4 State, with reasons, whether each pair of lines meet at

i only one point **ii** an infinite number of points

iii no point.

a $y = 3(x - 5)$ and $x - \frac{1}{3}y + 6 = 0$

b $\frac{y+1}{x-2} = -1$ and $y = -x + 1$

c $y = 4x - 8$ and $4x - 2y = 0$

d $x - y + 3 = 0$ and $3x - 3y + 9 = 0$

EXAM-STYLE QUESTION

5 Line L_1 has gradient 5 and intersects line L_2 at the point A(1, 0).

a Find the equation of L_1.

Line L_2 is perpendicular to L_1.

b Find the equation of L_2.

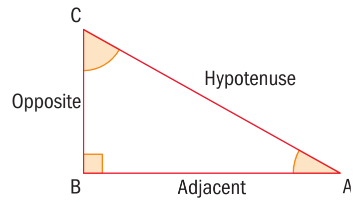
Point *A* lies on both lines.

3.3 The sine, cosine and tangent ratios

Trigonometry is the study of lengths and angles in triangles. This section looks at trigonometry in *right-angled* triangles.

In a **right-angled triangle** the side opposite the right angle is the **hypotenuse**, which is the **longest side**.

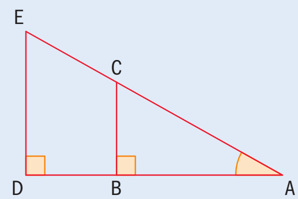
Some textbooks use 'right triangle' instead of right-angled triangle.



- AC is the hypotenuse
- AB is adjacent to angle A (Â)
- BC is opposite Â

Investigation – right-angled triangles

Draw a diagram of two triangles like this.



1 Measure the angles at E and C. What do you notice?

2 Measure the lengths AB and AD. Calculate the ratio $\frac{AD}{AB}$

3 Measure the lengths AE and AC. Calculate the ratio $\frac{AE}{AC}$

4 Measure the lengths DE and BC. Calculate the ratio $\frac{DE}{BC}$

What do you notice about your answers to questions 2 to 4?

In the diagram the right-angled triangles ABC and ADE have the same angles, and corresponding sides are in the same ratio.

The ratios $\frac{AB}{AC}$, $\frac{BC}{AC}$ and $\frac{BC}{AB}$ in triangle ABC are respectively equal to the ratios $\frac{AD}{AE}$, $\frac{DE}{AE}$ and $\frac{DE}{AD}$ in triangle ADE.

Two triangles with the same angles and corresponding sides in the same ratio are called **similar triangles**.

Therefore

$\frac{AB}{AC} = \frac{AD}{AE} = \frac{\text{Adjacent to } \hat{A}}{\text{Hypotenuse}}$

Note that both AB and AD are adjacent to Â, and AC and AE are the hypotenuses.

$\frac{BC}{AC} = \frac{DE}{AE} = \frac{\text{Opposite } \hat{A}}{\text{Hypotenuse}}$

Note that both BC and DE are opposite Â, and both AC and AE are the hypotenuses.

$\frac{BC}{AB} = \frac{DE}{AD} = \frac{\text{Opposite } \hat{A}}{\text{Adjacent to } \hat{A}}$

Note that both BC and DE are opposite Â, and both AB and AD are adjacent to Â.

Some textbooks call the two shorter sides of a right-angled triangle the ‘legs’ of the triangle.

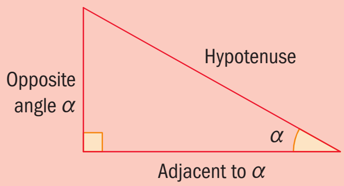
In any triangle **similar** to triangle ABC these ratios will remain the same.

→ Three trigonometric ratios in a right-angled triangle are defined as

$$\sin \alpha = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\cos \alpha = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\tan \alpha = \frac{\text{Opposite side}}{\text{Adjacent side}}$$



α is the Greek letter ‘alpha’.

- ‘sin α’ is read ‘sine of α’
- ‘cos α’ is read ‘cosine of α’
- ‘tan α’ is read ‘tangent of α’

You can use the acronym **SOHCAHTOA** to help you remember which ratio is which.

SOH as **S**in $\alpha = \frac{\mathbf{O}}{\mathbf{H}}$

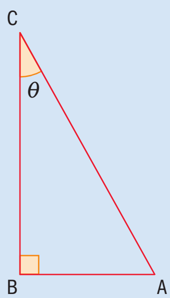
CAH as **C**os $\alpha = \frac{\mathbf{A}}{\mathbf{H}}$

TOA as **T**an $\alpha = \frac{\mathbf{O}}{\mathbf{A}}$

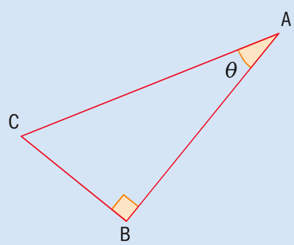
Example 9

For each triangle, write down the three trigonometric ratios for the angle θ in terms of the sides of the triangle.

a



b



Answers

a $\sin \theta = \frac{AB}{AC}$, $\cos \theta = \frac{BC}{AC}$, $\tan \theta = \frac{AB}{BC}$

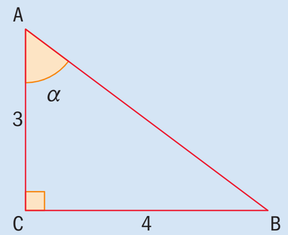
b $\sin \theta = \frac{BC}{AC}$, $\cos \theta = \frac{AB}{AC}$, $\tan \theta = \frac{BC}{AB}$

Example 10

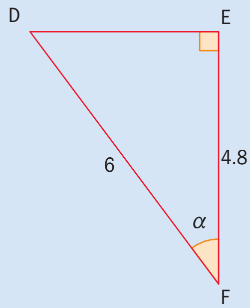
For each of these right-angled triangles find the value of

i $\sin \alpha$ **ii** $\cos \alpha$ **iii** $\tan \alpha$.

a



b



Answers

a $AB^2 = 3^2 + 4^2$

$AB = 5$

First find the hypotenuse. Use Pythagoras.

Now

i $\sin \alpha = \frac{BC}{AB}$ — *$\sin \alpha = \frac{opp}{hyp}$*

$\sin \alpha = \frac{4}{5}$

ii $\cos \alpha = \frac{AC}{AB}$ — *$\cos \alpha = \frac{adj}{hyp}$*

$\cos \alpha = \frac{3}{5}$

iii $\tan \alpha = \frac{BC}{AC}$ — *$\tan \alpha = \frac{opp}{adj}$*

$\tan \alpha = \frac{4}{3}$

b $DE^2 + 4.8^2 = 6^2$

First find DE.

