16 Basic differentiation and its applications

In this chapter you will learn:

- how to find the gradients of curves from first principles, a process called differentiation
- how to differentiate x^n
- how to differentiate $\sin x$, $\cos x$ and $\tan x$
- how to differentiate e^x and $\ln x$
- to find the equations of tangents and normals to curves at given points
- to find maximum and minimum points on curves.

Introductory problem

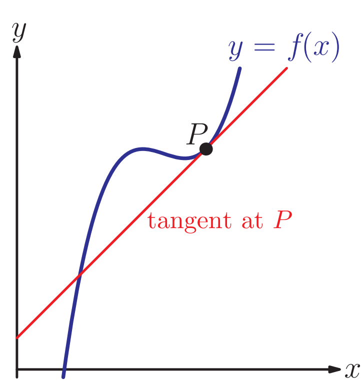
The cost of petrol used in a car, in £ per hour, is $\frac{12+v^2}{100}$ where v is measured in miles per hour and $v > 0$.

If Daniel wants to travel 50 miles as cheaply as possible, at what speed should he travel?

In real life, things change. Planets move, babies grow and prices rise. Calculus is the study of things that change, and one of its important tools is differentiation; the ability to find the rate at which the y-coordinate of a curve is changing when the x-coordinate changes. For a straight-line graph this is determined by the **gradient**, but it requires more work to apply the same idea to curves, where the gradient is different at different points.

16A Sketching derivatives

Our first task is to establish exactly what is meant by the gradient of a curve. We are clear on what is meant by the gradient of a straight line and we can use this idea to make a more general definition: the gradient of a curve at a point P is the gradient of the tangent to the curve at that point.



A **tangent** is a straight line which touches the curve without crossing it.

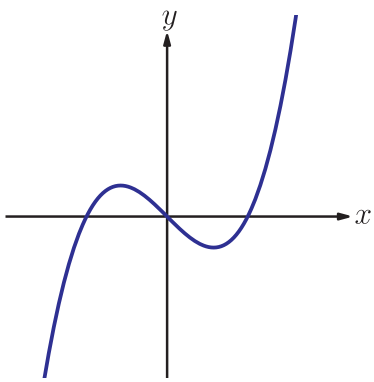
We have already met tangents in chapter 3.

Note that when we say that the tangent at P does not cross the curve we mean that this is only the case locally (close to the point P). The tangent might also intersect a different part of the curve.

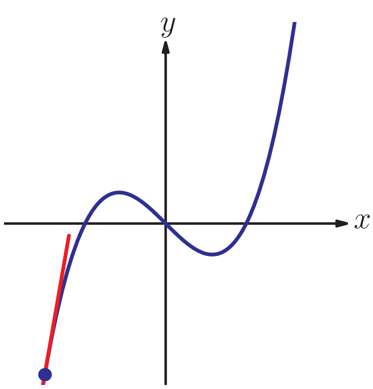
The **derivative** of a function, $f(x)$, is another function that gives the gradient of $y = f(x)$ at any point in the x domain.
It is often useful to be able to roughly sketch the derivative.

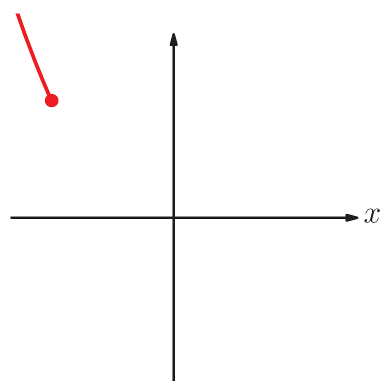
Worked example 16.1

Sketch the derivative of this function.



Imagine we are tracking a point moving along the curve from left to right; we will track the tangent to the curve at the moving point and form the graph of its gradient

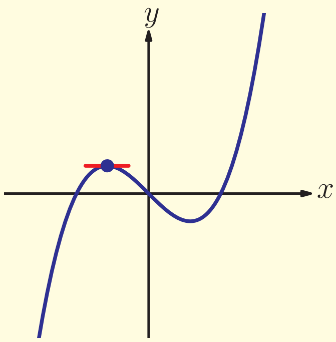




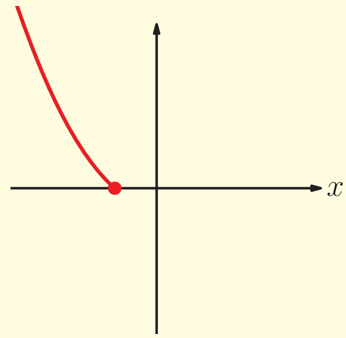
The curve is increasing from left to right, but more and more slowly...

... so the gradient is positive and falling

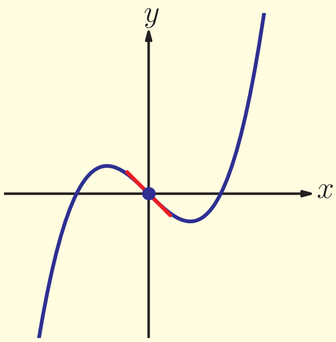
continued . . .



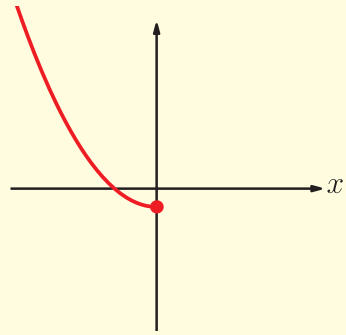
The tangent is horizontal...



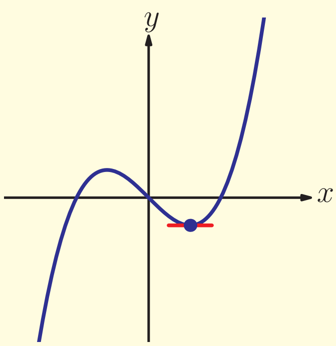
... so the gradient is zero



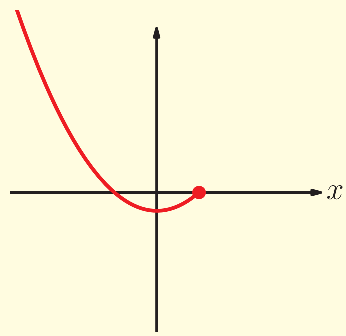
The curve is now decreasing. . . .



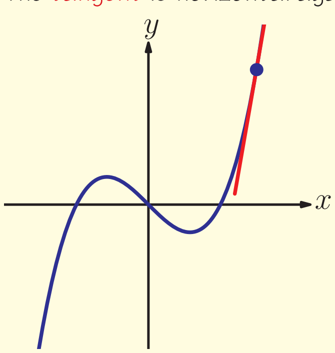
... so the gradient is negative



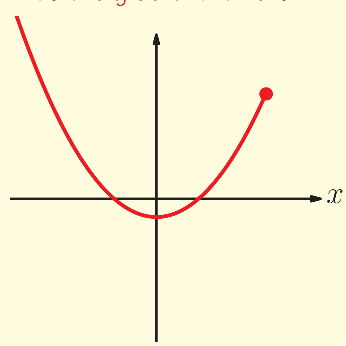
The tangent is horizontal again...



... so the gradient is zero



The curve is increasing, and does so faster and faster...

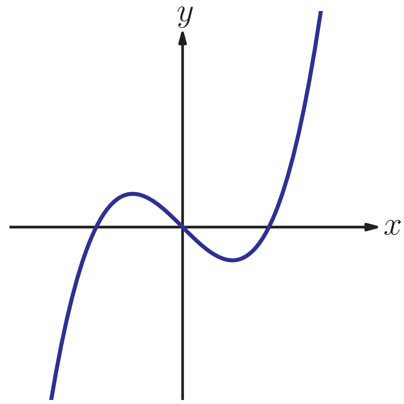


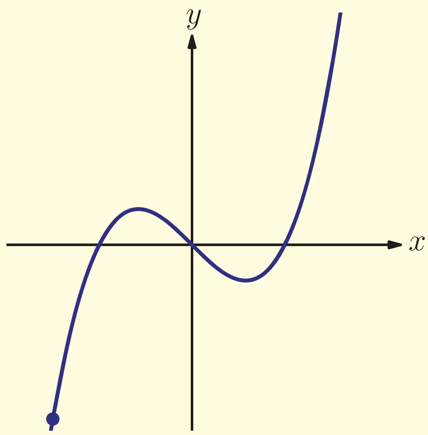
... so the gradient is positive and getting larger

We can also apply the same reasoning backwards.

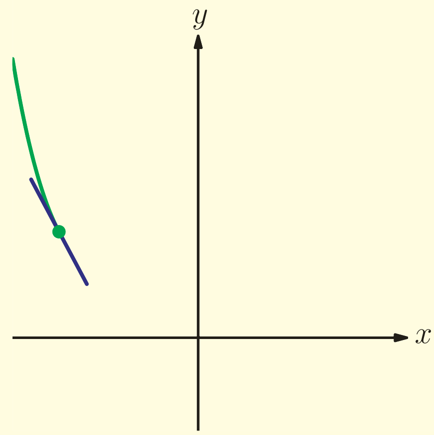
Worked example 16.2

You are given the derivative of a function. Sketch a possible graph of the original function.

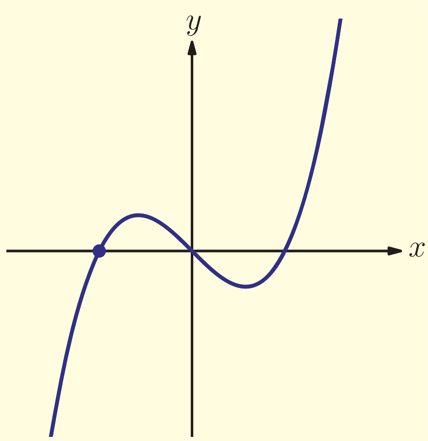




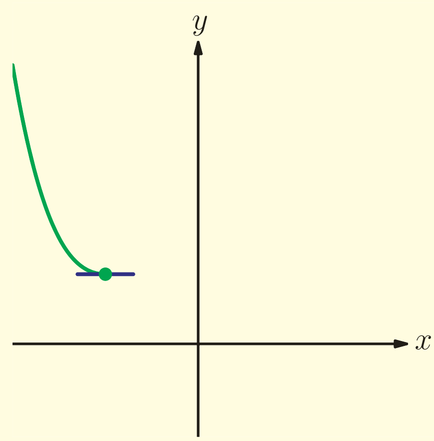
The gradient is negative...



... so the curve is decreasing.



The gradient is zero...



... so the tangent is horizontal.

