x = 4$

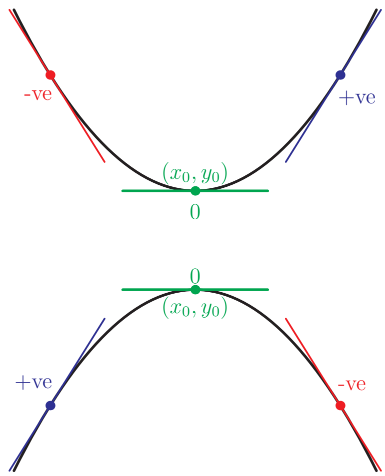
continued . . .

Remember to find the y-coordinate for each point

When $x = 1$:
$y = 2(1)^3 - 15(1)^2 + 24(1) + 8 = 19$
When $x = 4$:
$y = 2(4)^3 - 15(4)^2 + 24(4) + 8 = -8$
Therefore,
stationary points are (1, 19) and (4, −8)

The calculation in Worked example 16.15 does not tell us whether the stationary points we found are maximum or minimum points.

It can be seen from the diagrams that one way of testing for the nature of a stationary point is to look at the gradient either side of the point. You can do this by substituting nearby x-values into the expression for $\frac{dx}{dy}$. For a minimum point the gradient moves from negative to positive. For a maximum point the gradient moves from positive to negative.
We can also interpret these conditions by looking at the sign of the second derivative. Around a minimum point the curve is concave up, so $\frac{d^2y}{dx^2}$ is positive. Around a maximum point the curve is concave down and $\frac{d^2y}{dx^2}$ is negative.



See the end of Section 16D

This leads to the following test.

KEY POINT 16.11

Given a stationary point (x_0, y_0) of a function $y = f(x)$, if:

- $\frac{d^2y}{dx^2} < 0$, at x_0, then (x_0, y_0) is a *maximum*
- $\frac{d^2y}{dx^2} > 0$, at x_0, then (x_0, y_0) is a *minimum*
- $\frac{d^2y}{dx^2} = 0$, at x_0, then no conclusion can be drawn, so test the gradient either side of (x_0, y_0)

Worked example 16.16

Classify the stationary points of the function $y = 2x^3 - 15x^2 + 24x + 8$ from Worked example 16.15.

We have already found the stationary points

Stationary points are (1, 19) and (4, −8)

The nature of stationary points is determined by the value of the second derivative

$$\frac{d^2y}{dx^2} = 12x - 30$$

At $x = 1$:

$$\frac{d^2y}{dx^2} = 12(1) - 30 = -18 < 0$$

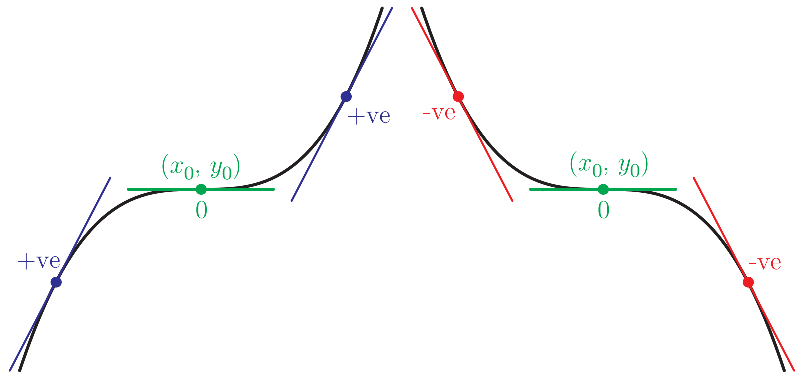
$\therefore (1, 19)$ is a maximum

At $x = 4$:

$$\frac{d^2y}{dx^2} = 12(4) - 30 = 18 > 0$$

$\therefore (4, -8)$ is a minimum

All local maximum points and local minimum points have $\frac{dy}{dx} = 0$, but the reverse is not true: A point with $\frac{dy}{dx} = 0$ does not have to be a maximum or a minimum point. There are two other possibilities:



These possibilities are called **points of inflexion**, and are labelled (x_0, y_0) on the above diagrams. Note that at those points the line with zero gradient actually crosses the curve. The gradient is either positive on both sides of a point of inflexion (positive point of inflexion), or negative on both sides (a negative point of inflexion).

In UK English, 'inflexion' may be spelt 'inflection'.