$DE = 3.6$

i $\sin \alpha = \frac{DE}{DF} = \frac{3.6}{6}$

$\sin \alpha = 0.6$

ii $\cos \alpha = \frac{EF}{DF} = \frac{4.8}{6}$

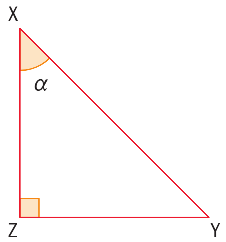
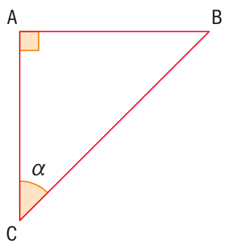
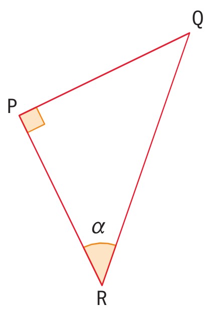
$\cos \alpha = 0.8$

iii $\tan \alpha = \frac{DE}{EF} = \frac{3.6}{4.8}$

$\tan \alpha = 0.75$

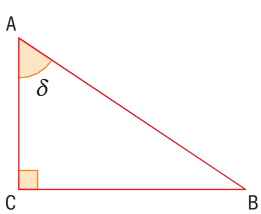
Exercise 3H

1 Copy and complete this table.

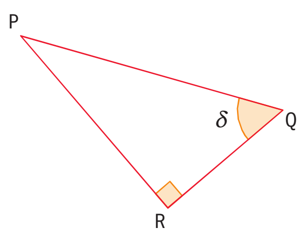
Triangle	Hypotenuse	Side opposite α	Side adjacent to α
			
			
			

2 Write down the three trigonometric ratios for the angle δ in terms of the sides of the triangle.

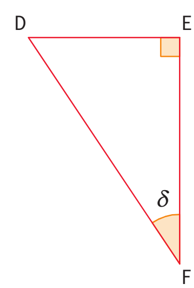
a



b

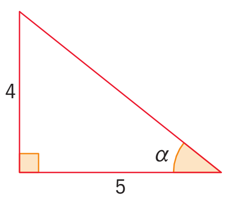


c

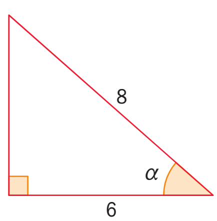


3 In each of these right-angled triangles all lengths are in cm.

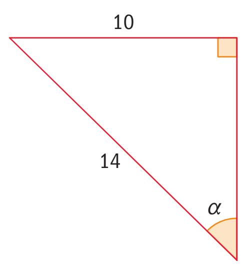
a



b



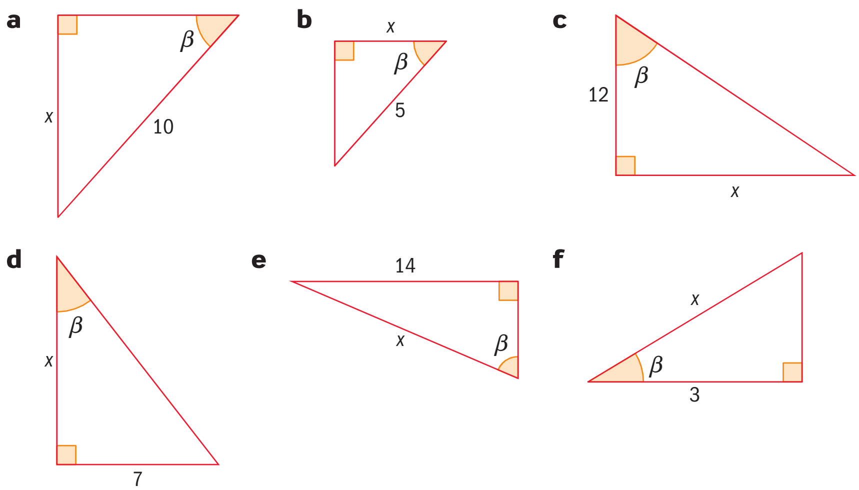
c



Find the exact value of

i $\sin\alpha$ **ii** $\cos\alpha$ **iii** $\tan\alpha$.

4 For each triangle write down a trigonometric equation to link angle β and the side marked x.



Finding the sides of a right-angled triangle

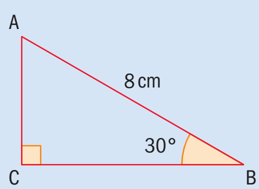
If you know the size of one of the acute angles and the length of one side in a right-angled triangle you can find

- the lengths of the other sides using trigonometric ratios
- the third angle using the sum of the interior angles of a triangle.

Label the sides opposite, adjacent and hypotenuse so you can identify which ones you know.

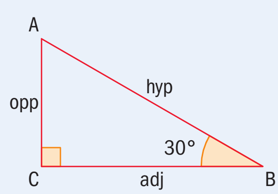
Example 11

Find the length of the unknown sides in triangle ABC.
Give your answer to 3sf.



Answer

To find BC:

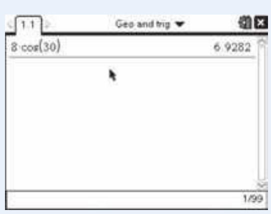


$\cos 30° = \frac{BC}{8}$

$BC = 8\cos 30°$

$BC = 6.93$ cm (to 3 sf)

Cosine links the unknown side BC (adjacent to the 30° angle) and the known side AB (the hypotenuse).
Use the GDC to solve for BC.



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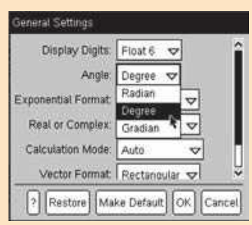
Remember to set your GDC in **degrees**. To change to **degree mode** press On and choose 5:Settings & Status | 2:Settings | 1:General

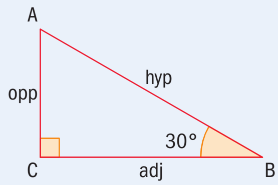


Use the tab key to move to Angle and select Degree. Press enter and then select 4:Current to return to the document.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

To find AC:

Method 1



$\sin 30° = \frac{AC}{8}$

Sine links the known and the unknown sides.

$AC = 8\sin 30°$

$= 4\,\text{cm}$

Solve for AC. Use the GDC:

Method 2

$AC^2 + BC^2 = AB^2$

$AC^2 + (8\cos 30°)^2 = 8^2$

$AC = \sqrt{8^2 - (8\cos 30°)^2}$

$= 4\,\text{cm}$

Use Pythagoras as you already know two sides of the triangle.

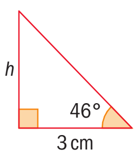
Solve for AC. Use the GDC:

You could also use tangent as you know the angle and the adjacent.

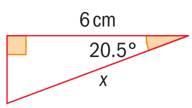
Exercise 3I

Find the lengths of the sides marked with letters. Give your answers correct to two decimal places.

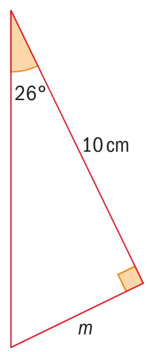
1



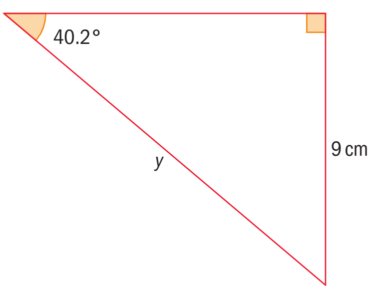
2



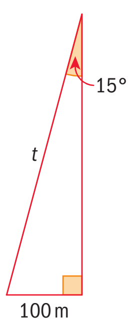
3



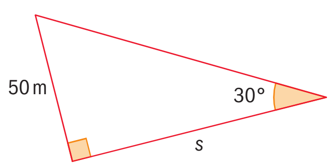
4



5



6



Example 12

In triangle DEF, $\hat{E} = 90°$, $\hat{F} = 50°$ and DE = 7 m

a Represent this information in a clear **labeled** diagram.

b Find the size of $\hat{D}$.

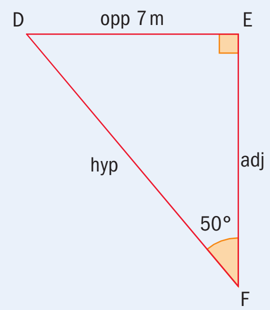
c Find EF.

d Find DF.

Give your answers to 3 sf.

Answers

a



Draw a diagram. Label the triangle in alphabetical order clockwise.

b $\hat{D} + 90° + 50° = 180°$

$\hat{D} = 40°$

The sum of the interior angles of a triangle is 180°.

c $\tan 50° = \frac{7}{EF}$

$EF = \frac{7}{\tan 50°}$

$= 5.87\text{ m}$

Tangent links the known and the unknown sides.
Use the GDC to solve for EF.