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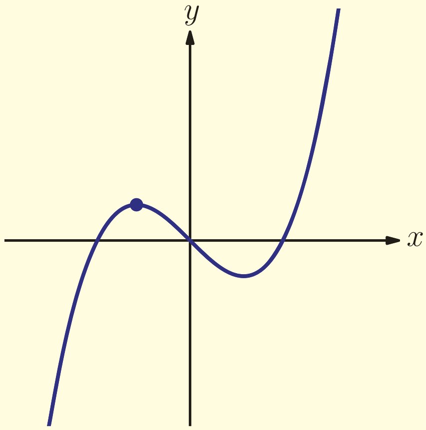
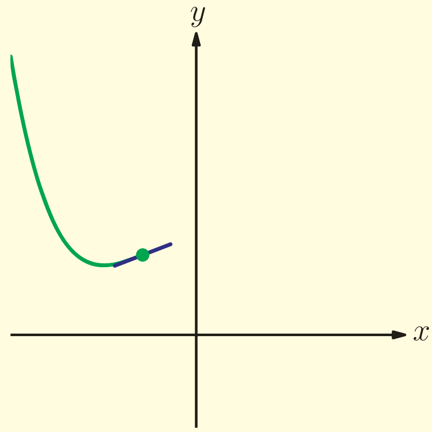
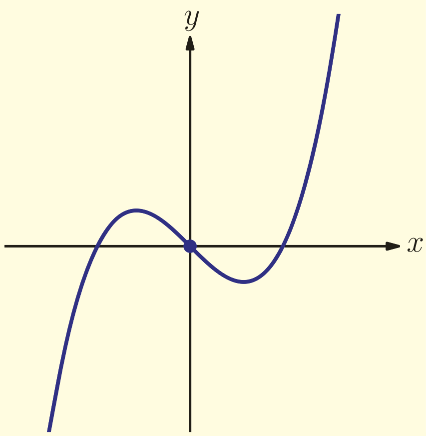


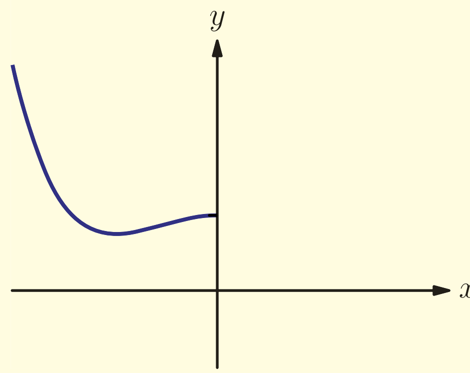
The gradient is positive...



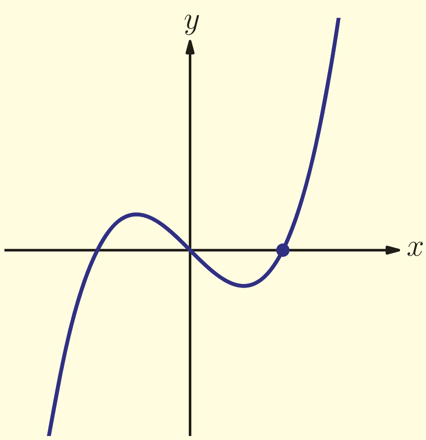
... so the curve is increasing.



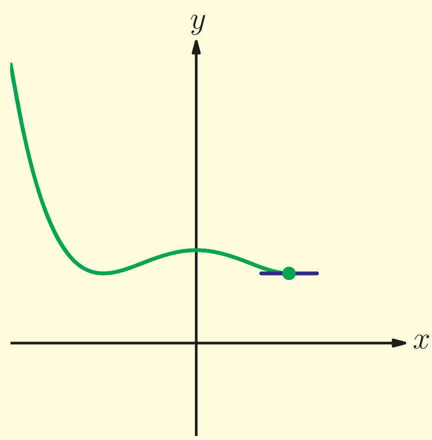
The gradient is zero...



.... so the tangent is horizontal.

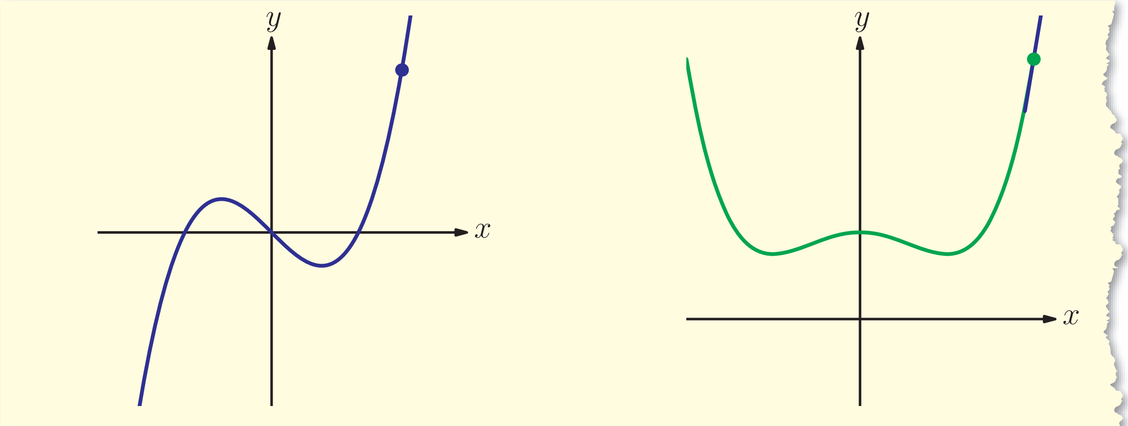


The gradient is zero...



... so the tangent is horizontal.

continued . . .



Notice in this example that there was more than one possible graph we could have drawn, depending on where we started the sketch. In chapter 17 you will learn more about this ambiguity when you 'undo' differentiation.

The relationship between a graph and its derivative can be summarised as follows:

KEY POINT 16.1

When the curve is increasing the gradient is positive.

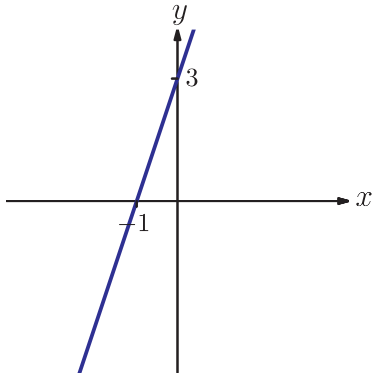
When the curve is decreasing the gradient is negative.

When the tangent is horizontal the gradient is zero; a point on the curve where this happens is called a **stationary point** or **turning point**.

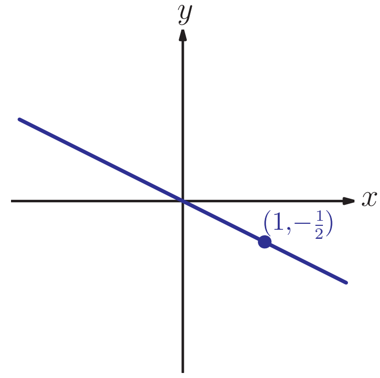
Exercise 16A

1. Sketch the derivatives of the following showing intercept with the x-axis:

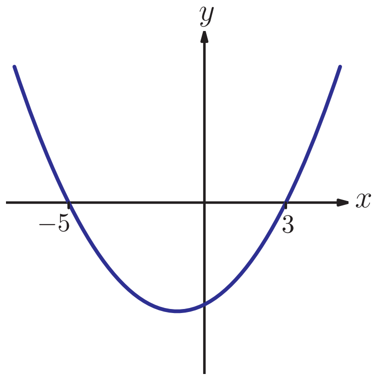
(a) (i)



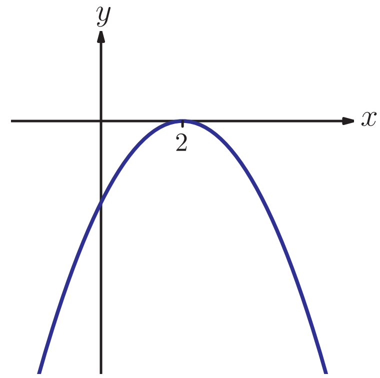
(ii)



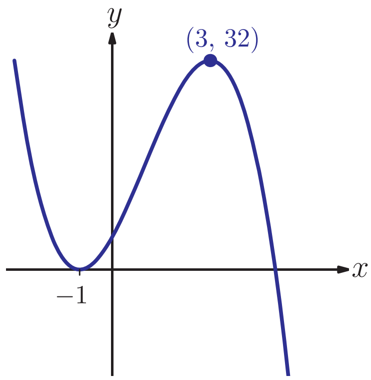
(b) (i)



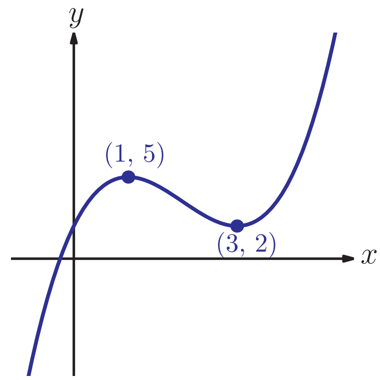
(ii)



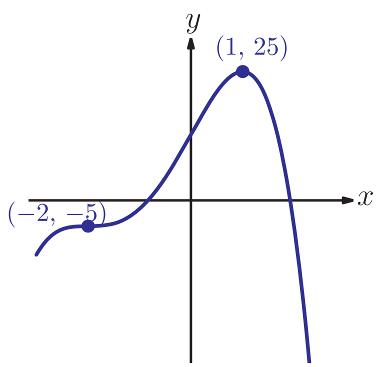
(c) (i)



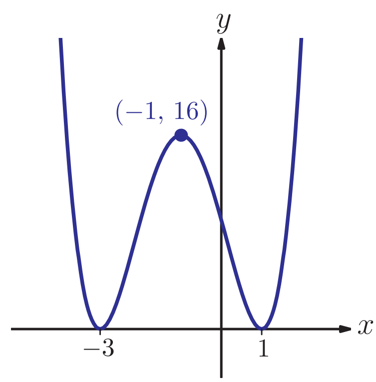
(ii)



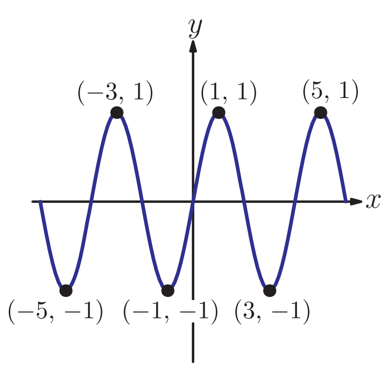
(d) (i)



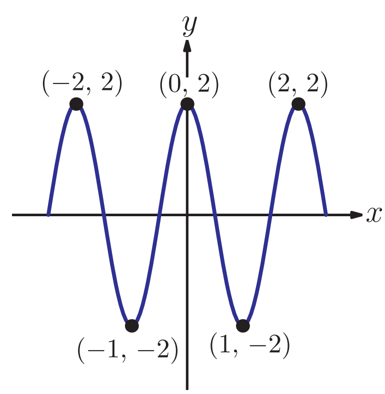
(ii)



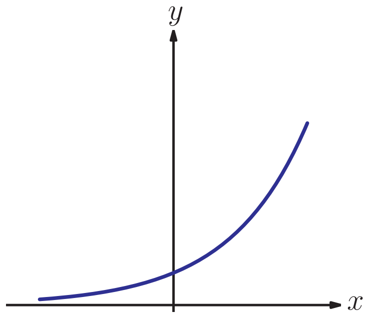
(e) (i)



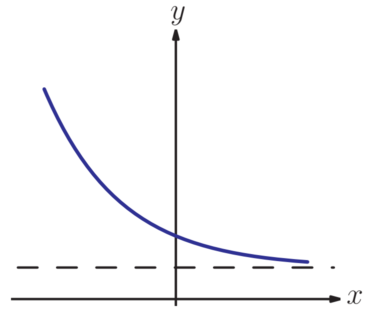
(ii)



(f) (i)

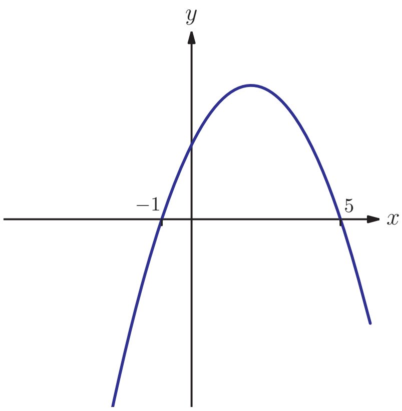


(ii)

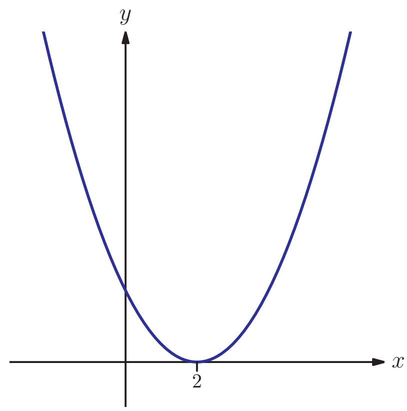


2. Each of the following represents a graph of a function's derivative. Sketch a possible graph for the original function, indicating any stationary points.