Worked example 16.17

Find the coordinates and nature of the stationary points of $y = 3 + 4x^3 - x^4$.

Stationary points have $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 12x^2 - 4x^3$

For stationary points $\frac{dy}{dx} = 0$:

$12x^2 - 4x^3 = 0$

$\Rightarrow 4x^2(3 - x) = 0$

$\Rightarrow x = 0$ or $x = 3$

Find *y*-coordinates

When $x = 0$:

$y = 3 + 4(0)^3 - (0)^4 = 3$

When $x = 3$:

$y = 3 + 4(3)^3 - (3)^4 = 30$

Therefore, stationary points are: $(0, 3)$ and $(3, 30)$

The nature of the stationary points is determined by the second derivative

Find the nature of these points:

$\frac{d^2y}{dx^2} = 24x - 12x^2$

At $x = 0$:

$\frac{d^2y}{dx^2} = 24(0) - 12(0)^2 = 0$

As $\frac{d^2y}{dx^2} = 0$ we need to check the gradient either side of the stationary point

Therefore, examine $\frac{dy}{dx}$:

At $x = -1$:

$\frac{dy}{dx} = 12(-1)^2 - 4(-1)^3 = 16 > 0$

At $x = 1$:

$\frac{dy}{dx} = 12(1)^2 - 4(1)^3 = 8 > 0$

$\therefore (0, 3)$ is a positive point of inflexion.

continued . . .

At $x=3$:

$$\frac{d^2y}{dx^2}=24(3)-12(3)^2=-36<0$$

$\therefore (3, 30)$ is a maximum

When $\frac{d^2y}{dx^2}=0$, the stationary point is NOT always a point of inflexion.

Worked example 16.18

Find the coordinates and nature of the stationary points of $f(x)=x^4$:

$f'(x)=4x^3$

Stationary points have $f'(x)=0$

For stationary points $f'(x)=0$

$4x^3=0$

$\Rightarrow x=0$

Find the y-coordinate

$f(0)=0$

Therefore, stationary point is:

$(0, 0)$

The nature is determined by $f''(x)$

Find the nature of this point:

$f''(x)=12x^2$

$f''(0)=0$

As $f''(0)=0$, we need to check the gradient on either side

Therefore, examine $f'(x)$:

$f'(-1)=4(-1)^3=-4<0$

$f'(1)=4(1)^3=4>0$

Therefore $(0, 0)$ is a minimum.

Exercise 16H

1. Find and classify the stationary points on the following curves:

   (a) (i) $y = x^3 - 5x^2$ (ii) $y = x^4 - 8x^2$

   (b) (i) $y = \sin x + \frac{x}{2}, \quad -\pi \le x \le \pi$

   (ii) $y = 2\cos x + 1,\ 0 \le x < 2\pi$

   (c) (i) $y = \ln x - \sqrt{x}$ (ii) $y = 2e^x - 5x$

2. Give an example to illustrate that the following statement is incorrect:

   'If $y = f(x)$ has exactly two stationary points, at x_1 and x_2, and $f(x_1) > f(x_2)$ then $(x_1, f(x_1))$ must be a local maximum.'

   Under what conditions is the statement true?

3. Find and classify the stationary points on the curve $y = x^3 + 3x^2 - 24x + 12$. *[6 marks]*

4. Find and classify the stationary points on the curve $y = x - \sqrt{x}$. *[6 marks]*

5. Find and classify the stationary points on the curve $y = \sin x + 4\cos x$ in the interval $0 < x < 2\pi$. *[6 marks]*

6. Show that the function $f(x) = \ln x + \frac{1}{x^k}$ has a stationary point with y-coordinate $\frac{\ln k + 1}{k}$. *[6 marks]*

7. Find the range of the function $f : x \mapsto 3x^4 - 16x^3 + 18x^2 + 6$. *[5 marks]*

8. Find the range of the function $f : x \mapsto e^x - 4x + 2$. *[5 marks]*

9. Find and classify in terms of k the stationary points on the curve $y = kx^3 + 6x^2$. *[6 marks]*

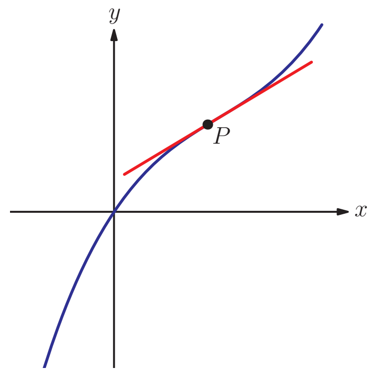
16I General points of inflexion

In the previous section we met stationary points of inflexion, but the idea of a point of inflexion is more general than this.