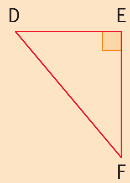
d $\sin 50° = \frac{7}{DF}$

$DF = \frac{7}{\sin 50°} = 9.14\text{ m}$

Sine links the known and the unknown sides.

Use the GDC to solve for DF.

The astronomer Aryabhata, born in India is about 476 CE, believed that the Sun, planets and stars circled the Earth in different orbits. He began to invent trigonometry in order to calculate the distances from planets to the Earth.



$\hat{D}$ can also be described as $E\hat{D}F$ or ∠FDE. Make sure you understand all these notations.

Exercise 3J

1 In triangle PQR, $\hat{R} = 90°$, $\hat{P} = 21°$, PR = 15 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{Q}$.

c Find QR.

2 In triangle STU, $\hat{T} = 90°$, $\hat{U} = 55°$, SU = 35 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{S}$.

c Find TU.

3 In triangle ZWV, $\hat{V} = 90°$, $\hat{W} = 15°$, WV = 30 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{Z}$.

c Find VZ.

Label the triangle in alphabetical order clockwise.

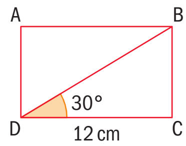
EXAM-STYLE QUESTIONS

4 In triangle LMN, $\hat{N} = 90°$, $\hat{L} = 33°$, LN = 58 cm.

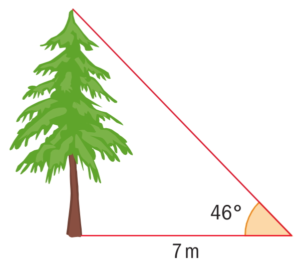
- **a** Represent this information in a clear and **labeled** diagram.
- **b** Write down the size of $\hat{M}$.
- **c** Find LM.

5 In rectangle ABCD, DC = 12 cm and the diagonal BD makes an angle of 30° with DC.

- **a** Find the length of BC.
- **b** Find the perimeter of the rectangle ABCD.
- **c** Find the area of the rectangle ABCD.



6 When the sun makes an angle of 46° with the horizon a tree casts a shadow of 7 m.
Find the height of the tree.



7 A ladder 7 metres long leans against a wall, touching a window sill, and makes an angle of 50° with the ground.

- **a** Represent this information in a clear and **labeled** diagram.
- **b** Find the height of the window sill above the ground.
- **c** Find how far the foot of the ladder is from the foot of the wall.

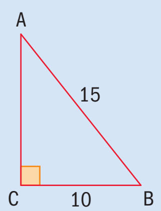
Finding the angles of a right-angled triangle

If you know the lengths of two sides in a right-angled triangle, you can find

- the length of the other side by using Pythagoras
- the size of the two acute angles by using the appropriate trigonometric ratios.

Example 13

Find the sizes of the two acute angles in this triangle.



Answer

Angle $\hat{B}$

$\cos\hat{B} = \frac{10}{15}$

$\hat{B} = \cos^{-1}\left(\frac{10}{15}\right)$

Cosine links adjacent and hypotenuse.

'$\cos^{-1}\left(\frac{10}{15}\right)$' means 'the angle with a cosine of $\frac{10}{15}$'.

$\cos^{-1}\left(\frac{10}{15}\right)$ is read 'inverse cosine of $\frac{10}{15}$'.

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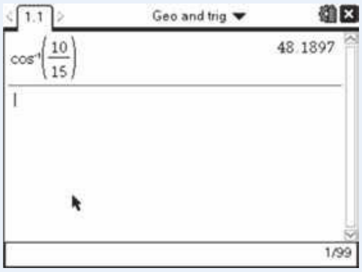
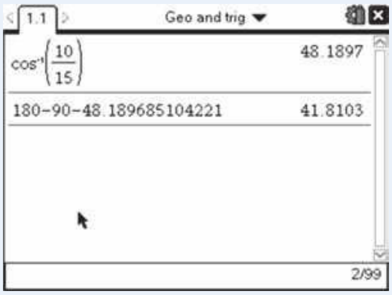
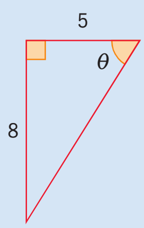
Therefore $\hat{B} = 48.2°$	*Use the GDC.* 
Angle $\hat{A}$ $90° + \hat{B} + \hat{A} = 180°$ $90° + 48.18\ldots + \hat{A} = 180°$ $\hat{A} = 41.8°$	*Use the angle sum of a triangle.* *Using the GDC:* 

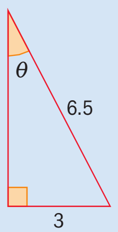
Example 14

Find the angle marked θ in each triangle.
Give your answers correct to the nearest degree.

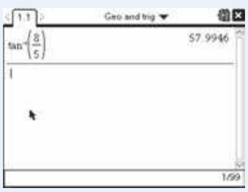
a



b



Answers

a $\tan\theta = \frac{8}{5}$	*Use tangent; it links the adjacent and the opposite.*
$\theta = \tan^{-1}\left(\frac{8}{5}\right)$	'$\theta = \tan^{-1}\left(\frac{8}{5}\right)$' *means 'the angle with a tangent of* $\frac{8}{5}$'.
$\theta = 58°$	*Use the GDC:* 

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b $\sin\theta = \dfrac{3}{6.5}$

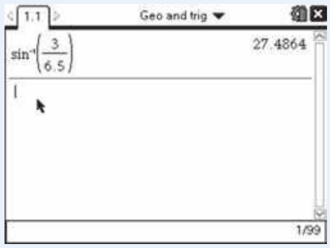
$\theta = \sin^{-1}\left(\dfrac{3}{6.5}\right)$

$\theta = 27°$

Use sine; it links the opposite and the hypotenuse.

'$\theta = \sin^{-1}\left(\dfrac{3}{6.5}\right)$' means 'the angle with a sine of $\dfrac{3}{6.5}$'.

Use the GDC:



Exercise 3K

Give your answers correct to 3 sf.

1 Explain the meaning of

a $\sin^{-1}(0.6)$ **b** $\tan^{-1}\left(\dfrac{1}{2}\right)$ **c** $\cos^{-1}\left(\dfrac{2}{3}\right)$.

2 Calculate

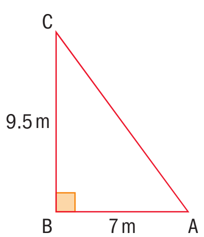
a $\sin^{-1}(0.6)$ **b** $\tan^{-1}\left(\dfrac{1}{2}\right)$ **c** $\cos^{-1}\left(\dfrac{2}{3}\right)$.

3 Find the **acute** angle α if

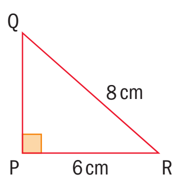
a $\sin\alpha = 0.2$ **b** $\cos\alpha = \dfrac{2}{3}$ **c** $\tan\alpha = 1$.

4 Find the sizes of the two acute angles in these triangles.

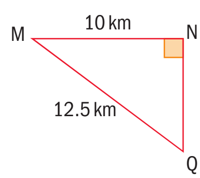
a



b



c



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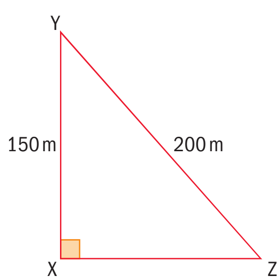

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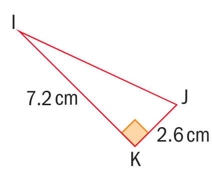

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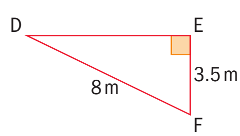


5 In triangle BCD, $\hat{D} = 90°$, BD = 54 cm, DC = 42 cm.

a Represent this information in a clear and **labeled** diagram.

b Find the size of $\hat{C}$.

6 In triangle EFG, $\hat{G} = 90°$, FG = 56 m, EF = 82 m.

a Represent this information in a clear and **labeled** diagram.