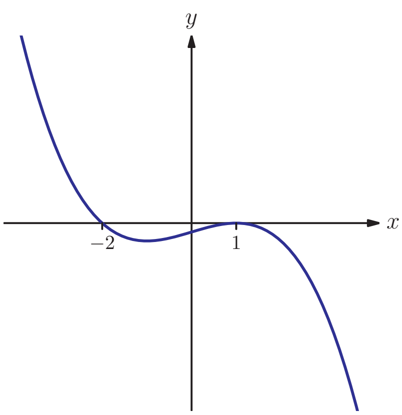
(a)



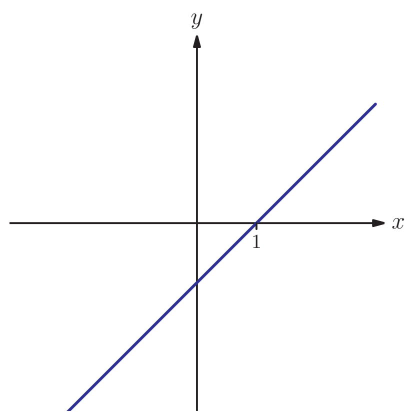
(b)



(c)



(d)



3. For each of the following statements decide if they are always true, sometimes true or always false.

(a) At a point where the derivative is positive, the original function is positive.

(b) If the original function is negative then the derivative is also negative.

(c) The derivative crossing the axis corresponds to a stationary point on the graph.

(d) When the derivative is zero, the graph is at a local maximum or minimum point.

(e) If the derivative function is always positive then part of the original function is above the x-axis.

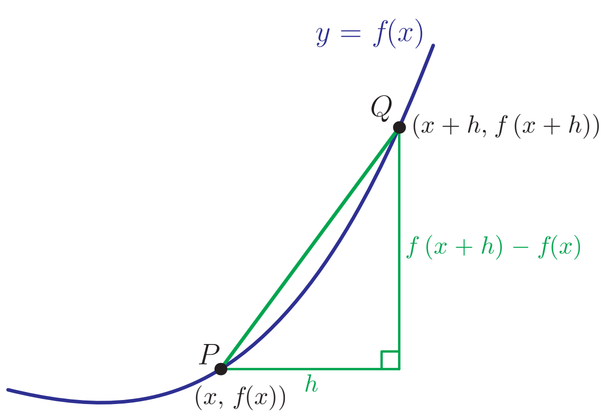
(f) At the lowest value of the original function, the derivative is zero.

16B Differentiation from first principles

You will probably find that drawing a tangent to a graph is very difficult to do accurately, and that your line actually crosses the curve at two points. The line segment between these two intersection points is called a **chord**. If the two points are close together, the gradient of the chord is very close to the gradient of the tangent. We can use this to establish a method for calculating the derivative for a given function.

Self-discovery worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM leads you through several examples of this method. Here we summarise the general procedure.

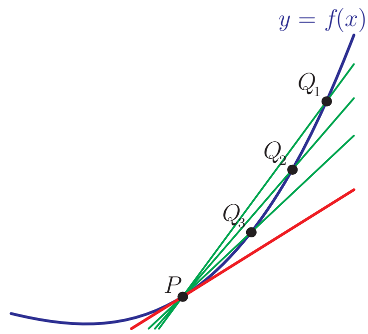
Consider a point $P(x, f(x))$ on the graph of the function $y = f(x)$ and move a distance h away from x to the point $Q(x+h, f(x+h))$.



We can find an expression for the gradient of the chord PQ:

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

As the point Q becomes closer and closer to P, the gradient of the chord PQ becomes a closer and closer approximation to the gradient of the tangent at P.



To denote this idea of the distance h approaching zero, we use $\lim_{h \to 0}$, which reads as 'the limit as h tends to 0'. This idea of a limit is very much like that encountered for asymptotes on graphs in chapters 2 and 4, where the graph tends to the asymptote (the limit) as x tends to ∞.

The process of finding $\lim_{h \to 0}$ of the gradient of the chord PQ is called **differentiation from first principles** and with this notation, we have the following definition:

KEY POINT 16.2

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is the **derivative of** $f(x)$. It can also be written as f', y' or $\frac{dy}{dx}$ where $y = f(x)$. The process of finding the derivative is called **differentiation**.

EXAM HINT

Differentiation from first principles means finding the derivative using this definition, rather than any of the rules we will meet in the later sections.

We can use this definition to find the derivative of simple polynomial functions.

Worked example 16.3

For the function $y = x^2$, find $\frac{dy}{dx}$ from first principles.

Use the formula

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

We do not want to let the denominator tend to zero so first simplify the numerator and hope the h in the denominator will cancel

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

Divide top and bottom by h

$$= \lim_{h\to 0} (2x + h)$$

Finally let $h \to 0$

$$= 2x$$

We can use the same method with other functions too, but it may require more complicated algebraic manipulation.

Worked example 16.4

Differentiate $f(x) = \sqrt{x}$ from first principles.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

We do not want to let the denominator tend to zero so manipulate the numerator to get a factor of h

We can get rid of the square roots by multiplying top and bottom of the fraction by $\sqrt{x+h} + \sqrt{x}$ and using the difference of two squares

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

We can now divide top and bottom by h...

$$= \frac{1}{\sqrt{h+x} + \sqrt{x}}$$

... and let $h \to 0$

$$\therefore f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{h+x}+\sqrt{x}}$$
$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}$$

Exercise 16B

1. Find the derivatives of the following functions from first principles:
   (a) (i) $f(x) = x^3$ (ii) $f(x) = x^4$
   (b) (i) $f(x) = -4x$ (ii) $f(x) = 3x^2$
   (c) (i) $f(x) = x^2 - 6$ (ii) $f(x) = x^2 - 3x + 4$

2. Prove from first principles that the derivative of $x^2 + 1$ is $2x$.
[4 marks]

3. Prove from first principles that the derivative of 8 is zero.
[4 marks]

4. Prove from first principles that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$.
[4 marks]

5. If k is a constant prove that the derivative of $kf(x)$ is $kf'(x)$.
[4 marks]

6. Prove from first principles that the derivative of $\frac{1}{\sqrt{x}}$ is $-\frac{1}{2x\sqrt{x}}$.
[5 marks]

16C Rules of differentiation

From Exercise 16B, and the results of Self-discovery worksheet 3 ‘Investigating derivatives of polynomials’ on the CD-ROM, some properties of differentiation are suggested:

KEY POINT 16.3

- If $y = x^n$ then:

$$\frac{dy}{dx} = nx^{n-1}$$

KEY POINT 16.3 continued...

- If we differentiate $kf(x)$ where k is a constant we get $kf'(x)$.
- Differentiation of the sum of various terms can proceed term by term.

Fill-in proof sheet 15 'Differentiating polynomials' on the CD-ROM proves these results for positive integer values; however, this result holds for all rational powers.

A special case is when $n = 0$. Since $x^0 = 1$, we can say that $\frac{dy}{dx} = 0x^{-1} = 0$. This is because the gradient of the graph $y = 1$ is zero everywhere; it is a horizontal line. In fact, the derivative of any constant is zero.

You often have to simplify an expression before differentiating, using the laws of algebra, in particular the laws of exponents.

If you need to review rules of exponents, see chapter 2.

Worked example 16.5

Find the derivative of the following functions:

(a) $f(x) = x^2\sqrt{x}$ (b) $g(x) = \frac{1}{\sqrt[3]{x}}$

First rewrite the function in the form x^n using the laws of exponents

(a) $f(x) = x^2\sqrt{x} = x^2x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{5}{2}}$

Differentiate

$f'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{\frac{3}{2}}$

Cube root can be written as a power

(b) $g(x) = \frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$

$g'(x) = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}}$

EXAM HINT

Note that you cannot differentiate products by differentiating each of the factors and multiplying them together – we will see in chapter 18 that there is a more complicated rule for dealing with products.

Worked example 16.6

Find the derivative of the following functions:

(a) $f(x) = 5x^3$

(b) $g(x) = x^4 - \frac{3}{2}x^2 + 5x - 4$

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

Differentiate x^3 then multiply by 5

(a) $f'(x) = 5 \times 3x^2 = 15x^2$

Differentiate each term separately

(b) $g'(x) = 4x^3 - \frac{3}{2} \times 2x + 5 = 4x^3 - 3x + 5$

We need to write this as a sum of terms of the form x^n

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

$= \frac{4x - 14}{x^{\frac{1}{2}}}$

$= 4x^{1-\frac{1}{2}} - 14x^{-\frac{1}{2}}$

$= 4x^{\frac{1}{2}} - 14x^{-\frac{1}{2}}$

Now differentiate each term separately

$h'(x) = 4 \times \frac{1}{2}x^{\frac{1}{2}-1} - 14\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$

$= 2x^{-\frac{1}{2}} + 7x^{-\frac{3}{2}}$

Exercise 16C

1. Differentiate the following:

(a) (i) $y = x^4$ (ii) $y = x$

(b) (i) $y = 3x^7$ (ii) $y = -4x^5$

(c) (i) $y = 10$ (ii) $y = -3$

(d) (i) $y = 4x^3 - 5x^2 + 2x - 8$ (ii) $y = 2x^4 + 3x^3 - x$

(e) (i) $y = \frac{1}{3}x^6$ (ii) $y = -\frac{3}{4}x^2$

(f) (i) $y = 7x - \frac{1}{2}x^3$ (ii) $y = 2 - 5x^4 + \frac{1}{5}x^5$

(g) (i) $y = x^{\frac{3}{2}}$ (ii) $y = x^{\frac{2}{3}}$

(h) (i) $y = 6x^{\frac{4}{3}}$ (ii) $y = \frac{3}{5}x^{\frac{5}{6}}$

(i) (i) $y = 3x^4 - x^2 + 15x^{\frac{2}{5}} - 2$ (ii) $y = x^3 - \frac{3}{5}x^{\frac{5}{3}} + \frac{4}{3}x^{\frac{1}{2}}$

(j) (i) $y = x^{-1}$ (ii) $y = -x^{-3}$

(k) (i) $y = x^{-\frac{1}{2}}$ (ii) $y = -8x^{-\frac{3}{4}}$

(l) (i) $y = 5x - \frac{8}{15}x^{-\frac{5}{2}}$ (ii) $y = -\frac{7}{3}x^{-\frac{3}{7}} + \frac{4}{3}x^{-6}$

2. Find $\frac{dy}{dx}$ for the following:

(a) (i) $y = \sqrt[3]{x}$ (ii) $y = \sqrt[5]{x^4}$

(b) (i) $y = \frac{3}{x^2}$ (ii) $y = -\frac{2}{5x^{10}}$

(c) (i) $y = \frac{1}{\sqrt{x}}$ (ii) $y = \frac{8}{3\sqrt[4]{x^3}}$

(d) (i) $y = x^2(3x - 4)$ (ii) $y = \sqrt{x}(x^3 - 2x + 8)$

(e) (i) $y = (x+2)(\sqrt[3]{x} - 1)$ (ii) $y = \left(x + \frac{2}{x}\right)^2$

(f) (i) $y = \frac{3x^5 - 2x}{x^2}$ (ii) $y = \frac{9x^2 + 3}{2\sqrt[3]{x}}$

3. Find $\frac{dy}{dx}$ if:

(a) (i) $x + y = 8$ (ii) $3x - 2y = 7$

(b) (i) $y + x + x^2 = 0$ (ii) $y - x^4 = 2x$

16D Interpreting derivatives and second derivatives

$\frac{dy}{dx}$ has two related interpretations:

- It is the gradient of the graph of y against x.
- It measures how fast y changes when x is changed – this is called the **rate of change** of y with respect to x.