One definition is that the tangent to the curve at a point of inflexion crosses the curve at the same point. This does not require the point to be a stationary point.

EXAM HINT

Although the red line actually crosses the graph at P, it is still referred to as the tangent, because it has the same gradient as the curve at P.



Geometrically, this can be interpreted as an 'S-bend', a curve which goes from decreasing gradient to increasing gradient (or vice versa). This means that the curve is concave down on one side of the point of inflexion and concave up on the other. We know that this corresponds to the second derivative changing from negative to positive (or vice versa).

See the end of Section 16D.

KEY POINT 16.12

At a point of inflexion $\frac{d^2y}{dx^2} = 0$.

Unfortunately, as in Worked example 16.18, just because a point has $\frac{d^2y}{dx^2} = 0$ it is not necessarily a point of inflexion. We have to determine the gradient on either side to be sure.

EXAM HINT

If a question states that a curve does have a point of inflexion and there is only one solution to the equation $\frac{d^2y}{dx^2} = 0$, you can then assume you have found the point of inflexion.

Worked example 16.19

Find the coordinates of the point of inflexion on the curve $y = x^3 - 3x^2 + 5x - 1$.

Find $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 3x^2 - 6x + 5$

$\frac{d^2y}{dx^2} = 6x - 6$

At a point of inflexion $\frac{d^2y}{dx^2} = 0$,

$6x - 6 = 0$

$x = 1$

Remember the other coordinate!

When $x = 1$, $y = 1 - 3 + 5 - 1 = 2$

So point of inflexion is at (1, 2)

Exercise 16I

1. Find the coordinates of the point of inflexion on the curve $y = e^x - x^2$. *[5 marks]*

2. The curve $y = x^4 - 6x^2 + 7x + 2$ has two points of inflexion. Find their coordinates. *[5 marks]*

3. Show that all points of inflexion on the curve $y = \sin x$ lie on the x-axis. *[6 marks]*

4. Find the coordinates of the points of inflexion on the curve $y = 2\cos x + x$ for $0 \leq x \leq 2\pi$. Justify carefully that these points are points of inflexion. *[5 marks]*

5. The point of inflexion on the curve $y = x^3 - ax^2 - bx + c$ is a stationary point of inflexion. Show that $b = 8a^2$. *[6 marks]*

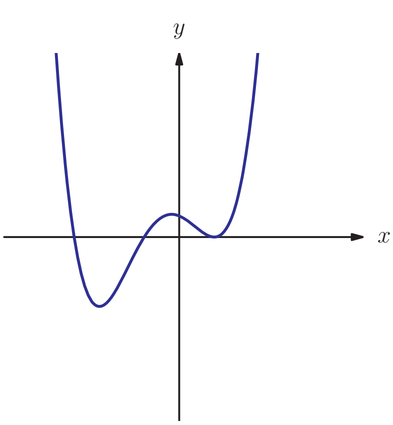
6. The graph shows $y = f'(x)$.

   On a copy of the diagram:

   (a) mark any point corresponding to a local minimum of $f(x)$ with an A

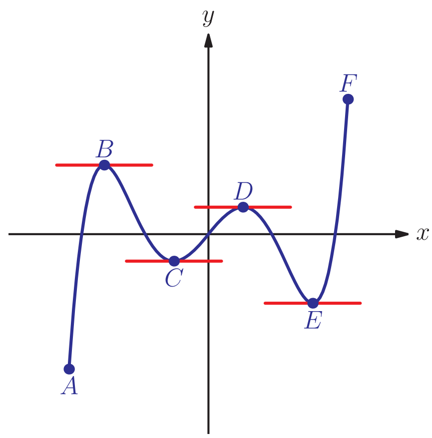
   (b) mark any point corresponding to a local maximum of $f(x)$ with a B