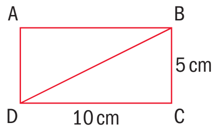
b Find the size of $\hat{F}$.

7 In triangle HIJ, $\hat{J} = 90°$, IJ = 18 m, HI = 25 m.

a Represent this information in a clear and **labeled** diagram.

b Find the size of $\hat{H}$.

8 In rectangle ABCD, BC = 5 cm and DC = 10 cm.



Find the size of the angle that the diagonal BD makes with the side DC.

9 The length and width of a rectangle are 20 cm and 13 cm respectively.
Find the angle between a diagonal and the shorter side of the rectangle.

10 A ladder 8 m long leans against a vertical wall.
The base of the ladder is 3 m away from the wall.
Calculate the angle between the wall and the ladder.

EXAM-STYLE QUESTIONS

11 a On a pair of Cartesian axes plot the points A(3, 0) and B(0, 4). Use the same scale on both axes.

b Draw the line AB.

c Find the size of the **acute** angle that the line AB makes with the x-axis.

12 a On a pair of Cartesian axes plot the points A(–1, 0) and B(1, 4). Use the same scale on both axes.

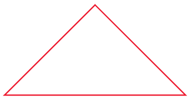
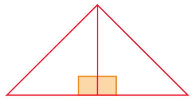
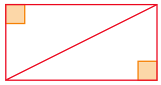
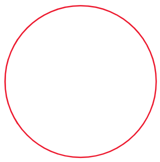
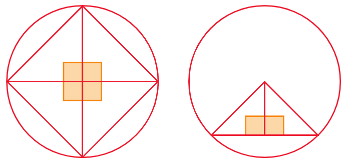
b Draw the line AB.

c Find the size of the **acute** angle that the line AB makes with the x-axis.

Finding right-angled triangles in other shapes

So far you have found unknown sides and angles in right-angled triangles. Next you will learn how to find unknown sides and angles in triangles that are not right-angled and in shapes such as rectangles, rhombuses and trapeziums.

The technique is to break down the shapes into smaller ones that contain right-angled triangles.

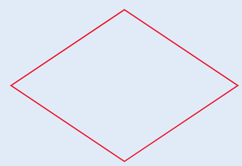
Name of shape	Shape	Where are the right-angled triangles?
Isosceles or equilateral triangles		
Rectangles or squares		
Circle		

Investigation – 2-D shapes

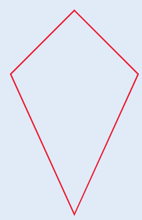
How can you break these shapes into smaller shapes so that at least one of them is a right-angled triangle?

To do this you need to know the properties of 2-D shapes.

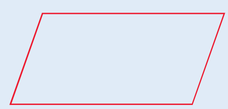
1 *Rhombus*
What is the property of the diagonals of a rhombus?
Make an accurate drawing of a rhombus on squared paper.
Draw the diagonals. How many right-angled triangles do you obtain? Are they congruent? Why?
Comment on your findings.



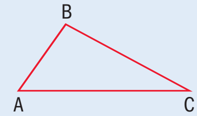
2 *Kite*
What is the property of the diagonals of a kite?
Make an accurate drawing of a kite on squared paper.
Draw the diagonals. How many right-angled triangles do you obtain?
Are they congruent? Why? Comment on your findings.



3 *Parallelogram*
Draw a parallelogram like this one on squared paper.
There is a rectangle that has the same base and height as this parallelogram. Draw dotted lines where you would cut the parallelogram and rearrange it to make a rectangle.
How many shapes do you obtain? How many of them are right-angled triangles? Comment on your findings.



4 *Triangle*
Draw a triangle like this one.
Every triangle has three heights, one for each base (or side).
Draw the height relative to AC (this is the line segment drawn from B to AC and perpendicular to AC). You will get two right-angled triangles that make up the triangle ABC. Under what conditions would these triangles be congruent? Comment on your findings.



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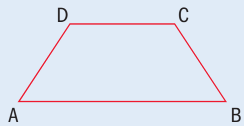
5 *Trapezium*

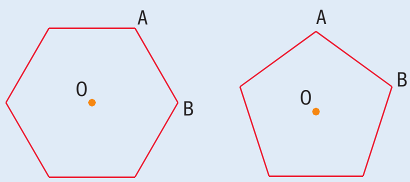
Draw a trapezium like this one.



Draw a line from D perpendicular to AB and a line from C perpendicular to AB. You will get two right-angled triangles. What is the condition for these triangles to be congruent?

6 *Regular polygon*

Here are a regular hexagon and a regular pentagon.



O is the center of each polygon.

For **each** polygon:

What type of triangle is ABO? Why? Draw a line from O perpendicular to the side AB to form two right-angled triangles. These two triangles are congruent.

Explain why.

A **regular polygon** has all sides equal lengths and all angles equal.

Example 15

Triangle ABC is isosceles. The two equal sides AB and BC are 10 cm long and each makes an angle of 40° with AC.

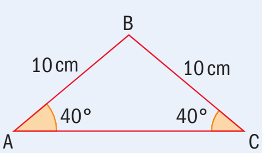
a Represent this information in a clear and labeled diagram.

b Find the length of AC.

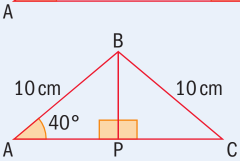
c Find the perimeter of triangle ABC.

Answers

a



b



In an isosceles triangle the perpendicular from the apex to the base ***bisects*** *the base, making two right-angled triangles.*

$$\cos 40° = \frac{AP}{10}$$

$\cos = \frac{adj}{hyp}$

AP = 10 cos 40°

Make AP the subject of the equation.

AC = 2 × 10 cos 40°

AC = 15.3 cm

Use the fact that AC = 2 × AP.

c Perimeter = AB + BC + CA

= 15.32 … + 2 × 10

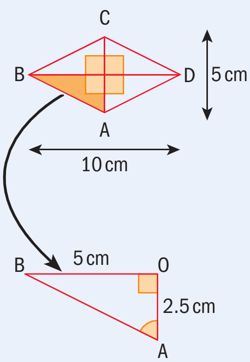
= 35.3 cm (to 3 sf)

Bisect means ‘cut in half’.

Example 16

The diagonals of a rhombus are 10 cm and 5 cm. Find the size of the **larger** angle of the rhombus.

Answer



Draw a diagram, showing the diagonals.

Let O be the point where the diagonals meet.

$\tan \text{angle OAB} = \dfrac{5}{2.5}$

In triangle ABO, angle OAB is greater than angle OBA (it is opposite the larger side). So find angle OAB.

$\text{Angle OAB} = \tan^{-1}\left(\dfrac{5}{2.5}\right)$

$\tan = \dfrac{opp}{adj}$

$\text{Angle BAD} = 2 \times \text{OAB}$

$= 2 \times \tan^{-1}\left(\dfrac{5}{2.5}\right)$

Angle BAD (or BCD) is the larger angle of the rhombus.

Angle BAD = 127° (to 3 sf)

The diagonals of the rhombus bisect each other at right angles.

'Angle OAB' and OÂB are alternative notation for Â.

Investigation – rhombus

1 Use a ruler and a pair of compasses to construct a rhombus with a side length of 6 cm.

2 Construct another rhombus with a side length of 6 cm that is not congruent to the one you drew in **1**.

3 How many different rhombuses with a side length of 6 cm could you construct? In what ways do they differ?

Exercise 3L

1 Triangle ABC is isosceles. The two equal sides AC and BC are 7 cm long and they each make an angle of 65° with AB.

- **a** Represent this information in a clear and labeled diagram.
- **b** Find the length of AB.
- **c** Find the perimeter of triangle ABC (give your answer correct to the nearest centimetre).