Remember that $\frac{dy}{dx}$ is itself a function – its value changes with x.

For example, if $y = x^2$ then $\frac{dy}{dx}$ is equal to 6 when $x = 3$, and it is equal to –2 when $x = -1$. This corresponds to the fact that the gradient of the graph of $y = x^2$ changes with x, or that the rate of change of y varies with x.

EXAM HINT

We can also write this using function notation:

If $f(x) = x^2$ then $f'(3) = 6$ and $f'(-1) = -2$

To calculate the gradient (or the rate of change) at any particular point, we simply substitute the value of x into the equation for the derivative.

Worked example 16.7

Find the gradient of the graph $y = 4x^3$ at the point where $x = 2$.

The gradient is given by the derivative, so find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 12x^2$$

Substitute the value for x

When $x = 2$: $\frac{dy}{dx} = 12 \times 2^2 = 48$

So the gradient is 48

EXAM HINT

Your calculator can find the gradient at a given point, but it cannot find the expression for the derivative. See Calculator sheet 8 on the CD-ROM.

If we know the gradient of a graph at a particular point, we can find the value of x at that point. This involves solving an equation.

The sign of the gradient tells us whether the function is increasing or decreasing.

Worked example 16.8

Find the values of x for which the graph of $y = x^3 - 7x + 1$ has gradient 5.

The gradient is given by the derivative

$$\frac{dy}{dx} = 3x^2 - 7$$

We know the value of $\frac{dy}{dx}$ so we can form an equation

$3x^2 - 7 = 5$

$\Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

$\Rightarrow x = 2 \text{ or } -2$

KEY POINT 16.4

If $\frac{dy}{dx}$ is positive the function is increasing – as x gets larger so does y.

If $\frac{dy}{dx}$ is negative the function is decreasing – as x gets larger y gets smaller.

In Section 16H we will discuss what happens when $\frac{dy}{dx} = 0$.

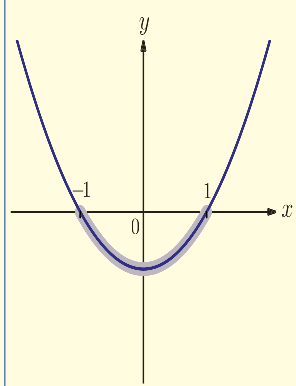
Worked example 16.9

Find the range of values of x for which the function $f(x) = 2x^3 - 6x$ is decreasing.

A decreasing function has negative gradient

$f'(x) < 0$
$\Rightarrow 6x^2 - 6 < 0$
$\Rightarrow x^2 - 1 < 0$

This is a quadratic inequality, so we need to look at the graph of $x^2 - 1$



$-1 < x < 1$

There is nothing special about the variables y and x. We can just as easily say that $\frac{dB}{dQ}$ is the gradient of the graph of B against Q or that $\frac{d(bananas)}{d(monkeys)}$ measures how fast bananas change when you change the variable monkeys. To emphasise which variables we are using, we call $\frac{dy}{dx}$ the **derivative of y with respect to x**.

You may wonder why it is so important to emphasise that we are differentiating with respect to x (or Q or *monkeys*). In this course we are only considering functions of one variable, but it is possible to generalise calculus to include functions which depend on several variables. This has many applications, particularly in physics and engineering.

Worked example 16.10

Given that $a = \sqrt{S}$, find the rate of change of a when $S = 9$.

The rate of change is given by the derivative

$a = S^{\frac{1}{2}}$

$\frac{da}{dS} = \frac{1}{2}S^{-\frac{1}{2}} = \frac{1}{2\sqrt{S}}$

Substitute the value for S

When $S = 9$: $\frac{da}{dS} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$\frac{d}{dx}$ is called an operator – it acts on functions to turn them into other functions. So when we differentiate $y = 3x^2$ what we are really doing is applying the $\frac{d}{dx}$ operator to both sides of the identity:

$$\frac{d}{dx}(y) = \frac{d}{dx}(3x^2)$$

$$\Rightarrow \frac{dy}{dx} = 6x$$

So $\frac{dy}{dx}$ is just $\frac{d}{dx}$ applied to y.

The $\frac{d}{dx}$ operator can also be applied to things which have already been differentiated. This is then called the **second derivative**.

KEY POINT 16.5

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is given the symbol $\frac{d^2y}{dx^2}$ or $f''(x)$ and it refers to the rate of change of the gradient.

We can differentiate again to find the third derivative $\left(\frac{d^3y}{dx^3} \text{ or } f'''(x)\right)$, fourth derivative $\left(\frac{d^4y}{dx^4} \text{ or } f^{(4)}(x)\right)$, and so on.

Worked example 16.11

Given that $f(x) = 5x^3 - 4x$:

(a) Find $f''(x)$.

(b) Find the rate of change of the gradient of the graph of $y = f(x)$ at the point where $x = -1$.

Differentiate $f(x)$ and then differentiate the result

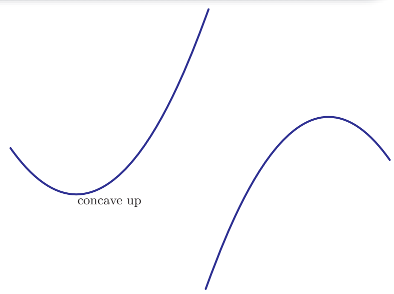
(a) $f'(x) = 15x^2 - 4$

$f''(x) = 30x$

The rate of change of the gradient means the second derivative

(b) $f''(-1) = -30$

We can use the second derivative to find out more about the shape of the graph. Remember that the second derivative is the rate of change of the gradient. So when the second derivative is positive, the gradient is increasing. This means that the graph is curving upwards; we say that it is **concave up**. When the second derivative is negative, the gradient is decreasing so the graph curves downwards; we say that it is **concave down**.



Exercise 16D

1. Write the following rates of change as derivatives:
   (a) The rate of change of z as t changes.
   (b) The rate of change of Q with respect to P.
   (c) How fast R changes when m is changed.
   (d) How quickly balloon volume (V) changes over time (t).
   (e) The rate of increase of the cost of apples (y) as the weight of the apple (x) increases.
   (f) The rate of change of the rate of change of z as y changes.
   (g) The second derivative of H with respect to m.

2. (a) (i) If $f = 5x^{\frac{1}{3}}$ what is the derivative of f with respect to x?
   (ii) If $p = 3q^5$ what is the derivative of p with respect to q?
   (b) (i) Differentiate $d = 3t + 7t^{-1}$ with respect to t.
   (ii) Differentiate $r = c + \frac{1}{c}$ with respect to c.
   (c) (i) Find the second derivative of $y = 9x^2 + x^3$ with respect to x.
   (ii) Find the second derivative of $z = \frac{3}{t}$ with respect to t.

You may think that it is contradictory to talk about the rate of change of y as x changes if we are fixing x to have a certain value. Think about x passing through this point.

3. (a) (i) If $y = 5x^2$, find $\frac{dy}{dx}$ when $x = 3$.

(ii) If $y = x^3 + \frac{1}{x}$, find $\frac{dy}{dx}$ when $x = 1.5$.

(b) (i) If $A = 7b + 3$, find $\frac{dA}{db}$ when $b = -1$.

(ii) If $f = \theta^2 + \theta^{-3}$, find $\frac{df}{d\theta}$ when $\theta = 0.1$.

(c) (i) Find the gradient of the graph of $A = x^3$ when $x = 2$.

(ii) Find the gradient of the tangent to the graph of $z = 2a + a^2$ when $a = -6$.

(d) (i) How quickly does $f = 4T^2$ change as T changes when $T = 3$?

(ii) How quickly does $g = y^4$ change as y changes when $y = 2$?

(e) (i) What is the rate of increase of W with respect to p when p is -3 if $W = -p^2$?

(ii) What is the rate of change of L with respect to c when $c = 6$ if $L = 7\sqrt{c} - 8$?

4. (a) (i) If $y = ax^2 + (1-a)x$ where a is a constant, find $\frac{dy}{dx}$.

(ii) If $y = x^3 + b^2$ where b is a constant, find $\frac{dy}{dx}$.

(b) (i) If $Q = \sqrt{ab} + \sqrt{b}$ where b is a constant, find $\frac{dQ}{da}$.

(ii) If $D = 3(av)^2$ where a is a constant, find $\frac{dD}{dv}$.

5. (a) (i) If $y = x^3 - 5x$, find $\frac{d^2y}{dx^2}$ when $x = 9$.

(ii) If $y = 8 + 2x^4$, find $\frac{d^2y}{dx^2}$ when $x = 4$.

(b) (i) If $S = 3A^2 + \frac{1}{A}$, find $\frac{d^2S}{dA^2}$ when $A = 1$.

(ii) If $J = v - \sqrt{v}$, find $\frac{d^2J}{dv^2}$ when $v = 9$.

(c) (i) Find the second derivative of B with respect to n if $B = 8n$ and $n = 2$.

(ii) Find the second derivative of g with respect to r if $g = r^7$ and $r = 1$.

6. (a) (i) If $y = 3x^3$ and $\frac{dy}{dx} = 36$, find x.

(ii) If $y = x^4 + 2x$ and $\frac{dy}{dx} = 5$, find x.

(b) (i) If $y = 2x + \frac{8}{x}$ and $\frac{dy}{dx} = -31$, find y.

(ii) If $y = \sqrt{x} + 3$ and $\frac{dy}{dx} = \frac{1}{6}$ find y.

7. (a) (i) Find the interval in which $x^3 - 4x$ is an increasing function.

(ii) Find the interval in which $x^3 - 3x^2$ is a decreasing function.

(b) (i) Find the interval in which $3x + \frac{2}{x}$ is a decreasing function.

(ii) Find the interval in which $x - \sqrt{x}$ is an increasing function.

(c) (i) Find the interval in which the graph of $y = x^3 - 4x + 3$ is concave up.

(ii) Find the interval in which the graph of $y = x^3 + 6x^2 - 1$ is concave up.

(d) (i) Find the set of values of x for which the graph of $f(x) = x^4 - 6x^3 + 12x^2$ is concave down.

(ii) Find the set of values of x for which the graph of $f(x) = x^4 - 54x^2$ is concave down.

8. Find all points of the graph of $y = x^3 - 2x^2 + 1$ where the gradient equals the y-coordinate. *[5 marks]*

9. In what interval on the graph of $y = 7x - x^2 - x^3$ is the gradient decreasing? *[5 marks]*

10. In what interval on the graph of $y = \frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 - 3x + 6$ is the gradient increasing? *[6 marks]*

11. Find an alternative expression for $\frac{d^n}{dx^n}(x^n)$.

16E Trigonometric functions

Using the techniques from Section 16A we can sketch the derivative of the graph of $y = \sin x$. The result is a graph that looks just like $y = \cos x$. On Fill-in proof sheet 17 'Differentiating trigonometric functions' on the CD-ROM you can see why this is the case. Results for $y = \cos x$ and $y = \tan x$ can be established in a similar manner giving these results:

In Section 18C we will prove the derivative of tan x using the quotient rule.

