COMPLEX NUMBERS

Name:

1. (2 points) If w is a complex cube root of 1, find the value of $w^4 + w^8$.

2. (3 points) Show that $2 + i$ is a root of the equation

$$3x^3 - 14x^2 + 23x - 10 = 0$$

Find the other two roots of the equation. State the value of the sum and the value of the product of these roots.

3. (3 points)
   (a) Find the roots of the equation $z^2 + 4z + 7 = 0$, giving your answers in the form $p \pm i\sqrt{q}$, where p and q are integers.
   (b) Show these roots on an Argand diagram.
   (c) Find for each root the modulus and the argument (in radians).

4. (5 points) Let $z_1 = -30 + 15i$.
   (a) Find $arg(z_1)$, giving your answer in radians to two decimal places.

   The complex numbers z_2 and z_3 are given by $z_2 = -3 + pi$ and $z_3 = q + 3i$, where p and q are real constants and $p > q$.
   (b) Given that $z_2 z_3 = z_1$, find the value of p and the value of q.
   (c) Using your values of p and q, plot the points corresponding to z_1, z_2 and z_3 on an Argand diagram,
   (d) Verify that $2z_2 + z_3 - z_1$ is real and find its value.

5. (3 points) Solve the equation

$$z^5 = 4 + 4i$$

giving your answers in the form $z = re^{ik\pi}$, where r is the modulus of z and k is a rational number with $0 \leq k \leq 2$. Show your solutions on an Argand diagram.

6. (6 points) Find each of the roots of the equation $z^5 - 1 = 0$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.
   (a) Given that α is a complex root of this equation with the smallest positive argument, show that the roots of $z^5 - 1 = 0$ can be written as $\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4$.
   (b) Show that $\alpha^4 = \alpha^*$ and hence, or otherwise, write $z^5 - 1$ as a product of real linear and quadratic factors, giving the coefficients in terms of integers and cosines.
   (c) Show also that:
   $$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$$
   and hence, or otherwise, find $\cos(\frac{2\pi}{5})$, giving your answer in terms of surds.

7. (4 points) Show that $e^{\pi i/6}$ is a root of the equation $z^3 = i$. Find the other two roots and mark on an Argand diagram the points representing the three roots. Show that these three roots are also roots of the equation:
$$z^6 + 1 = 0$$
and write down the remaining three roots. Hence, or otherwise, express $z^6 + 1$ as the product of three quadratic factors each with coefficients in integer or surd form.

8. (6 points) Let $z = \cos\theta + i\sin\theta$.
   (a) Use the binomial theorem to show that the real part of z^4 is
   $$cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
   (b) Obtain a similar expression for the imaginary part of z^4 in terms of θ.
   (c) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ.
   (d) Use your answers to parts **a** and **c** to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.
   (e) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$, where k, m, n and p are integers to be determined.

9. (3 points) Find the roots of the equation $(z-4)^3 = 8i$ in the form $a + bi$, where a and b are real numbers. Mark, on an Argand diagram, the points A, B and C representing these three roots and find the area of $\triangle ABC$.

10. (3 points) The equation $z^3 + pz^2 + 40z + q = 0$, where p and q are real, has a root $3 + i$. Write down another root of the equation. Hence, or otherwise, find the values of p and q.

11. (6 points) Write down the fifth roots of unity in the form $\cos\theta + i\sin\theta$, where $0 \leq \theta \leq 2\pi$.
   (a) Hence, or otherwise, find the fifth roots of i in a similar form.
   (b) By writing the equation $(z-1)^5 = z^5$ in the form

$$\left(\frac{z-1}{z}\right)^5 = 1$$

   show that its roots are

$$\frac{1}{2}(1 + i\cot\frac{1}{5}k\pi) \qquad k = 1, 2, 3, 4$$

12. (6 points) Derive expressions for the three cube roots of unity in the form $re^{i\theta}$. Represent the roots on an Argand diagram.
   Let ω denote one of the non-real roots. Show that the other non-real root is ω^2. Show also that:

$$1 + \omega + \omega^2 = 0$$

   Given that:

$$\alpha = p + q \quad \beta = p + q\omega \quad \gamma = p + q\omega^2$$

   where p and q are real.
   (a) Find, in terms of p, $\alpha\beta + \beta\gamma + \gamma\alpha$.
   (b) Show that $\alpha\beta\gamma = p^3 + q^3$.
   (c) Find a cubic equation, with coefficients in terms of p and q, whose roots are α, β, γ.