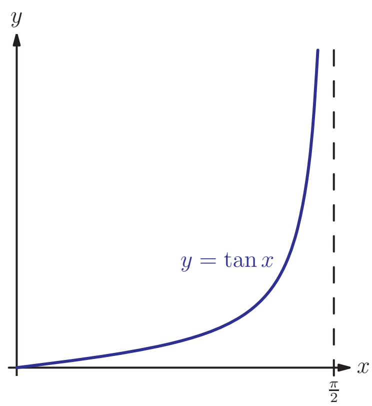
   (c) mark any point corresponding to a point of inflexion of $f(x)$ with a C. *[4 marks]*



16J Optimisation

We can now start to use differentiation to maximise or minimise quantities. In Section 16H we saw how to find stationary points (the points with zero gradient) and how to decide whether they are local maximum or local minimum points. We also noted that a stationary point does not necessarily give the largest or smallest value of the function over the whole domain. For example, on the diagram, points B and D are local maximum points, but the largest value of the function occurs at point F, which is an end point of the domain.





Some functions do not have maximum or minimum values at all. This can happen when the graph has an asymptote. We say that the function is not continuous throughout its domain. For example, the value of tan x increases without a limit as x increases towards $\frac{\pi}{2}$, so tan x does not have a maximum value.

If we wish to minimise or maximise A by changing B we do so in four stages:

1. Find the relationship between A and B.
2. Solve the equation $\frac{dA}{dB} = 0$.
3. Decide whether it is a maximum, minimum or point of inflexion by considering $\frac{d^2A}{dB^2}$.
4. Check whether the end points of the domain are actually global maximum or minimum points, and check that there are no vertical asymptotes.

Often the first stage of this process is the most difficult and there are many questions where we have to use a geometric context to make this link. Thankfully in many questions this relationship is given to you.

Worked example 16.20

The height of a swing (h) in metres at a time t seconds is given by $h = 2 - 1.5\sin t$ for $0 < t < 3$. Find the minimum and maximum height of the swing.

Find stationary points

$\frac{dh}{dt} = -1.5\cos t = 0$ at a stationary point

$\Rightarrow \cos t = 0$

$0 < t < 3 \therefore t = \frac{\pi}{2}$ (only one solution)

Classify stationary points

$\frac{d^2h}{dt^2} = 1.5\sin t$

When $t = \frac{\pi}{2}$, $\frac{d^2h}{dt^2} = 1.5 > 0$ so $t = \frac{\pi}{2}$ is a local minimum. This minimum height is

$h = 2 - 1.5\sin\frac{\pi}{2} = 0.5$ metres

Check end points

Check there are no vertical asymptotes

When $t = 0$, $h = 2$m

When $t = 3$, $h = 1.79$m (3SF)

So maximum height is 2 m.

Exercise 16J

1. What are the minimum and maximum values of the expression e^x for $0 \le x \le 1$? *[4 marks]*

2. A rectangle has width x metres and length $30 - x$ metres.
   (a) Find the maximum area of the rectangle.
   (b) Show that as x changes the perimeter stays constant and find the value of this perimeter. *[5 marks]*

3. Find the maximum and minimum values of the function $y = x^3 - 9x$ if $-2 \le x \le 5$. *[5 marks]*

4. What are the maximum and minimum values of $f(x) = e^x - 3x$ if $0 \le x \le 2$? *[5 marks]*

5. What are the minimum and maximum values of $y = \sin x + 2x$ for $0 \le x \le 2\pi$? *[5 marks]*

6. Find the minimum value of the sum of a positive real number and its reciprocal. *[5 marks]*

7. A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1 - \frac{1}{\sqrt{w}}$ ms^{-1} for a time of $\frac{5}{w}$ s. What weight will achieve the maximum distance travelled? *[6 marks]*

8. The time in minutes (t) taken to melt 100 g of butter depends upon the percentage of the butter which is made of saturated fats (p) as in the following function:

$$t = \frac{p^2}{10\,000} + \frac{p}{100} + 2$$

Find the maximum and minimum times to melt 100 g of butter. *[6 marks]*

9. The volume of water in millions of litres (V) in a new tidal lake is modelled by $V = 60\cos t + 100$ where t is the time in days after being completed.

(a) What is the smallest volume of the lake?

(b) A hydroelectric plant produces an amount of electricity proportional to the rate of flow of water. In the first 6 days when is the plant producing maximum electricity? *[6 marks]*

10. The owner of a fast-food shop finds that there is a relationship between the amount of salt s (g/tray) added to the fries and his weekly sales of fries F (100s of portions):

$$F(s) = 4s + 1 - s^2,\ \ 0 \le s \le 4.2$$

(a) Find the amount of salt he should put on his fries to maximise his sales.
The total cost C (\$ per tray) associated with the sales of fries is given by:

$$C(s) = 0.3 + 0.2F(s) + 0.1s$$

(b) Find the amount of salt he should put on his fries to minimise his costs.