KEY POINT 16.6

Differentiating trigonometric functions gives:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

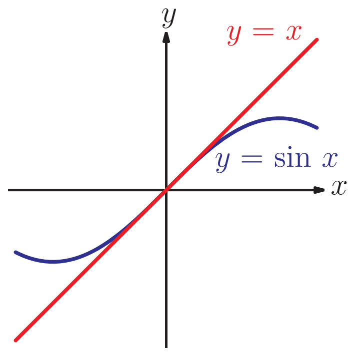
Reciprocal trigonometric functions were covered in Section 12D.

EXAM HINT

Whenever you are doing calculus you MUST work in radians.

It is possible to do calculus using degrees, or any other unit for measuring angles, but using radians gives the simplest rules, which is why they are the unit of choice for almost all mathematicians.

These rules only work if x is measured in radians since they are based upon the result that $\sin x \approx x$ for very small values of x. You can check on your calculator that $\sin x \approx x$ for radians but not for degrees. The result can also be seen on the graph and is proved on Fill-in proof sheet 16 'The small angle approximations' on the CD-ROM.



All rules of differentiation from Section 16C still apply.

Worked example 16.12

Differentiate $y = 3\tan x - 2\cos x$.

Differentiate using the rules in key point 16.6. Note that $\sec^2 x$ can also be written as $\frac{1}{\cos^2 x}$

$$\frac{dy}{dx} = 3(\sec^2 x) - 2(-\sin x)$$
$$= 3\sec^2 x + 2\sin x$$

Exercise 16E

1. Differentiate the following:
   (a) (i) $y = 3\sin x$ (ii) $y = 2\cos x$
   (b) (i) $y = 2x - 5\cos x$ (ii) $y = \tan x + 5$
   (c) (i) $y = \frac{\sin x + 2\cos x}{5}$ (ii) $y = \frac{1}{2}\tan x - \frac{1}{3}\sin x$

2. Find the gradient of $f(x) = \sin x + x^2$ at the point $x = \frac{\pi}{2}$. *[5 marks]*

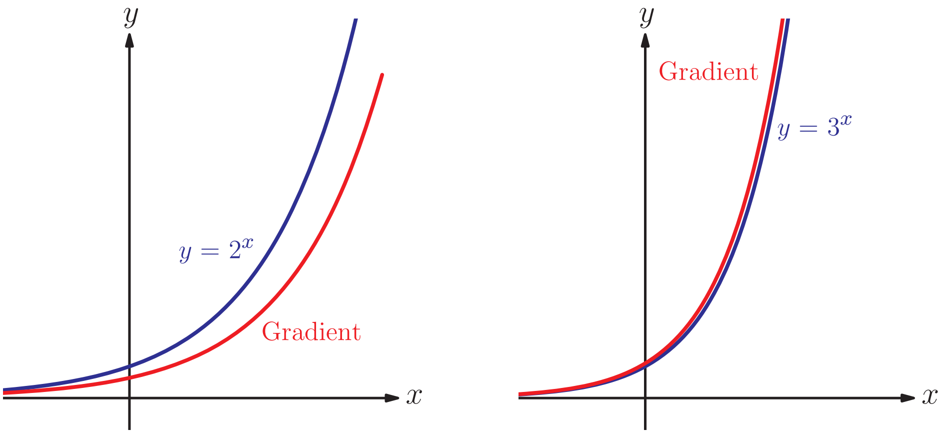
3. Find the gradient of $g(x) = \frac{1}{4}\tan x - 3\cos x - x^3$ at the point $x = \frac{\pi}{6}$. *[5 marks]*

4. Given $h(x) = \sin x + \cos x \quad 0 \le x < 2\pi$, find the values of x for which $h'(x) = 0$. *[6 marks]*

5. Given $y = \frac{1}{4}\tan x + \frac{1}{x^2} \quad 0 < x \le 2\pi$ solve the equation $\frac{dy}{dx} = 1 - \frac{2}{x^3}$. *[6 marks]*

16F The exponential and natural logarithm functions

Use your calculator to plot the graphs of $y = 2^x$ and $y = 3^x$ and their derivatives. The results looks like another exponential function.



It appears that there is a number somewhere between two and three where the derivative of the graph would be exactly the same as the original exponential. It turns out that this is the graph of $y = e^x$ where $e = 2.718\ldots$ It is the same as the base of the natural logarithm defined in Section 2E.

We will see how to differentiate exponential functions with bases other than e in Section 20D.

KEY POINT 16.7

$$\frac{d}{dx}(e^x) = e^x$$

The natural logarithm function $y = \ln x$ behaves in a surprising way, having a derivative of a completely different form.

KEY POINT 16.8

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

This result is proved on Fill-in proof sheet 18 ‘Differentiating logarithmic functions graphically’ on the CD-ROM.

Exercise 16F

1. Differentiate the following:

   (a) (i) $y = 3e^x$ (ii) $y = \frac{2e^x}{5}$

   (b) (i) $y = -2\ln x$ (ii) $y = \frac{1}{3}\ln x$

   (c) (i) $y = \frac{\ln x}{5} - 3x + 4e^x$ (ii) $y = 4 - \frac{e^x}{2} + 3\ln x$

2. (i) Find the exact value of the gradient of the graph of $f(x) = \frac{1}{2}e^x - 7\ln x$ at the point $x = \ln 4$.

   (ii) Find the exact value of the gradient of the graph $f(x) = e^x - \frac{\ln x}{2}$ when $x = \ln 3$. *[4 marks]*

3. Find the value of x where the gradient of $f(x) = 5 - 2e^x$ is –6. *[4 marks]*

4. Find the value of x where the gradient of $g(x) = x^2 - 12\ln x$ is 2. *[4 marks]*

There is an easier way to do some parts of Question 5 using a method from Section 18A. For now, you will have to use your algebra skills!

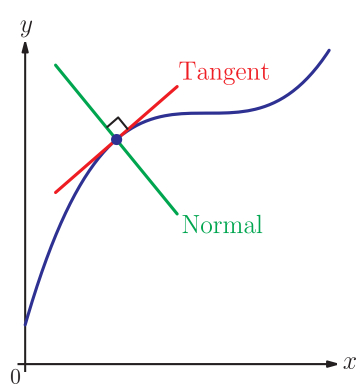
5. Differentiate:

   (a) (i) $y = \ln x^3$ (ii) $y = \ln 5x$

   (b) (i) $y = e^{x+3}$ (ii) $y = e^{x-3}$

   (c) (i) $y = e^{2\ln x}$ (ii) $y = e^{3\ln x + 2}$

   (d) (i) $y = \log_3 x$ (ii) $y = 4\log_6 x$



16G Tangents and normals

The tangent to a curve at a given point is a straight line which touches the curve and has the same gradient at that point. Finding the equation of the tangent at a point relies on knowing the gradient of the function at that point. This can be found by differentiating the function. We then have both the gradient of the line and a point on it and we can use the standard procedure for finding the equation of a straight line.

See Prior learning section W on the CD-ROM.

Normals are lines which pass through the graph and are perpendicular to the tangent. They have many uses, such as finding centres of curvature of graphs and working out how light is reflected from curved mirrors. However, in the International Baccalaureate® you are only likely to be asked to calculate their equations. To do this you use the fact that if two lines with gradients m_1 and m_2 are perpendicular, $m_1 m_2 = -1$.

Worked example 16.13

(a) Find the equation of the tangent to the graph of the function $f(x) = \cos x + e^x$ at the point $x = 0$.

(b) Find the equation of the normal to the graph of the function $g(x) = x^3 - 5x^2 - x^{\frac{3}{2}} + 22$ at (4, −2).

In each case give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

We need the gradient, which is $f'(0)$

(a) $f'(x) = -\sin x + e^x$

$\therefore f'(0) = -\sin 0 + e^0 = 1$

To find the equation of a straight line we also need coordinates of one point

The tangent passes through the point on the graph where $x = 0$. Its y-coordinate is $f(0)$

When $x = 0$,

$$y = f(0) = \cos 0 + e^0 = 1 + 1 = 2$$

Put all the information into the equation of a line

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 1(x - 0)$$
$$\Rightarrow y = x + 2$$
$$\Rightarrow y - x - 2 = 0$$

The normal is perpendicular to the tangent, so we need the gradient of the tangent first

(b) $f'(x) = 3x^2 - 10x - \frac{3}{2}x^{\frac{1}{2}}$

Find the gradient at $x = 4$

$$\therefore f'(4) = 3(4)^2 - 10(4) - \frac{3}{2}(4)^{\frac{1}{2}} = 48 - 40 - 3 = 5$$

For perpendicular lines, $mm_1 = -1$

Therefore gradient of normal,

$$m = \frac{-1}{5}$$

We are given both x and y-coordinates of the point, so put all the information into the equation of a line

$$y - y_1 = m(x - x_1)$$
$$y - (-2) = \frac{-1}{5}(x - 4)$$
$$\Rightarrow 5y + 10 = -x + 4$$
$$\Rightarrow x + 5y + 6 = 0$$

The procedure for finding the equations of tangents and normals can be summarised as follows:

EXAM HINT

Your calculator may be able to find the equation of a tangent at a given point.

KEY POINT 16.9

For the point on the curve $y = f(x)$ with $x = a$:

- the gradient of the tangent is $f'(a)$
- the gradient of the normal is $-\frac{1}{f'(a)}$
- the coordinates of the point are $x_1 = a,\ y_1 = f(a)$.

To find the equation of the tangent or the normal use $y - y_1 = m(x - x_1)$ with the appropriate gradient.

You may not be given the coordinates of the point where the tangent touches the curve, but asked to find them given another point.

Worked example 16.14

The tangent at point P on the curve $y = x^2 + 1$ passes through the origin. Find the possible coordinates of P.

We want to find the equation of the tangent at P, so use unknowns for its coordinates

Let P have coordinates (p, q)

As P lies on the curve, (p, q) satisfies $y = x^2 + 1$

Then $q = p^2 + 1$

The gradient of the tangent is given by $\frac{dy}{dx}$ when $x = p$

$\frac{dy}{dx} = 2x$

When $x = p$: $\frac{dy}{dx} = 2p$

$\therefore m = 2p$

Write the equation of the tangent, remembering it passes through (p, q)

Equation of the tangent:

$y - q = 2p(x - p)$

$\Rightarrow y - (p^2 + 1) = 2p(x - p)$

→

continued . . .