2 The diagonals of a rhombus are 12 cm and 7 cm. Find the size of the **smaller** angle of the rhombus.

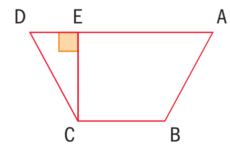
3 The size of the larger angle of a rhombus is 120° and the longer diagonal is 7 cm.

a Represent this information in a clear and labeled diagram.

b Find the length of the shorter diagonal.

EXAM-STYLE QUESTIONS

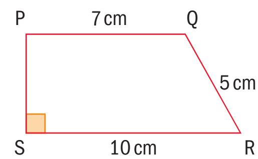
4 In the diagram ABCD is a trapezium where AD || BC, CD = BA = 6 m, BC = 12 m and DA = 16 m



a Show that DE = 2 m.

b Find the size of $\hat{D}$.

5 In the diagram PQRS is a trapezium, PQ || SR, PQ = 7 cm, RS = 10 cm, QR = 5 cm and $\hat{S} = 90°$



a Find the height, PS, of the trapezium.

b Find the area of the trapezium.

c Find the size of angle SRQ.

6 The length of the shorter side of a rectangular park is 400 m. The park has a straight path 600 m long joining two opposite corners.

a Represent this information in a clear and labeled diagram.

b Find the size of the angle that the path makes with the longer side of the park.

7 a On a pair of Cartesian axes, plot the points A(3, 2), C(–1, –4), and D(–1, 2). Use the same scale on both axes.
B is a point such that ABCD is a rectangle.

b i Plot B on your diagram.

ii Write down the coordinates of the point B.

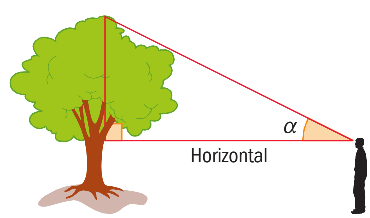
c Write down the length of

i AB ii BC.

d Hence find the size of the angle that a diagonal of the rectangle makes with one of the shorter sides.

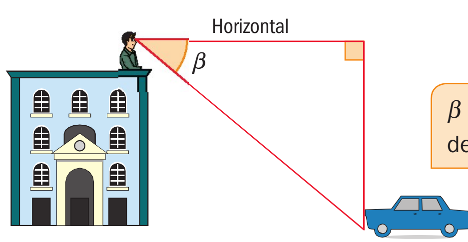
Angles of elevation and depression

→ The **angle of elevation** is the angle you lift your eyes through to look at something above.



α is the angle of elevation.

→ The **angle of depression** is the angle you lower your eyes through to look at something below.



β is the angle of depression.

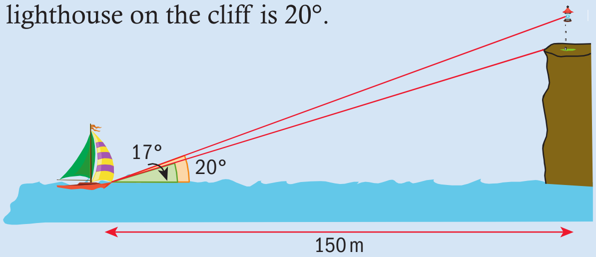
Notice that both the angle of elevation and the angle of depression are measured from the **horizontal**.

Example 17

From a yacht, 150 metres out at sea, the angle of elevation of the top of a cliff is 17°. The angle of elevation to the top of a lighthouse on the cliff is 20°. This information is shown in the diagram.

a Find the height of the cliff.

b Hence find the height of the lighthouse.



Answers

a Let x be the height of the cliff

$\tan 17° = \frac{x}{150}$

$x = 45.9$ m (to 3 sf)

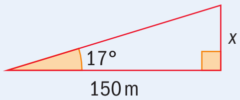
b Let y be the distance from the top of the lighthouse to the sea.

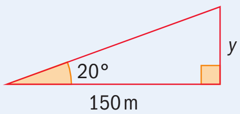
$\tan 20° = \frac{y}{150}$

$y = 54.5955\ldots$m

height of the lighthouse $= y - x$

$= 8.74$ m (to 3 sf)





Use the unrounded value of x to find y − x.

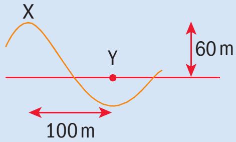
Example 18

A boy standing on a hill at X can see a boat on a lake at Y as shown in the diagram. The vertical distance from X to Y is 60 m and the horizontal distance is 100 m.

Find:

a the shortest distance between the boy and the boat

b the angle of depression of the boat from the boy.



Answers

a $XY^2 = 100^2 + 60^2$

$XY = 117$ m (to 3 sf)

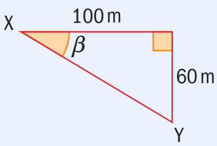
b $\tan \beta = \frac{60}{100}$

The angle of depression

$= 31.0°$ (to 3 sf)

Use Pythagoras.

Use $\tan = \frac{opp}{adj}$



The **shortest distance** is the length *XY*.

Exercise 3M

> Draw a diagram for each question.

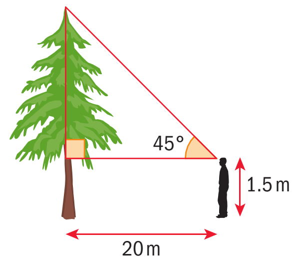
1 Find the angle of elevation of the top of a tree 13 m high from a point 25 m away on level ground.

2 A church spire 81 metres high casts a shadow 63 metres long. Find the angle of elevation of the sun.

3 The angle of depression from the top of a cliff to a ship at sea is 14°. The ship is 500 metres from shore. Find the height of the cliff.

4 Find the angle of depression from the top of a cliff 145 metres high to a ship at sea 1.2 kilometres from the shore.

5 A man whose eye is 1.5 metres above ground level stands 20 metres from the base of a tree. The angle of elevation to the top of the tree is 45°. Calculate the height of the tree.



6 The height of a tree is 61.7 metres and the angle of elevation to the top of the tree from ground level is 62.4°. Calculate the distance from the tree to the point at which the angle was measured.

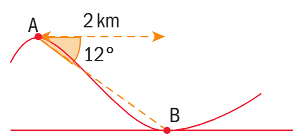
EXAM-STYLE QUESTION

7 The angle of depression of town B from town A is 12°.

a Find the angle of elevation of town A from town B.

The horizontal distance between the towns is 2 km.

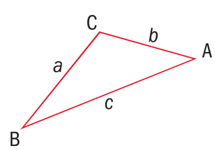
b Find the vertical distance between the towns. Give your answer correct to the nearest metre.



3.4 The sine and cosine rules

The sine and cosine rules are formulae that will help you to find unknown sides and angles in a triangle. They enable you to use trigonomentry in triangles that are **not** right-angled.

The formula and notation are simpler if you label triangles like this.



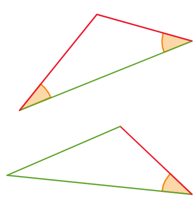
> - The side opposite $\hat{A}$ is a.
> - The side opposite $\hat{B}$ is b.
> - The side opposite $\hat{C}$ is c.
>
> Also notice that
> - $\hat{A}$ is between sides b and c.
> - $\hat{B}$ is between sides a and c.
> - $\hat{C}$ is between sides a and b.

The sine rule

If you have this information about a triangle:

- two angles and one side, or
- two sides and a non-included angle,

then you can find the other sides and angles of the triangle.

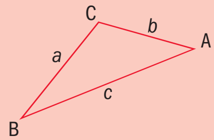


Sine rule

In any triangle ABC with angles $\hat{A}$, $\hat{B}$ and $\hat{C}$, and opposite sides a, b and c respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The sine rule is in the Formula booklet.

Example 19

In triangle ABC, $b = 16$ cm, $c = 10$ cm and $\hat{B} = 135°$.

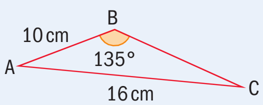
a Represent the given information in a labeled diagram.

b Find the size of angle C.

c Hence find the size of angle A.

