COMPLEX NUMBERS

Name:

1. (4 points) Let w be a root of the equation $z^7 - 1 = 0$. Show that if w is not real, then:

$$w^6 + w^5 + w^4 + w^3 + w^2 + w + 1 = 0$$

Write down all possible values of w in the form $e^{i\theta}$, where $-\pi < \theta \leq \pi$.

2. (6 points) The complex numbers z and w are such that:

$$z = -2 + 5i \qquad zw = 14 + 23i$$

(a) (3 points) Find w in the form $a + bi$, where a and b are real.

(b) (1 point) Display z and w on the same Argand diagram.

(c) (2 points) Write down the complex number that represents the midpoint of the line joining the points z and zw.

3. (6 points) Find a complex number z which satisfies the equation:

$$z + 2z^* = \frac{15}{2-i}$$

where z^* denotes the complex conjugate of z.

4. (9 points)

   (a) (2 points) Verify that $z_1 = 1 + e^{\pi i/5}$ is a root of the equation $(z-1)^5 = -1$.

   (b) (3 points) Find the other four roots of the equation and mark them on an Argand diagram.

   (c) (4 points) By considering the Argand diagram, or otherwise, find:

      i. $arg(z_1)$ in terms of π.

      ii. $|z_1|$ in the form $a\cos\frac{\pi}{b}$, where a and b are integers to be determined.

5. (5 points) Consider the polynomial $P(x) = x^3 + x^2 - 4x + 6$.

   (a) Show that $1 + i$ is a root of $P(x)$.

   (b) Write down the other complex root of $P(x)$.

   (c) Factorize $P(x)$ completely.

6. (9 points) Consider the complex number $z = \cos\theta + i\sin\theta$.

(a) Using de Moivre's theorem show that:

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

(b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that:

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(c) Hence solve the equation

$$\cos 4\theta + 4\cos 2\theta = 5$$

for $\theta \in [0, 2\pi]$

7. (9 points) Solve the following equation:

$$z^2 + \frac{1}{\sqrt{2}}z - \frac{i\sqrt{3}}{8} = 0$$

Express your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

8. (8 points) Solve the following system of equations:

$$3z + (1 - i)w = 7 - 4i$$
$$(2 - i)z + iw = 8 + 4i$$

9. (6 points) Three consecutive terms of an increasing arithmetic sequence are $x + 4$, $2x^2$ and $2x + 6$. Find the possible values of x.

10. (8 points) Consider the polynomial $P(x) = x^3 + px^2 + qx - 4$. Suppose that $x - 1$ is a factor of $P(x)$.

    (a) (2 points) Show that $p + q = 3$

    (b) (3 points) Show that $P(x) = (x - 1)(x^2 + (p + 1)x + 4)$

    (c) (3 points) Find the range of values of p for which $P(x)$ has three (not necessarily distinct) real roots.

11. (10 points)

(a) Prove, using mathematical induction, that:

$$1 + 4 + 9 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(b) Hence find an expression, in terms of n, for:

$$\log 2 + 2\log 4 + 3\log 8 + ... + n\log 2^n$$

(c) Calculate:

$$\log 2 + 2\log 4 + 3\log 8 + ... + 10\log 1024$$