(c) The profit made on his fries is given by the difference between the sales and the costs.
How much salt should he add to maximise his profit? *[8 marks]*

11. A car tank is being filled with petrol such that the volume in the tank in litres (V) over time in minutes (t) is given by

$$V = 300(t^2 - t^3) + 4 \ \text{ for } 0 < t < 0.5$$

(a) How much petrol was initially in the tank?

(b) After 30 seconds the tank was full. What is the capacity of the tank?

(c) At what time is petrol flowing in at the greatest rate? *[8 marks]*

12. x is the surface area of leaves on a tree in m^2. Because leaves may be shaded by other leaves, the amount of energy produced by the tree is given by $2-\frac{x}{10}$ kJ per square metre of leaves.

(a) Find an expression for the total energy produced by the tree.

(b) What area of leaves provides the maximum energy for the tree?

(c) Leaves also use energy. The total energy requirement is given by $0.01x^3$. The net energy produced is the difference between the energy produced by the leaves and the energy used by the leaves. For what range of x do the leaves produce more energy than they use?

(d) Show that the maximum net energy is produced when the tree has leaves with a surface area of $\frac{10(\sqrt{7}-1)}{3}$. *[12 marks]*

Summary

- The **gradient** of a function at the point P is the gradient of the **tangent** to the function's graph at that point.
- To find the gradient of a function we can **differentiate** from first principles:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ (also denoted by } \frac{d}{dx}f(x))$$

- For the point on the curve $y=f(x)$ with $x=a$:

  – the gradient of the tangent is $f'(a)$

  – the gradient of the **normal** is $-\frac{1}{f'(a)}$

- If $f(x)=x^n$, then $f'(x)=nx^{n-1}$.
- The **derivative** of a sum is the sum of the derivatives of each term.
- If we differentiate $kf(x)$ where k is a constant we get $kf'(x)$.
- The **derivatives** of the trigonometric functions are:

$$\frac{d}{dx}(\sin x)=\cos x \qquad \frac{d}{dx}(\cos x)=-\sin x \qquad \frac{d}{dx}(\tan x)=\sec^2 x$$

- The derivatives of the exponential and natural logarithm functions are:

$$\frac{d}{dx}(e^x)=e^x \qquad \frac{d}{dx}(\ln x)=\frac{1}{x}$$

- **Stationary points** of a function are points where the gradient is zero, i.e.

$$\frac{dy}{dx}=0$$

- Stationary points can be one of four types:
  - local maximum
  - local minimum
  - positive point of inflexion
  - negative point of inflexion.
- The **second derivative** can be used to test which of these occurs. At a stationary point (x_0, y_0), if
  - $\frac{d^2y}{dx^2} < 0$ at x_0 then (x_0, y_0) is a maximum
  - $\frac{d^2y}{dx^2} > 0$ at x_0 then (x_0, y_0) is a minimum
  - $\frac{d^2y}{dx^2} = 0$ at x_0 then no conclusion can be drawn, so check the sign of the gradient either side of (x_0, y_0).
- **Points of inflexion** can also have a non-zero gradient.
- At a point of inflexion $\frac{d^2y}{dx^2} = 0$.
- Global maximum or minimum points may also occur at the endpoint of a graph.

Introductory problem revisited

The cost of petrol used in a car, in £ per hour, is $\frac{12 + v^2}{100}$ where v is measured in miles per hour and $v > 0$. If Daniel wants to travel 50 miles as cheaply as possible, at what speed should he travel?

If we have the cost per hour and we want the total cost we must find the total time. But the time taken is $\frac{50}{v}$ hours, so the total cost is $C = \frac{50}{v}\left(12 + \frac{v^2}{100}\right) = \frac{600}{v} + \frac{v}{2}$.

If we wish to find a minimum value of C by changing v we can do this by setting $\frac{dC}{dv} = 0$:

$$-\frac{600}{v^2} + \frac{1}{2} = 0$$
$$\Rightarrow v^2 = 1200$$

$v = 34.6\,\text{mph}$ (3SF) (Taking the positive root since $v > 0$)

To see if we have found a minimum we find $\frac{d^2C}{dv^2} = 1200v^{-3}$ which is positive for any positive v, so the point is a local minimum.