Answers

a



b $\frac{16}{\sin 135°} = \frac{10}{\sin \hat{C}}$ — *Substitute in the sine rule.*

$16 \sin \hat{C} = 10 \sin 135°$ — *Cross multiply.*

$\sin \hat{C} = \frac{10 \sin 135°}{16}$ — *Make sin $\hat{C}$ the subject of the formula.*

$\hat{C} = 26.2°$ (to 3 sf) — *Use your GDC.*

c $\hat{A} + \hat{B} + \hat{C} = 180°$

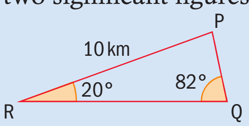
$\hat{A} + 135° + 26.227\ldots = 180°$ — *Use your GDC.*

$\hat{A} = 18.8°$ (to 3 sf)

Cross multiply.
$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

Example 20

In triangle PQR, find the length of RQ. Give your answer correct to two significant figures.



Answer

$\hat{P} = 78°$ — *RQ is opposite angle P so first find the size of angle P.*

$\frac{RQ}{\sin 78°} = \frac{10}{\sin 82°}$ — *Substitute in the sine rule.*

$RQ = \frac{10 \sin 78°}{\sin 82°}$ — *Make RQ the subject of the formula.*

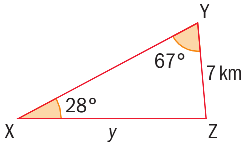
$= 9.9$ km (to 2 sf) — *Use your GDC.*

Ptolemy (c. 90–168 CE), in his 13-volume work *Almagest*, wrote sine values for angles form 0° to 90°. He also included theorems similar to the sine rule.

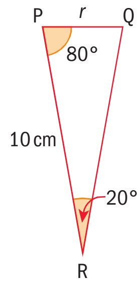
Exercise 3N

1 Find the sides marked with letters.

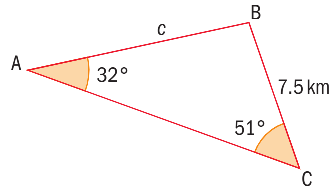
a



b



c



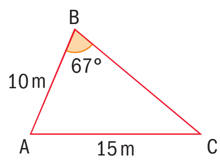
2 In triangle ABC, AC = 12 cm, $\hat{A} = 30°$ and $\hat{B} = 46°$.
Find the length of BC.

3 In triangle ABC, $\hat{A} = 15°$, $\hat{B} = 63°$ and AB = 10 cm. Find the length of BC.

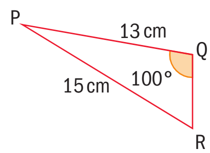
4 In triangle PQR, PR = 15 km, $\hat{P} = 25°$ and $\hat{Q} = 60°$. Find the length of QR.

5 In each triangle, find the angle indicated.

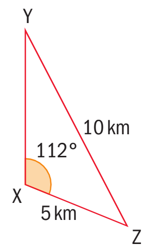
a $\hat{C}$



b $\hat{R}$



c $\hat{Y}$



6 In triangle ABC, BC = 98 m, AB = 67 m and $\hat{A} = 85°$.
Find the size of $\hat{C}$.

7 In triangle PQR, PQ = 5 cm, QR = 6.5 cm and $\hat{P} = 70°$.
Find the size of $\hat{R}$.

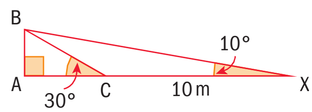
EXAM-STYLE QUESTION

8 In the diagram, $\hat{A} = 90°$, CX = 10 m, $A\hat{C}B = 30°$ and $\hat{X} = 10°$

a Write down the size of angle BCX.

b Find the length of BC.

c Find the length of AB.

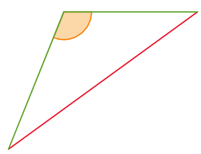


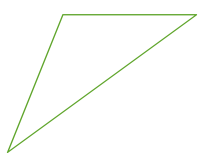
The cosine rule

If you have this information about a triangle:

- two sides and the included angle, or
- the three sides,

then you can find the other side and angles of the triangle.





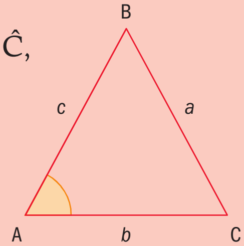
→ Cosine rule

In any triangle ABC with angles $\hat{A}$, $\hat{B}$ and $\hat{C}$, and opposite sides a, b and c respectively:

$$a^2 = b^2 + c^2 - 2bc\cos \hat{A}$$

This formula can be rearranged to

$$\cos \hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$



These formulae are in the Formula booklet. The first version of the formula is useful when you need to find a side. The second version of the formula is useful when you need to find an angle.

Example 21

In triangle ABC, AC = 8.6 m, AB = 6.3 m and $\hat{A} = 50°$.
Find the length of BC.

Answer

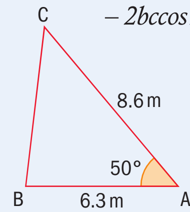
$BC^2 = 8.6^2 + 6.3^2 - 2 \times 8.6 \times 6.3 \times \cos 50°$

$BC^2 = 43.9975\ldots$

BC = 6.63 m (to 3 sf)

Sketch the triangle.

Use $a^2 = b^2 + c^2 - 2bc\cos \hat{A}$



The cosine rule applies to **any** triangle. For a right-angled triangle $A = 90°$. What does the formula look like? Do you recognize it? Is the cosine rule a generalization of Pythagoras' theorem?

Example 22

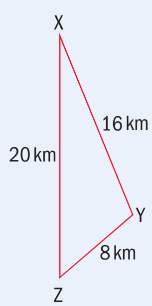
X, Y and Z are three towns. X is 20 km due north of Z.
Y is to the east of line XZ. The distance from Y to X is 16 km and the distance from Z to Y is 8 km.

a Represent this information in a clear and labeled diagram.

b Find the size of angle X.

Answers

a

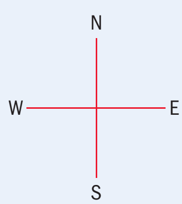


b $\cos \hat{X} = \frac{20^2 + 16^2 - 8^2}{2 \times 20 \times 16}$

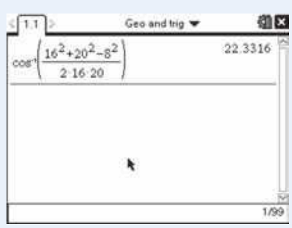
$\hat{X} = \cos^{-1}\left(\frac{20^2 + 16^2 - 8^2}{2 \times 20 \times 16}\right)$

$= 22.3$ (to 3 sf)

Remember:



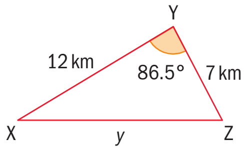
Use $\cos \hat{X} = \frac{y^2 + z^2 - x^2}{2yz}$



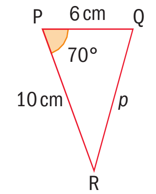
Exercise 30

1 Find the sides marked with letters.

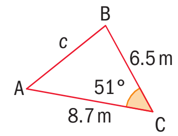
a



b

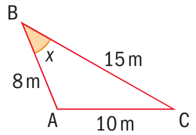


c

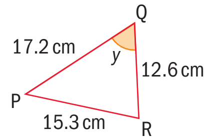


2 Find the angles marked with letters.

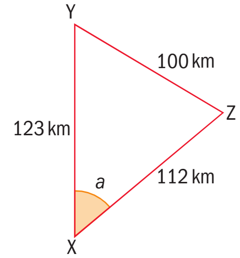
a



b



c



3 In triangle ABC, CB = 120 m, AB = 115 m and $\hat{B} = 110°$.
Find the length of side AC.

4 In triangle PQR, RQ = 6.9 cm, PR = 8.7 cm and $\hat{R} = 53°$.
Find the length of side PQ.

5 In triangle XYZ, XZ = 12 m, XY = 8 m, YZ = 10 m.
Find the size of angle X.

EXAM-STYLE QUESTIONS

6 X, Y and Z are three towns. X is 30 km due south from Y.
Z is to the east of the line joining XY.
The distance from Y to Z is 25 km and the distance from X to Z is 18 km.

a Represent this information in a clear and labeled diagram.

b Find the size of angle Z.