Tangent passes through the origin, so set $x=0, y=0$

Passes through $(0,0)$:
$0-(p^2+1)=2p(0-p)$
$\Rightarrow -p^2-1=-2p^2$
$\Rightarrow p^2=1$
$\Rightarrow p=1$ or -1

We can now find q

When $p=1$, $q=2$
When $p=-1$, $q=2$
So the coordinates of P are (1, 2) or (–1, 2)

Exercise 16G

1. Find the equations of the tangent and normal to the following:
   (a) $y=\dfrac{x^2+4}{\sqrt{x}}$ at $x=4$
   (b) $y=3\tan x-2\sqrt{2}\sin x$ at $x=\dfrac{\pi}{4}$
   (c) $y=3-\dfrac{1}{5}e^x$ at $x=2\ln 5$

2. Find the coordinates of the point on the curve $y=\sqrt{x}+3x$ where the gradient is 5. *[4 marks]*

3. Find the equation of the tangent to the curve $y=e^x+x$ which is parallel to $y=3x$. *[4 marks]*

4. Find the x-coordinates of the points on the curve $y=x^3-3x^2$ where the tangent is parallel to the normal of the point at (1, –1). *[6 marks]*

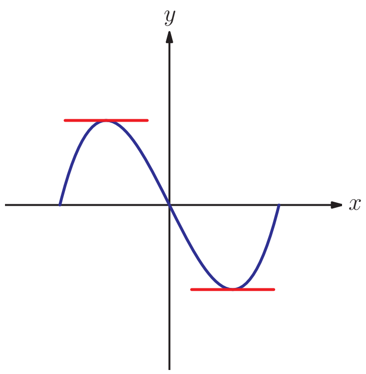
5. Find the coordinates of the point where the tangent to the curve $y=x^3-3x^2$ at $x=2$ meets the curve again. *[6 marks]*

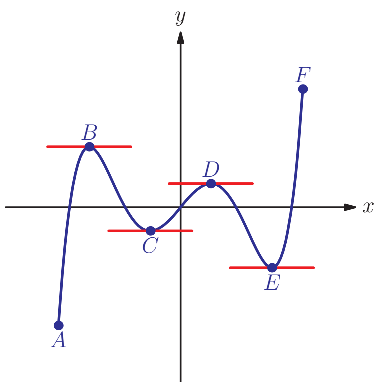
6. Find the coordinates of the point on the curve $y=(x-1)^2$ where the normal passes through the origin. *[5 marks]*

7. Points P and Q lie on the graph of $f(x)=2\sin x$ and have x-coordinates $\dfrac{\pi}{6}$ and $\dfrac{\pi}{4}$.
   (a) Evaluate $f'\left(\dfrac{\pi}{6}\right)$.
   (b) Find the angle between the tangent at P and the chord PQ, giving your answer to the nearest tenth of a degree. *[11 marks]*

8. A tangent is drawn on the graph $y=\dfrac{k}{x}$ at the point where $x=a$, $(a>0)$. The tangent intersects the y-axis at P and the x-axis at Q. If O is the origin show that the area of the triangle OPQ is independent of a. *[8 marks]*

9. Show that the tangent to the curve $y = x^3 - x$ at the point with x-coordinate a meets the curve again at a point with x-coordinate $-2a$. *[6 marks]*





16H Stationary points

In real life people are interested in maximising their profits, or minimising the drag on a car. We can use calculus to describe such things mathematically as points on a graph.

The gradient at both the maximum and minimum point on the above graph is zero and therefore:

KEY POINT 16.10

To find local maximum and local minimum points, we solve the equation $\frac{dy}{dx} = 0$.

We use the phrase **local maximum** and **local minimum** because it is possible that the largest or smallest value of the whole function occurs at the endpoint of the graph, or that there are other points which also have gradient of zero. The points that we have found are just the largest or smallest values of y in that part of the graph.

Points which have a gradient of zero are called **stationary points**.

Worked example 16.15

Find the coordinates of the stationary points of $y = 2x^3 - 15x^2 + 24x + 8$.

Stationary points have $\frac{dy}{dx} = 0$ so we need to differentiate

$\frac{dy}{dx} = 6x^2 - 30x + 24$

Then form an equation

For stationary points $\frac{dy}{dx} = 0$:

$6x^2 - 30x + 24 = 0$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 4)(x - 1) = 0$

$\Rightarrow x = 1$ or $x = 4$

continued . . .

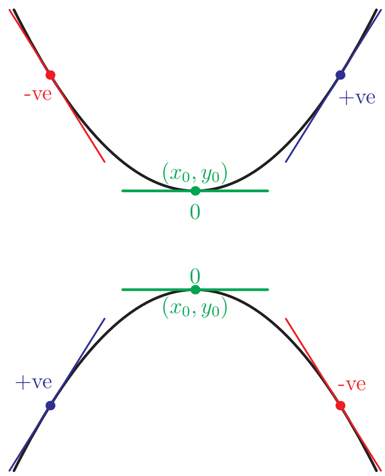
Remember to find the y-coordinate for each point

When $x = 1$:
$y = 2(1)^3 - 15(1)^2 + 24(1) + 8 = 19$
When $x = 4$:
$y = 2(4)^3 - 15(4)^2 + 24(4) + 8 = -8$
Therefore,
stationary points are (1, 19) and (4, −8)

The calculation in Worked example 16.15 does not tell us whether the stationary points we found are maximum or minimum points.

It can be seen from the diagrams that one way of testing for the nature of a stationary point is to look at the gradient either side of the point. You can do this by substituting nearby x-values into the expression for $\frac{dx}{dy}$. For a minimum point the gradient moves from negative to positive. For a maximum point the gradient moves from positive to negative.
We can also interpret these conditions by looking at the sign of the second derivative. Around a minimum point the curve is concave up, so $\frac{d^2y}{dx^2}$ is positive. Around a maximum point the curve is concave down and $\frac{d^2y}{dx^2}$ is negative.

This leads to the following test.



See the end of Section 16D

KEY POINT 16.11

Given a stationary point (x_0, y_0) of a function $y = f(x)$, if:

- $\frac{d^2y}{dx^2} < 0$, at x_0, then (x_0, y_0) is a *maximum*
- $\frac{d^2y}{dx^2} > 0$, at x_0, then (x_0, y_0) is a *minimum*
- $\frac{d^2y}{dx^2} = 0$, at x_0, then no conclusion can be drawn, so test the gradient either side of (x_0, y_0)

Worked example 16.16

Classify the stationary points of the function $y = 2x^3 - 15x^2 + 24x + 8$ from Worked example 16.15.

We have already found the stationary points

Stationary points are $(1, 19)$ and $(4, -8)$

The nature of stationary points is determined by the value of the second derivative

$$\frac{d^2y}{dx^2} = 12x - 30$$

At $x = 1$:

$$\frac{d^2y}{dx^2} = 12(1) - 30 = -18 < 0$$

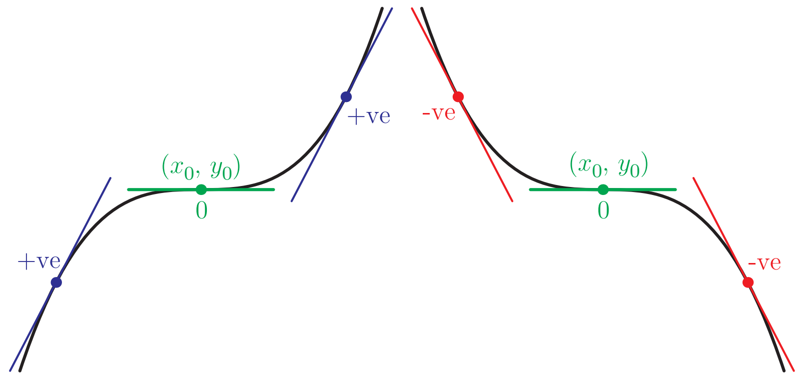
$\therefore (1, 19)$ is a maximum

At $x = 4$:

$$\frac{d^2y}{dx^2} = 12(4) - 30 = 18 > 0$$

$\therefore (4, -8)$ is a minimum

All local maximum points and local minimum points have $\frac{dy}{dx} = 0$, but the reverse is not true: A point with $\frac{dy}{dx} = 0$ does not have to be a maximum or a minimum point. There are two other possibilities:



These possibilities are called **points of inflexion**, and are labelled (x_0, y_0) on the above diagrams. Note that at those points the line with zero gradient actually crosses the curve. The gradient is either positive on both sides of a point of inflexion (positive point of inflexion), or negative on both sides (a negative point of inflexion).

In UK English, 'inflexion' may be spelt 'inflection'.

Worked example 16.17

Find the coordinates and nature of the stationary points of $y = 3 + 4x^3 - x^4$.

Stationary points have $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 12x^2 - 4x^3$

For stationary points $\frac{dy}{dx} = 0$:

$12x^2 - 4x^3 = 0$

$\Rightarrow 4x^2(3 - x) = 0$

$\Rightarrow x = 0$ or $x = 3$

Find y-coordinates

When $x = 0$:

$y = 3 + 4(0)^3 - (0)^4 = 3$

When $x = 3$:

$y = 3 + 4(3)^3 - (3)^4 = 30$

Therefore, stationary points are:

(0, 3) and (3, 30)

The nature of the stationary points is determined by the second derivative

Find the nature of these points:

$\frac{d^2y}{dx^2} = 24x - 12x^2$

At $x = 0$:

$\frac{d^2y}{dx^2} = 24(0) - 12(0)^2 = 0$

As $\frac{d^2y}{dx^2} = 0$ we need to check the gradient either side of the stationary point

Therefore, examine $\frac{dy}{dx}$:

At $x = -1$:

$\frac{dy}{dx} = 12(-1)^2 - 4(-1)^3 = 16 > 0$

At $x = 1$:

$\frac{dy}{dx} = 12(1)^2 - 4(1)^3 = 8 > 0$

$\therefore (0, 3)$ is a positive point of inflexion.

continued . . .

At $x = 3$:

$\frac{d^2y}{dx^2} = 24(3) - 12(3)^2 = -36 < 0$

$\therefore (3, 30)$ is a maximum

When $\frac{d^2y}{dx^2} = 0$, the stationary point is NOT always a point of inflexion.

Worked example 16.18

Find the coordinates and nature of the stationary points of $f(x) = x^4$:

$f'(x) = 4x^3$

Stationary points have $f'(x) = 0$

For stationary points $f'(x) = 0$

$4x^3 = 0$

$\Rightarrow x = 0$

Find the y-coordinate

$f(0) = 0$

Therefore, stationary point is:

$(0, 0)$

The nature is determined by $f''(x)$

Find the nature of this point:

$f''(x) = 12x^2$

$f''(0) = 0$

As $f''(0) = 0$, we need to check the gradient on either side

Therefore, examine $f'(x)$:

$f'(-1) = 4(-1)^3 = -4 < 0$

$f'(1) = 4(1)^3 = 4 > 0$

Therefore $(0, 0)$ is a minimum.

Exercise 16H

1. Find and classify the stationary points on the following curves:
   (a) (i) $y = x^3 - 5x^2$ (ii) $y = x^4 - 8x^2$
   (b) (i) $y = \sin x + \frac{x}{2}, \quad -\pi \le x \le \pi$
   (ii) $y = 2\cos x + 1,\ 0 \le x < 2\pi$
   (c) (i) $y = \ln x - \sqrt{x}$ (ii) $y = 2e^x - 5x$

2. Give an example to illustrate that the following statement is incorrect:
   'If $y = f(x)$ has exactly two stationary points, at x_1 and x_2, and $f(x_1) > f(x_2)$ then $(x_1, f(x_1))$ must be a local maximum.'
   Under what conditions is the statement true?

3. Find and classify the stationary points on the curve $y = x^3 + 3x^2 - 24x + 12$. *[6 marks]*

4. Find and classify the stationary points on the curve $y = x - \sqrt{x}$. *[6 marks]*

5. Find and classify the stationary points on the curve $y = \sin x + 4\cos x$ in the interval $0 < x < 2\pi$. *[6 marks]*

6. Show that the function $f(x) = \ln x + \frac{1}{x^k}$ has a stationary point with y-coordinate $\frac{\ln k + 1}{k}$. *[6 marks]*

7. Find the range of the function $f : x \mapsto 3x^4 - 16x^3 + 18x^2 + 6$. *[5 marks]*

8. Find the range of the function $f : x \mapsto e^x - 4x + 2$. *[5 marks]*

9. Find and classify in terms of k the stationary points on the curve $y = kx^3 + 6x^2$. *[6 marks]*

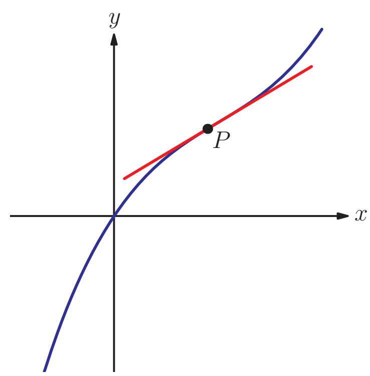
16I General points of inflexion

In the previous section we met stationary points of inflexion, but the idea of a point of inflexion is more general than this.

