Mixed examination practice 18

Short questions

1. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = x^2 \arcsin x$

(b) $xe^y = 4y^2$ *[7 marks]*

2. Differentiate $f(x) = \arccos(1 - x^2)$. *[4 marks]*

3. Find the exact value of the gradient of the curve with equation $y = \frac{1}{4 - x^2}$ when $x = \frac{1}{2}$. *[5 marks]*

4. Find the equation of the normal to the curve with equation $4x^2 + xy^2 - 3y^3 = 56$ at the point (–5, 2). *[7 marks]*

5. Given that $y = \arctan(x^2)$ find $\frac{d^2y}{dx^2}$. *[5 marks]*

6. Find the gradient of the curve with equation $4\sin x \cos y + \sec^2 y = 5$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. *[6 marks]*

7. The graph of $y = xe^{-kx}$ has a stationary point when $x = \frac{2}{5}$. Find the value of k. *[4 marks]*

8. A curve has equation $f(x) = \frac{a}{b + e^{-cx}}, a \neq 0, b, c > 0$.

(a) Show that $f''(x) = \frac{ac^2e^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3}$.

(b) Find the coordinates of the point on the curve where $f''(x) = 0$.

(c) Show that this is a point of inflexion. *[8 marks]*

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9. Find the coordinates of stationary points on the curve with equation $(y - 2)^2 e^x = 4x$. *[7 marks]*

Long questions

1. A curve has equation $y = \dfrac{x^2}{1-2x}$.

(a) Write down the equation of the vertical asymptote of the curve.

(b) Use differentiation to find the coordinates of stationary points on the curve.

(c) Determine the nature of the stationary points.

(d) Sketch the graph of $y = \dfrac{x^2}{1-2x}$. *[15 marks]*

2. The function f is defined by $f(x) = \dfrac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that $f'(x) = \dfrac{2x - x^2 \ln 2}{2^x}$.

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(b) (i) Find the exact value of x satisfying the equation $f'(x) = 0$.

(ii) Show that this value gives a maximum value for $f(x)$.

(c) Find the x-coordinates of the two points of inflexion on the graph of f.

[12 marks]

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3. Let $f(x) = \arccos(\sqrt{1-9x^2})$ for $0 < x < \dfrac{1}{3}$.

(a) Show that $f'(x) = \dfrac{3}{\sqrt{1-9x^2}}$.

(b) Show that $f''(x) > 0$ for all $x \in \left]0, \dfrac{1}{3}\right[$.

(c) Let $g(x) = \arccos(kx)$. If $g'(x) = -pf'(x)$ for $0 < x < \dfrac{1}{3}$, find the values of p and k. *[12 marks]*

4. A curve is given by the implicit equation $x^2 - xy + y^2 = 12$.

(a) Find the coordinates of the stationary points on the curve.

(b) Show that at the stationary points, $(x - 2y)\dfrac{d^2y}{dx^2} = 2$.

(c) Hence determine the nature of the stationary points. *[16 marks]*

5. If $f(x) = \sec x$, $0 \le x \le \pi$ the inverse function is $f^{-1}(x) = \text{arcsec}\, x$.

(a) Write down the domain of $\text{arcsec}\, x$.

(b) Sketch the graph of $y = \text{arcsec}\, x$.

(c) Show that the derivative of $\sec x$ is $\sec x \tan x$.

(d) Find the derivative of $\text{arcsec}\, x$ with respect to x, justifying carefully the sign of your answer. *[12 marks]*

20 Further applications of calculus

Introductory problem

A forest fire spreads in a circle at the speed of 12 km/h. How fast is the area affected by the fire increasing when its radius is 68 km?

In this chapter you will learn:

- to write real world problems as equations involving variables and their derivatives
- how to relate different rates of change
- to apply calculus to problems involving motion (kinematics)
- to find volumes of shapes rotated around an axis
- to maximise or minimise functions with constraints.

Did you know that if you are in a sealed box you cannot measure your velocity but you can measure your acceleration? Or that Newton's second law says that force is the rate of change of momentum? These are two examples where a rate of change is easier to find than the underlying variable. To get from this rate of change to the underlying variable requires the use of integration. This chapter will look at various applications of the calculus you have met in the previous four chapters, with a particular emphasis on real-world applications of rates of change.

20A Related rates of change

When blowing up a balloon we can control the amount of gas in the balloon (V), but we may want to know how fast the radius (r) is increasing. These are two different rates of change, but they are linked – the faster the gas fills the balloon the faster the radius will increase. We need to link two derivatives: $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$. This is done by using the chain rule and the geometric context.

Worked example 20.1

A spherical balloon is being inflated with air at a rate of 200 cm^3 per minute. At what rate is the radius increasing when the radius is 8 cm?

Define variables

V = volume of air in balloon in cm^3

r = radius of balloon in cm

t = time in minutes

Write the given rate of change and the required rate of change

$$\frac{dV}{dt} = 200$$

$$\frac{dr}{dt} = ?$$

Relate these rates of change using the chain rule

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

So we need to find $\frac{dV}{dr}$

Use geometric context

Since the balloon is spherical, $V = \frac{4}{3}\pi r^3$,

$$\Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$$

Put into the chain rule

$$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 200, \ r = 8$$

$$\therefore 200 = 256\pi \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = 0.249 \text{ (3SF)}$$

So radius is increasing at about 0.249 cm/minute

EXAM HINT

Don't use units in the working, as long as the units in the information are consistent. Always give units with your final answer.

The rate required may be linked to several other variables.