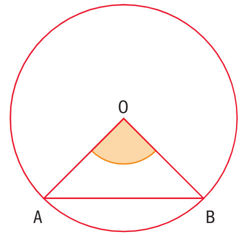
7 Alison, Jane and Stephen are together at point A. Jane walks 12 m due south from A and reaches point J. Stephen looks at Jane, turns through 110°, walks 8 m from A and reaches point S.

a Represent this information in a clear and labeled diagram.

b Find how far Stephen is from Jane.

c Find how far **north** Stephen is from Alison.

8 The diagram shows a circle of radius 3 cm and center O.
A and B are two points on the circumference.
The length AB is 5 cm.
A triangle AOB is drawn inside the circle.
Calculate the size of angle AOB.



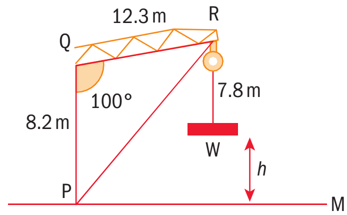
EXAM-STYLE QUESTION

9 The diagram shows a crane PQR that carries a flat box W. PQ is vertical, and the floor PM is horizontal.
Given that PQ = 8.2 m, QR = 12.3 m, $P\hat{Q}R = 100°$ and RW = 7.8 m, calculate

a PR

b the size of angle PRQ

c the height, *h*, of W above the floor, PM.



Extension material on CD:
Worksheet 3 - Cosine and sine rule proofs

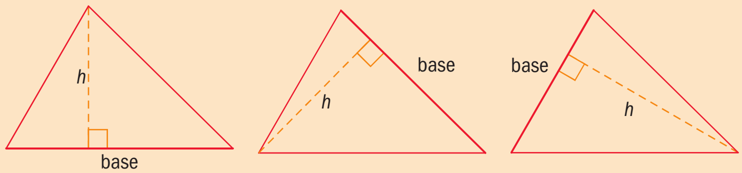
Area of a triangle

If you know one side of a triangle, the base *b* and the corresponding height *h*, you can calculate the area of the triangle using the formula

$$A = \frac{1}{2}(b \times h)$$

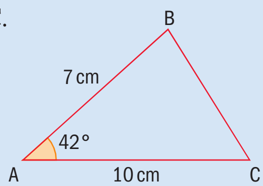
If you do not know the height, you can still calculate the area of the triangle as in the next example.

Remember that a triangle has three heights, one height per side.

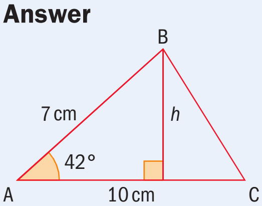


Example 23

Calculate the area of triangle ABC.



Answer



Use the formula

$A = \frac{1}{2}(b \times h)$ *with AC as the base,*

$b = 10$

Draw the height, h, the perpendicular to AC from B.

$\sin 42° = \frac{h}{7} \Rightarrow h = 7\sin 42°$

Substitute in the formula for the area of a triangle.

$Area = \frac{1}{2} \times b \times h$

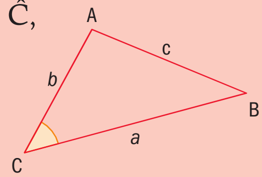
$= \frac{1}{2}(10 \times 7\sin 42°)$

$= 23.4\,\text{cm}^2$ (to 3 sf)

You can use the same method for any triangle.

→ In any triangle ABC with angles Â, B̂ and Ĉ, and opposite sides a, b and c respectively, this rule applies:

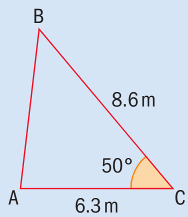
Area of triangle $= \frac{1}{2}ab\sin\hat{C}$



This formula is in the formula booklet.

Example 24

Calculate the area of the triangle ABC.



Answer

Area of triangle ABC =

$\frac{1}{2} \times 8.6 \times 6.3 \times \sin 50°$

$= 20.8\,\text{m}^2$ (to 3sf)

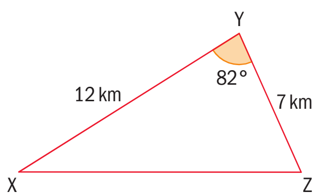
Substitute in the formula

Area $= \frac{1}{2}ab\sin\hat{C}$

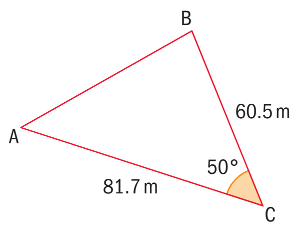
Exercise 3P

1 Calculate the area of each triangle.

a



b

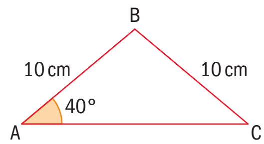


In the first century CE, Hero (or Heron) of Alexandria developed a different method for finding the area of a triangle using the lengths of the triangle's sides.

2 Here is triangle ABC.

a Find the size of angle B.

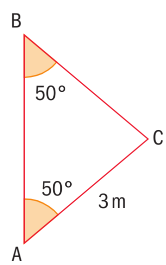
b Calculate the area of triangle ABC.



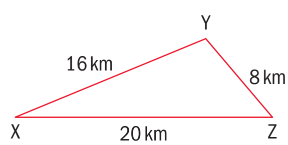
3 Here is triangle ABC.

a Write down the size of angle C.

b Find the area of triangle ABC.



4 Calculate the area of triangle XYZ.



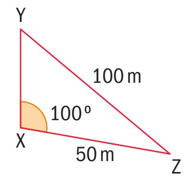
First find the size of one of the angles.

EXAM-STYLE QUESTIONS

5 The diagram shows a triangular field XYZ.
XZ is 50 m, YZ is 100 m and angle X is 100°.

a Find angle Z.

b Find the area of the field. Give your answer correct to the nearest $10\,m^2$.



6 The area of an isosceles triangle ABC is $4\,cm^2$. Angle B is 30° and AB = BC = x cm.

a Write down, in terms of x, an expression for the area of the triangle.

b Find the value of x.

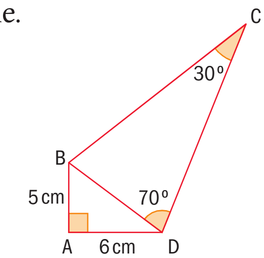
7 In the diagram, AB = 5 cm, AD = 6 cm, $B\hat{A}D = 90°$, $B\hat{C}D = 30°$, $B\hat{D}C = 70°$.

a Find the length of DB.

b Find the length of DC.

c Find the area of triangle BCD.

d Find the area of the quadrilateral ABCD.



Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

Give answers correct to 3 sf.

1 Line L_1 passes through the points A(1, 3) and B(5, 1).

a Find the gradient of the line AB.

Line L_2 is parallel to line L_1 and passes through the point (0, 4).

b Find the equation of the line L_2.

2 Line L_1 passes through the points A(0, 6) and B(6, 0).

a Find the gradient of the line L_1.

b Write down the gradient of all lines perpendicular to L_1.

c Find the equation of a line L_2 perpendicular to L_1 and passing through O(0,0).