One definition is that the tangent to the curve at a point of inflexion crosses the curve at the same point. This does not require the point to be a stationary point.

EXAM HINT

Although the red line actually crosses the graph at P, it is still referred to as the tangent, because it has the same gradient as the curve at P.



See the end of Section 16D.

Geometrically, this can be interpreted as an 'S-bend', a curve which goes from decreasing gradient to increasing gradient (or vice versa). This means that the curve is concave down on one side of the point of inflexion and concave up on the other. We know that this corresponds to the second derivative changing from negative to positive (or vice versa).

KEY POINT 16.12

At a point of inflexion $\frac{d^2y}{dx^2} = 0$.

EXAM HINT

If a question states that a curve does have a point of inflexion and there is only one solution to the equation $\frac{d^2y}{dx^2} = 0$, you can then assume you have found the point of inflexion.

Unfortunately, as in Worked example 16.18, just because a point has $\frac{d^2y}{dx^2} = 0$ it is not necessarily a point of inflexion. We have to determine the gradient on either side to be sure.

Worked example 16.19

Find the coordinates of the point of inflexion on the curve $y = x^3 - 3x^2 + 5x - 1$.

Find $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 3x^2 - 6x + 5$

$\frac{d^2y}{dx^2} = 6x - 6$

At a point of inflexion $\frac{d^2y}{dx^2} = 0$,

$6x - 6 = 0$

$x = 1$

Remember the other coordinate!

When $x = 1$, $y = 1 - 3 + 5 - 1 = 2$

So point of inflexion is at (1, 2)

Exercise 16I

1. Find the coordinates of the point of inflexion on the curve $y = e^x - x^2$. *[5 marks]*

2. The curve $y = x^4 - 6x^2 + 7x + 2$ has two points of inflexion. Find their coordinates. *[5 marks]*

3. Show that all points of inflexion on the curve $y = \sin x$ lie on the x-axis. *[6 marks]*

4. Find the coordinates of the points of inflexion on the curve $y = 2\cos x + x$ for $0 \le x \le 2\pi$. Justify carefully that these points are points of inflexion. *[5 marks]*

5. The point of inflexion on the curve $y = x^3 - ax^2 - bx + c$ is a stationary point of inflexion. Show that $b = 8a^2$. *[6 marks]*

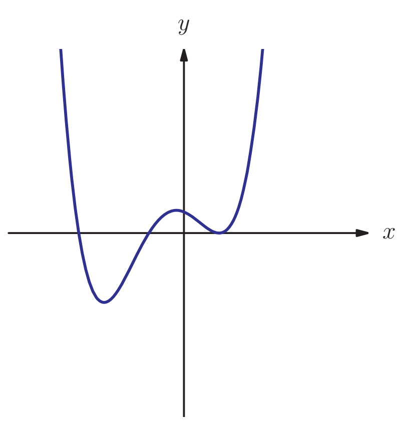
6. The graph shows $y = f'(x)$.

   On a copy of the diagram:

   (a) mark any point corresponding to a local minimum of $f(x)$ with an A

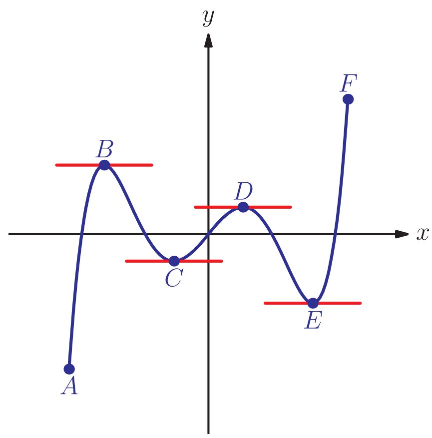
   (b) mark any point corresponding to a local maximum of $f(x)$ with a B

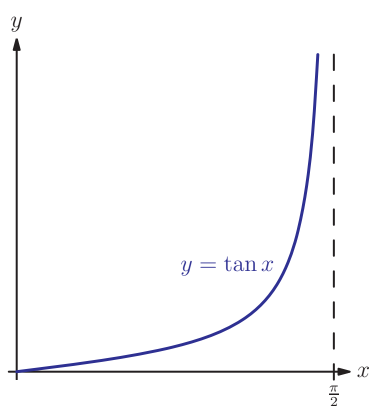
   (c) mark any point corresponding to a point of inflexion of $f(x)$ with a C. *[4 marks]*



16J Optimisation

We can now start to use differentiation to maximise or minimise quantities. In Section 16H we saw how to find stationary points (the points with zero gradient) and how to decide whether they are local maximum or local minimum points. We also noted that a stationary point does not necessarily give the largest or smallest value of the function over the whole domain. For example, on the diagram, points B and D are local maximum points, but the largest value of the function occurs at point F, which is an end point of the domain.





Some functions do not have maximum or minimum values at all. This can happen when the graph has an asymptote. We say that the function is not continuous throughout its domain. For example, the value of tan x increases without a limit as x increases towards $\frac{\pi}{2}$, so tan x does not have a maximum value.

If we wish to minimise or maximise A by changing B we do so in four stages:

1. Find the relationship between A and B.
2. Solve the equation $\frac{dA}{dB} = 0$.
3. Decide whether it is a maximum, minimum or point of inflexion by considering $\frac{d^2A}{dB^2}$.
4. Check whether the end points of the domain are actually global maximum or minimum points, and check that there are no vertical asymptotes.

Often the first stage of this process is the most difficult and there are many questions where we have to use a geometric context to make this link. Thankfully in many questions this relationship is given to you.

Worked example 16.20

The height of a swing (h) in metres at a time t seconds is given by $h = 2 - 1.5\sin t$ for $0 < t < 3$. Find the minimum and maximum height of the swing.

Find stationary points

$\frac{dh}{dt} = -1.5\cos t = 0$ at a stationary point

$\Rightarrow \cos t = 0$

$0 < t < 3 \therefore t = \frac{\pi}{2}$ (only one solution)

Classify stationary points

$\frac{d^2h}{dt^2} = 1.5\sin t$

When $t = \frac{\pi}{2}, \frac{d^2h}{dt^2} = 1.5 > 0$ so $t = \frac{\pi}{2}$ is a local minimum. This minimum height is

$h = 2 - 1.5\sin\frac{\pi}{2} = 0.5$ metres

Check end points

Check there are no vertical asymptotes

When $t = 0$, $h = 2$m

When $t = 3$, $h = 1.79$m (3SF)

So maximum height is 2 m.

Exercise 16J

1. What are the minimum and maximum values of the expression e^x for $0 \le x \le 1$? *[4 marks]*

2. A rectangle has width x metres and length $30 - x$ metres.
   (a) Find the maximum area of the rectangle.
   (b) Show that as x changes the perimeter stays constant and find the value of this perimeter. *[5 marks]*

3. Find the maximum and minimum values of the function $y = x^3 - 9x$ if $-2 \le x \le 5$. *[5 marks]*

4. What are the maximum and minimum values of $f(x) = e^x - 3x$ if $0 \le x \le 2$? *[5 marks]*

5. What are the minimum and maximum values of $y = \sin x + 2x$ for $0 \le x \le 2\pi$? *[5 marks]*

6. Find the minimum value of the sum of a positive real number and its reciprocal. *[5 marks]*

7. A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1-\dfrac{1}{\sqrt{w}}$ ms^{-1} for a time of $\dfrac{5}{w}$ s. What weight will achieve the maximum distance travelled? *[6 marks]*

8. The time in minutes (t) taken to melt 100 g of butter depends upon the percentage of the butter which is made of saturated fats (p) as in the following function:

$$t = \frac{p^2}{10\,000} + \frac{p}{100} + 2$$

Find the maximum and minimum times to melt 100 g of butter. *[6 marks]*

9. The volume of water in millions of litres (V) in a new tidal lake is modelled by $V = 60\cos t + 100$ where t is the time in days after being completed.

(a) What is the smallest volume of the lake?

(b) A hydroelectric plant produces an amount of electricity proportional to the rate of flow of water. In the first 6 days when is the plant producing maximum electricity? *[6 marks]*

10. The owner of a fast-food shop finds that there is a relationship between the amount of salt s (g/tray) added to the fries and his weekly sales of fries F (100s of portions):

$$F(s) = 4s + 1 - s^2, \quad 0 \le s \le 4.2$$

(a) Find the amount of salt he should put on his fries to maximise his sales.
The total cost C (\$ per tray) associated with the sales of fries is given by:

$$C(s) = 0.3 + 0.2F(s) + 0.1s$$

(b) Find the amount of salt he should put on his fries to minimise his costs.

(c) The profit made on his fries is given by the difference between the sales and the costs.
How much salt should he add to maximise his profit? *[8 marks]*

11. A car tank is being filled with petrol such that the volume in the tank in litres (V) over time in minutes (t) is given by

$$V = 300(t^2 - t^3) + 4 \quad \text{for } 0 < t < 0.5$$

(a) How much petrol was initially in the tank?

(b) After 30 seconds the tank was full. What is the capacity of the tank?

(c) At what time is petrol flowing in at the greatest rate? *[8 marks]*

12. x is the surface area of leaves on a tree in m^2. Because leaves may be shaded by other leaves, the amount of energy produced by the tree is given by $2 - \frac{x}{10}$ kJ per square metre of leaves.

(a) Find an expression for the total energy produced by the tree.

(b) What area of leaves provides the maximum energy for the tree?

(c) Leaves also use energy. The total energy requirement is given by $0.01x^3$. The net energy produced is the difference between the energy produced by the leaves and the energy used by the leaves. For what range of x do the leaves produce more energy than they use?

(d) Show that the maximum net energy is produced when the tree has leaves with a surface area of $\frac{10(\sqrt{7}-1)}{3}$. *[12 marks]*

Summary

- The **gradient** of a function at the point P is the gradient of the **tangent** to the function's graph at that point.
- To find the gradient of a function we can **differentiate** from first principles:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ (also denoted by } \frac{d}{dx} f(x))$$
- For the point on the curve $y = f(x)$ with $x = a$:
  – the gradient of the tangent is $f'(a)$
  – the gradient of the **normal** is $-\frac{1}{f'(a)}$
- If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- The **derivative** of a sum is the sum of the derivatives of each term.
- If we differentiate $kf(x)$ where k is a constant we get $kf'(x)$.
- The **derivatives** of the trigonometric functions are:
$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$
- The derivatives of the exponential and natural logarithm functions are:
$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$
- **Stationary points** of a function are points where the gradient is zero, i.e.
$$\frac{dy}{dx} = 0$$

- Stationary points can be one of four types:
  - local maximum
  - local minimum
  - positive point of inflexion
  - negative point of inflexion.
- The **second derivative** can be used to test which of these occurs. At a stationary point (x_0, y_0), if
  - $\frac{d^2y}{dx^2} < 0$ at x_0 then (x_0, y_0) is a maximum
  - $\frac{d^2y}{dx^2} > 0$ at x_0 then (x_0, y_0) is a minimum
  - $\frac{d^2y}{dx^2} = 0$ at x_0 then no conclusion can be drawn, so check the sign of the gradient either side of (x_0, y_0).
- **Points of inflexion** can also have a non-zero gradient.
- At a point of inflexion $\frac{d^2y}{dx^2} = 0$.
- Global maximum or minimum points may also occur at the endpoint of a graph.