Worked example 20.2

As a conical icicle melts the rate of decrease of height h is 1 cm^{-1} and the rate of decrease of the radius of the base, r, is 0.1 cm h^{-1}. At what rate is the volume (V) of the icicle decreasing when the height is 30 cm and the base radius is 4 cm?

Write the given rates of change and the required rates of change
Remember that decrease means negative derivative

$$\frac{dh}{dt} = -1$$

$$\frac{dr}{dt} = -0.1$$

$$\frac{dV}{dt} = ?$$

Use geometry to link the variables

$$V = \frac{1}{3}\pi r^2 h$$

Differentiate both sides with respect to t, requiring the product rule and the chain rule

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2\right)h + \left(\frac{1}{3}\pi r^2\right)\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r\frac{dr}{dt}h + \frac{1}{3}\pi r^2\frac{dh}{dt}$$

$$r = 30,\ h = 4,\ \frac{dr}{dt} = -0.1,\ \frac{dh}{dt} = -1$$

Put in given values

$$\therefore \frac{dV}{dt} = \frac{2}{3}\pi \times 4 \times (-0.1) \times 30 + \frac{1}{3}\pi \times 4^2 \times (-1)$$

$$= -41.9 \text{ cm}^3\text{h}^{-1}$$

The volume is decreasing at 41.9 cm^3 per hour

Exercise 20A

1. In each case, find an expression for $\dfrac{dz}{dx}$ in terms of x.

(a) (i) $z = 4y^2,\ y = 3x^2$ (ii) $z = y^2, y = x^3 + 1$

(b) (i) $z = \cos y,\ y = 3x^2$ (ii) $z = \tan y,\ y = x^2 + 1$

2. (a) (i) Given that $z = y^2 + 1$ and $\dfrac{dy}{dx} = 5$, find $\dfrac{dz}{dx}$ when $y = 5$.

(ii) Given that $z = 2y^3$ and $\dfrac{dy}{dx} = -2$, find $\dfrac{dz}{dx}$ when $y = 1$.

(b) (i) If $w = \sin x$ and $\frac{dw}{dt} = -3$, find $\frac{dx}{dt}$ when $x = \frac{\pi}{3}$.

(ii) If $P = \tan h$ and $\frac{dP}{dx} = 2$, find $\frac{dh}{dx}$ when $h = \frac{\pi}{4}$.

(c) (i) Given that $V = 12r^3$, $\frac{dr}{dt} = 1$ and $\frac{dV}{dt} = 4$, find the possible values of r.

(ii) Given that $H = 3S^{-2}$, find the value of S for which $\frac{dH}{dx} = 3$ and $\frac{dS}{dx} = 4$.

3. (a) (i) Given that $V = 3r^2h$, find $\frac{dV}{dt}$ when $r = 3$, $h = 2$, $\frac{dr}{dt} = 2$ and $\frac{dh}{dt} = -1$.

(ii) Given that $N = kx^4$, find $\frac{dN}{dt}$ when $x = 2$, $k = 5$, $\frac{dk}{dt} = 1$ and $\frac{dx}{dt} = 1$.

(b) (i) Given that $m = \frac{S}{n}$ and that $S = 100$, $\frac{dS}{dt} = 20$, $n = 50$ and $\frac{dn}{dt} = 4$, find $\frac{dm}{dt}$.

(ii) Given that $\rho = \frac{m}{V}$ and that $m = 24$, $\frac{dm}{dt} = 2$, $V = 120$ and $\frac{dV}{dt} = 6$, find $\frac{d\rho}{dt}$.

4. A circular stain is spreading so that the radius is increasing at the constant rate of 1.5 cm s^{-1}. Find the rate of increase of the area when the radius is 12 cm. *[5 marks]*

5. The area of a square is increasing at the constant rate of 50 $cm^2\,s^{-1}$. Find the rate of increase of the side of the square when the length of the side is 12.5 cm. *[5 marks]*

6. The surface area of a closed cylinder is given by $A = 2\pi r^2 + 2\pi rh$, where h is the height and r is the radius of the base. At the time when the surface area is increasing at the rate of 20π cm^2 s^{-1} the radius is 4 cm, the height is 1 cm and is decreasing at the rate of 2 cm s^{-1}. Find the rate of change of radius at this time. *[6 marks]*

7. A spherical balloon is being inflated at a constant rate of 500 cm^3 s^{-1}. The radius at time t seconds is r cm.

Find the radius of the balloon at the time when it is increasing at the rate of 0.5 cm s^{-1}. *[6 marks]*

8. A ship is 5 km east and 7 km North of a lighthouse. It is moving North at a rate of 12 kmh^{-1} and East at a rate of 16 kmh^{-1}. At what rate is its distance from the lighthouse changing? *[7 marks]*

20B Kinematics

Kinematics is the study of movement – especially position, speed and acceleration. We first need to define some terms carefully:

Time is normally given the symbol t. We can normally define $t = 0$ at any convenient time.

In a 400 m race athletes run a single lap so, despite running 400 m they have returned to where they started. This distance is how much ground someone has covered, whilst the **displacement** is how far away they are from a particular position. The symbol s is normally used to represent displacement.

The rate of change of displacement with respect to time is called **velocity**, and it is normally given the symbol v.

KEY POINT 20.1

Velocity is given by: $v = \frac{ds}{dt}$.

Speed is the magnitude of the velocity: $|v|$.

In the IB you will only have to deal with motion in one dimension. However, motion is often in two or three dimensions. To deal with this requires a combination of vectors and calculus called (unsurprisingly) vector calculus.