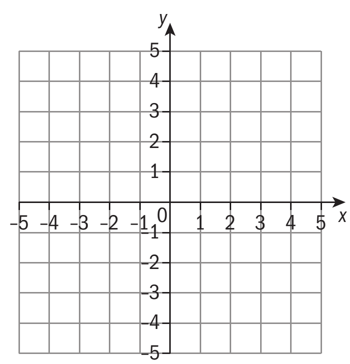
3 Consider the line L with equation $y = 2x + 3$.

a Write down the coordinates of the point where

i L meets the x-axis **ii** L meets the y-axis.

b Draw L on a grid like this one.

c Find the size of the acute angle that L makes with the x-axis.



4 Consider the line L_1 with equation $y = -2x + 6$.

a The point $(a, 4)$ lies on L_1. Find the value of a.

b The point $(12.5, b)$ lies on L_1. Find the value of b.

Line L_2 has equation $3x - y + 1 = 0$.

c Find the point of intersection between L_1 and L_2.

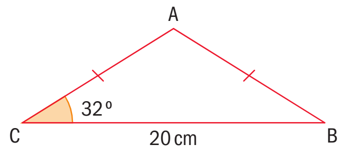
5 The height of a vertical cliff is 450 m. The angle of elevation from a ship to the top of the cliff is 31°. The ship is x metres from the bottom of the cliff.

a Draw a diagram to show this information.

b Calculate the value of x.

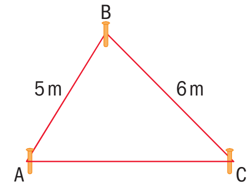
EXAM-STYLE QUESTIONS

6 In the diagram, triangle ABC is isosceles.
AB = AC, CB = 20 cm and angle ACB is 32°.
Find **a** the size of angle CAB
b the length of AB
c the area of triangle ABC.



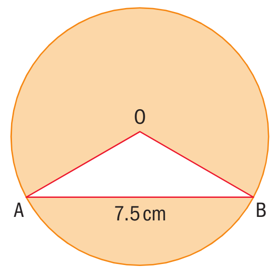
7 A gardener pegs out a rope, 20 metres long, to form a triangular flower bed as shown in this diagram.

a Write down the length of AC.
b Find the size of the angle BAC.
c Find the area of the flower bed.



8 The diagram shows a circle with diameter 10 cm and center O. Points A and B lie on the circumference and the length of AB is 7.5 cm. A triangle AOB is drawn inside the circle.

a Find the size of angle AOB.
b Find the area of triangle AOB.
c Find the shaded area.



Paper 2 style questions

EXAM-STYLE QUESTIONS

1 **a** On a pair of axes plot the points A(–2, 5), B(2, 2) and C(8, 10).
Use the same scale on both axes.
The quadrilateral ABCD is a rectangle.

b **i** Plot D on the pair of axes used in **a**.
ii Write down the coordinates of D.

c Find the gradient of line BC.

d Hence write down the gradient of line DC.

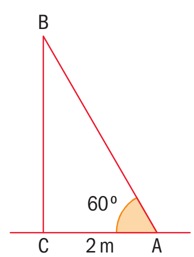
e Find the equation of line DC in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.

f Find the length of **i** DC **ii** BC.

g Find the size of the angle DBC.

2 The diagram shows a ladder AB. The ladder rests on the horizontal ground AC. The ladder is touching the top of a vertical telephone pole CB. The angle of elevation of the top of the pole from the foot of the ladder is 60°. The distance from the foot of the ladder to the foot of the pole is 2 m.

a Calculate the length of the ladder.
b Calculate the height of the pole.

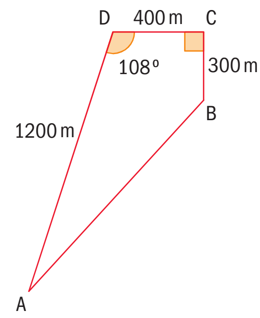


The ladder is moved in the same vertical plane so that its foot remains on the ground and its top touches the pole at a point P which is 1.5 m below the top of the pole.

c Write down the length of CP.
d Find the new distance from the foot of the ladder to the foot of the pole.
e Find the size of the new angle of elevation of the top of the pole from the foot of the ladder.

EXAM-STYLE QUESTION

3 The diagram shows a cross-country running course. Runners start and finish at point A.



a Find the length of BD.

b Find the size of angle BDC, giving your answer correct to two decimal places.

c Write down the size of angle ADB.

d Find the length of AB.

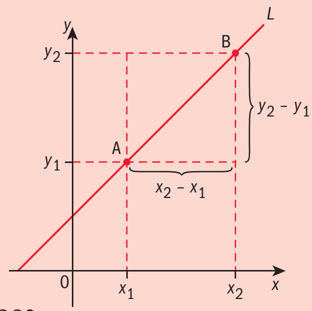
e **i** Find the perimeter of the course.

ii Rafael runs at a constant speed of $3.8\,\text{m}\,\text{s}^{-1}$. Find the time it takes Rafael to complete the course. Give your answer correct to the nearest minute.

f Find the area of the quadrilateral ABCD enclosed by the course. Give your answer in km^2.

CHAPTER 3 SUMMARY

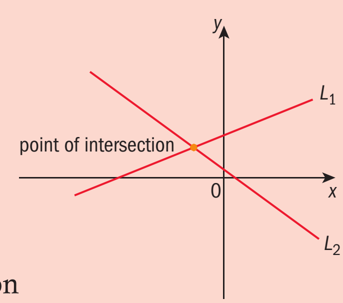
Gradient of a line



- If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points that lie on line L, the gradient of L is $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- **Parallel lines** have the **same gradient**. This means that
  - if two lines are parallel then they have the same gradient
  - if two lines have the same gradient then they are parallel.
- Two lines are **perpendicular** if, and only if, they make an angle of 90°. This means that
  - if two lines are perpendicular then they make an angle of 90°
  - if two lines make an angle of 90° then they are perpendicular.
- Two lines are **perpendicular** if the product of their gradients is –1.

Equations of lines

- The equation of a straight line can be written in the form
  i $y = mx + c$, where m is the **gradient** and c is the y-**intercept** (the y-coordinate of the point where the line crosses the y-axis).
  ii $ax + by + d = 0$ where a, b and $d \in \mathbb{Z}$.
- The equation of any vertical line is of the form $x = k$ where k is a constant.
- The equation of any horizontal line is of the form $y = k$ where k is a constant.
- If two lines are parallel then they have the same gradient and do not intersect.
- If two lines L_1 and L_2 are not parallel then they intersect at just one point. To find the point of intersection write $m_1x_1 + c_1 = m_2x_2 + c_2$ and solve for x.



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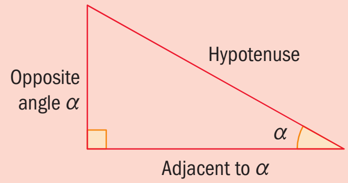
The sine, cosine, and tangent ratios

- Three trigonometric ratios in a right-angled triangle are defined as

$$\sin \alpha = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\cos \alpha = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\tan \alpha = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

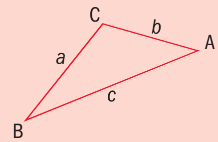


- The **angle of elevation** is the angle you lift your eyes through to look at something above.
- The **angle of depression** is the angle you lower your eyes through to look at something below.

The sine and cosine rules

- In any triangle ABC with angles A, B and C, and opposite sides a, b and c respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

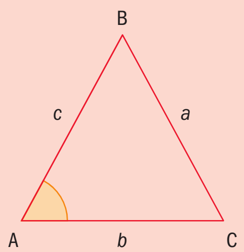


- In any triangle ABC with angles A, B and C, and opposite sides a, b and c respectively:

$$a^2 = b^2 + c^2 - 2bc\cos \hat{A}$$

This formula can be rearranged to

$$\cos \hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$



- In any triangle ABC with angles A, B and C, and opposite sides a, b and c respectively, this rule applies:

$$\text{Area of triangle} = \frac{1}{2}ab\sin \hat{C}$$