Next, to see if it is global minimum we must consider the end points. Although v is never actually zero as it gets close to it, the $\frac{600}{v}$ term gets very large. When v gets very large the $\frac{v}{2}$ term gets very large. Therefore we have found the global minimum.

Mixed examination practice 16

Short questions

1. Find the equation of the tangent to the curve $y = e^x + 2\sin x$ at the point where $x = \frac{\pi}{2}$. *[5 marks]*

2. Find the equation of the normal to the curve $y = (x-2)^3$ when $x = 2$. *[5 marks]*

3. $f(x)$ is a quadratic function taking the form $x^2 + bx + c$. If $f(1) = 2$ and $f'(2) = 12$ find the values of b and c. *[5 marks]*

4. Find the coordinates of the point of inflexion on the graph of $y = \frac{x^3}{6} - x^2 + x$. *[6 marks]*

5. Find and classify the stationary points on the curve $y = \tan x - \frac{4x}{3}$. *[6 marks]*

6. Let f be a cubic polynomial function. Given that $f(0) = 2, f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$. *[2 marks]*

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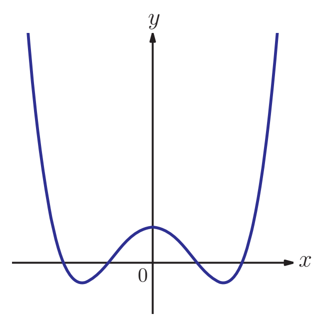
7. The graph shows $y = f'(x)$:

On a sketch of this graph:

(a) Mark points corresponding to a local minimum of $f(x)$ with an A.

(b) Mark points corresponding to a local maximum of $f(x)$ with a B.

(c) Mark points corresponding to a point of inflexion of $f(x)$ with a C. *[6 marks]*



8. On the curve $y = x^3$ a tangent is drawn from the point (a, a^3), $a > 0$ and a normal is drawn from the point $(-a, -a^3)$. The tangent and the normal meet on the y-axis. Find the value of a. *[6 marks]*

Long questions

1. The line $y = 24(x-1)$ is tangent to the curve $y = ax^3 + bx^2 + 4$ at $x = 2$.

(a) Use the fact that the tangent meets the curve to show that $2a + b = 5$.

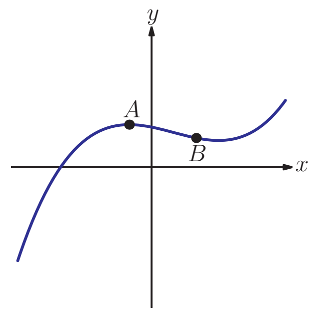
(b) Use the fact that the tangent has the same gradient as the curve to find another relationship between a and b.

(c) Hence find the values of a and b.

(d) The line meets the curve again. Find the coordinates of the other point of intersection. *[12 marks]*

2. The graph shows part of $y = x^3 - x^2 - x + 3$.

The point A is a local maximum and the point B is a point of inflexion.



(a) (i) Find the coordinates of A.

(ii) Find the coordinates of B.

(b) (i) Find the equation of the line containing both A and B.

(ii) Find the x coordinate of the points on the curve at which the tangent is parallel to this line. *[10 marks]*

3. (a) Sketch and label the curves $y = x^2$ for $-2 \le x \le 2$, and $y = -\frac{1}{2}\ln x$ for $0 < x \le 2$.

(b) Find the x-coordinate of P, the point of intersection of the two curves.

(c) If the tangents to the curves at P meet the y-axis at Q and R, calculate the area of the triangle PQR.

(d) Prove that the two tangents at the points where $x = a, a > 0$, on each curve are always perpendicular.

[14 marks]

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4. The population of bacteria (P) in thousands at a time t in hours is modelled by:

$$P = 10 + e^t - 3t,\ t \ge 0$$

(a) (i) Find the initial population of bacteria.

(ii) At what time does the number of bacteria reach 14 million?

(b) (i) Find $\frac{dP}{dt}$.

(ii) Find the time at which the bacteria are growing at a rate of 6 million per hour.

(c) (i) Find $\frac{d^2P}{dt^2}$ and explain the physical significance of this quantity.

(ii) Find the minimum number of bacteria, justifying that it is a minimum.

[12 marks]