1. Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

(Total 4 marks)

2. The complex numbers $z_1 = 2 - 2\mathrm{i}$ and $z_2 = 1 - \mathrm{i}\sqrt{3}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b-\sqrt{3}}$, where $a, b \in \mathbb{Z}^+$;

(3)

(b) calculate $\mathrm{A\hat{O}B}$ in terms of π.

(3)
(Total 6 marks)

3. Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$;

(2)

(b) $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0, z \neq -1$.

(5)
(Total 7 marks)

4. Consider the complex number $\omega = \frac{z+\mathrm{i}}{z+2}$, where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $\omega = \mathrm{i}$, determine z in the form $z = r \operatorname{cis} \theta$.

(6)

(b) Prove that $\omega = \frac{(x^2 + 2x + y^2 + y) + \mathrm{i}(x + 2y + 2)}{(x+2)^2 + y^2}$.

(3)

(c) **Hence** show that when $\mathrm{Re}(\omega) = 1$ the points (x, y) lie on a straight line, l_1, and write down its gradient.

(4)

(d) Given $\arg(z) = \arg(\omega) = \frac{\pi}{4}$, find $|z|$.

(6)
(Total 19 marks)

5. Consider the complex numbers $z = 1 + 2\text{i}$ and $w = 2 + a\text{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$;

(3)

(b) $\text{Re}(zw) = 2\,\text{Im}(zw)$.

(3)
(Total 6 marks)

6. If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + \text{i} \arg(z),\ 0 \le \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(2)

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1 - \text{i})$;

(iii) $L(-1 + \quad \text{i})$.

(5)

(c) Hence show that the property $L(z_1 z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2.

(2)
(Total 9 marks)

7. Find, in its simplest form, the argument of $(\sin\theta + \text{i}\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

8. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \text{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\dfrac{z_2}{z_1}$ in the form $a + \text{i}b$.

(4)
(Total 12 marks)

9. Given that $(a + b\mathrm{i})^2 = 3 + 4\mathrm{i}$ obtain a pair of simultaneous equations involving a and b. Hence find the two square roots of $3 + 4\mathrm{i}$.

(Total 7 marks)

10. (a) Factorize $z^3 + 1$ into a linear and quadratic factor.

(2)

Let $\gamma = \dfrac{1+\mathrm{i}\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

(9)

(Total 11 marks)

11. Given that $|z| = \sqrt{10}$, solve the equation $5z + \dfrac{10}{z^*} = 6 - 18\mathrm{i}$, where z^* is the conjugate of z.

(Total 7 marks)

12. Solve the simultaneous equations

$$\mathrm{i}z_1 + 2z_2 = 3$$
$$z_1 + (1 - \mathrm{i})z_2 = 4$$

giving z_1 and z_2 in the form $x + \mathrm{i}y$, where x and y are real.

(Total 9 marks)

13. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$.

(2)

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(3)

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(3)

(d) **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(6)

(e) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$.

(8)
(Total 22 marks)