Introductory problem revisited

The cost of petrol used in a car, in £ per hour, is $\frac{12+v^2}{100}$ where v is measured in miles per hour and $v > 0$. If Daniel wants to travel 50 miles as cheaply as possible, at what speed should he travel?

If we have the cost per hour and we want the total cost we must find the total time. But the time taken is $\frac{50}{v}$ hours, so the total cost is $C = \frac{50}{v}\left(12 + \frac{v^2}{100}\right) = \frac{600}{v} + \frac{v}{2}$.

If we wish to find a minimum value of C by changing v we can do this by setting $\frac{dC}{dv} = 0$:

$$-\frac{600}{v^2} + \frac{1}{2} = 0$$
$$\Rightarrow v^2 = 1200$$

$v = 34.6\,\text{mph}$ (3SF) (Taking the positive root since $v > 0$)

To see if we have found a minimum we find $\frac{d^2C}{dv^2} = 1200v^{-3}$ which is positive for any positive v, so the point is a local minimum.

Next, to see if it is global minimum we must consider the end points. Although v is never actually zero as it gets close to it, the $\frac{600}{v}$ term gets very large. When v gets very large the $\frac{v}{2}$ term gets very large. Therefore we have found the global minimum.

Mixed examination practice 16

Short questions

1. Find the equation of the tangent to the curve $y = e^x + 2\sin x$ at the point where $x = \frac{\pi}{2}$. *[5 marks]*

2. Find the equation of the normal to the curve $y = (x-2)^3$ when $x = 2$. *[5 marks]*

3. $f(x)$ is a quadratic function taking the form $x^2 + bx + c$. If $f(1) = 2$ and $f'(2) = 12$ find the values of b and c. *[5 marks]*

4. Find the coordinates of the point of inflexion on the graph of $y = \frac{x^3}{6} - x^2 + x$. *[6 marks]*

5. Find and classify the stationary points on the curve $y = \tan x - \frac{4x}{3}$. *[6 marks]*

6. Let f be a cubic polynomial function. Given that $f(0) = 2$, $f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$. *[2 marks]*

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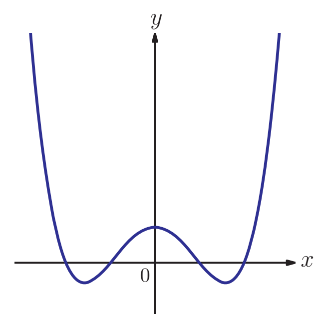
7. The graph shows $y = f'(x)$:



On a sketch of this graph:

(a) Mark points corresponding to a local minimum of $f(x)$ with an A.

(b) Mark points corresponding to a local maximum of $f(x)$ with a B.

(c) Mark points corresponding to a point of inflexion of $f(x)$ with a C. *[6 marks]*

8. On the curve $y = x^3$ a tangent is drawn from the point (a, a^3), $a > 0$ and a normal is drawn from the point $(-a, -a^3)$. The tangent and the normal meet on the y-axis. Find the value of a. *[6 marks]*

Long questions

1. The line $y = 24(x-1)$ is tangent to the curve $y = ax^3 + bx^2 + 4$ at $x = 2$.

(a) Use the fact that the tangent meets the curve to show that $2a + b = 5$.

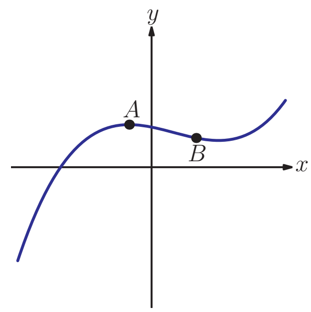
(b) Use the fact that the tangent has the same gradient as the curve to find another relationship between a and b.

(c) Hence find the values of a and b.

(d) The line meets the curve again. Find the coordinates of the other point of intersection. *[12 marks]*

2. The graph shows part of $y = x^3 - x^2 - x + 3$.

The point A is a local maximum and the point B is a point of inflexion.



(a) (i) Find the coordinates of A.

(ii) Find the coordinates of B.

(b) (i) Find the equation of the line containing both A and B.

(ii) Find the x coordinate of the points on the curve at which the tangent is parallel to this line. *[10 marks]*

3. (a) Sketch and label the curves $y = x^2$ for $-2 \le x \le 2$, and $y = -\frac{1}{2}\ln x$ for $0 < x \le 2$.

(b) Find the x-coordinate of P, the point of intersection of the two curves.

(c) If the tangents to the curves at P meet the y-axis at Q and R, calculate the area of the triangle PQR.

(d) Prove that the two tangents at the points where $x = a, a > 0$, on each curve are always perpendicular.

[14 marks]

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4. The population of bacteria (P) in thousands at a time t in hours is modelled by:

$$P = 10 + e^t - 3t,\ t \ge 0$$

(a) (i) Find the initial population of bacteria.

(ii) At what time does the number of bacteria reach 14 million?

(b) (i) Find $\frac{dP}{dt}$.

(ii) Find the time at which the bacteria are growing at a rate of 6 million per hour.

(c) (i) Find $\frac{d^2P}{dt^2}$ and explain the physical significance of this quantity.

(ii) Find the minimum number of bacteria, justifying that it is a minimum.

[12 marks]

17 Basic integration and its applications

In this chapter you will learn:

- to reverse the process of differentiation (this process is called integration)
- to find the equation of a curve given its derivative and a point on the curve
- to integrate sin x, cos x and tan x
- to integrate e^x and $\frac{1}{x}$
- to find the area between a curve and the x- or y-axis
- to find the area enclosed between two curves.

Introductory problem

The amount of charge stored in a capacitor is given by the area under the graph of current (I) against time (t). When it contains alternating current the relationship between I and t is given by $I = \sin t$. When it contains direct current the relationship between I and t is given by $I = k$. What value of k means that the amount of charge stored in the capacitor from $t = 0$ to $t = \pi$ is the same whether alternating or direct current is used?

As in many areas of mathematics, as soon as we learn a new process we must then learn how to undo it. However, it turns out that undoing the process of differentiation opens up the possibility of answering a seemingly unconnected problem: what is the area under a curve?

17A Reversing differentiation

We saw in the last chapter how differentiation gives us the gradient of a curve or the rate of change of one quantity with another. What then if we already know the function describing a curve's gradient, or the expression for a rate of change, and wish to find the original function? Our only way of proceeding is to 'undo' the differentiation that has already taken place and this process of reverse differentiation is known as **integration**.

Let us look at two particular cases to get a feel for this process. Each time we will be given $\frac{dy}{dx}$ and need to answer the question 'What was differentiated to give this?'

If $\frac{dy}{dx} = 2x$ then the original function y must have contained x^2 as we know that differentiation decreases the power by 1. Differentiating x^2 gives exactly $2x$, so we have found that if $\frac{dy}{dx} = 2x$ then $y = x^2$.

If $\frac{dy}{dx} = x^{\frac{1}{2}}$ then the original function y must have contained $x^{\frac{3}{2}}$. Differentiating $x^{\frac{3}{2}}$ will give $y = \frac{3}{2}x^{\frac{1}{2}}$ and we do not want the $\frac{3}{2}$. However, if we multiply the $x^{\frac{3}{2}}$ by $\frac{2}{3}$ then when we differentiate the coefficient cancels to 1, so if $\frac{dy}{dx} = x^{\frac{1}{2}}$ then $y = \frac{2}{3}x^{\frac{3}{2}}$.

Writing out 'if $\frac{dy}{dx} = x^{\frac{1}{2}}$ then $y = \frac{2}{3}x^{\frac{3}{2}}$, is descriptive but rather laborious and so the notation used for integration is:

$$\int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}$$

Here, the dx simply states that the integration is taking place with respect to the variable x in exactly the same way that in $\frac{dy}{dx}$ it states that the differentiation is taking place with respect to x. We could equally well write, for example, $\int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}}$.

The integration symbol comes from the old English way of writing the letter 'S'. Originally it stood for the word 'Sum' (or rather, ∫um). As you will see in later sections, the integral does indeed represent a sum of infinitesimally small quantities.

Exercise 17A

You may have heard of the term 'differential equation'. These are the simplest types of differential equation.

1. Find a possible expression for y in terms of x:

(a) (i) $\frac{dy}{dx} = 3x^2$ (ii) $\frac{dy}{dx} = 5x^4$

(b) (i) $\frac{dy}{dx} = -\frac{1}{x^2}$ (ii) $\frac{dy}{dx} = -\frac{4}{x^5}$

(c) (i) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ (ii) $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$

(d) (i) $\frac{dy}{dx} = 10x^4$ (ii) $\frac{dy}{dx} = 12x^2$

17B Constant of integration

We have seen how to integrate some functions of the form x^n by reversing the process of differentiation but the process as carried out above was not complete.

Let us consider again the first example where we stated that: $\int 2x\,dx = x^2$.

Were there any other possible answers here?

We could have given $\int 2x\,dx = x^2 + 1$ or $\int 2x\,dx = x^2 - \frac{3}{5}$.

Both of these are just as valid as our original answer; we know that when we differentiate the constant ($+1$ or $-\frac{3}{5}$) we just get 0. We could therefore have given any constant; without further information we cannot know what this constant on the original function was before it was differentiated.

Hence our complete answers to the integrals considered in Section 17A are:

$$\int 2x\,dx = x^2 + c$$

$$\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$$

where the c is an unknown **constant of integration**.

We will see later that, given further information, we can find this constant.

We will see how to find the constant of integration in Section 17F.

Exercise 17B

1. Give three possible functions which when differentiated with respect to x give the following:

   (a) $3x^3$

   (b) 0

2. Find the integrals:

(a) (i) $\int 7x^4\,dx$ (ii) $\int \frac{1}{3}x^2\,dx$

(b) (i) $\int \frac{1}{2t^2}\,dt$ (ii) $\int \frac{8}{y^3}\,dy$

17C Rules of integration

To find integrals so far we have used the idea of reversing differentiation for each specific function. Let us now think about applying the reverse process to the general rule of differentiation.

We know that for $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$ or in words:

To differentiate x^n multiply by the old power then decrease the power by 1.

We can express the reverse of this process as follows.

To integrate x^n increase the power by 1 then divide by the new power.

Using integral notation:

We will see how to integrate x^{-1} in Section 17D.

KEY POINT 17.1

The general rule for integrating x^n for any rational power $n \neq -1$ is:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$$

EXAM HINT

The + c is a part of the answer, and you must write it every time.

Note the condition n $\neq -1$ which ensures that we are not dividing by zero.

It is worth remembering the formula below for integrating a constant: $\int k\,dx = kx + c$, which is a special case of the rule in Key point 17.1

$$\int k\,dx = \int kx^0\,dx = \frac{k}{1}x^1 + c$$

In Key point 16.3, we saw that if we differentiate $kf(x)$ we get $kf'(x)$; we can reverse this logic to show that:

KEY POINT 17.2

To integrate multiples of functions:

$$\int kf(x)\,dx = k\int f(x)\,dx$$

EXAM HINT

This rule only works if k is a constant.

Since we can differentiate term by term (also in Key point 16.4) then we can also split up integrals of sums.

KEY POINT 17.3

For the sum of integrals:

$$\int f(x)+g(x)\,dx=\int f(x)\,dx+\int g(x)\,dx$$

EXAM HINT

Be warned! You cannot integrate products or quotients by integrating each part separately.

By combining Key points 17.2 and 17.3 with $k=-1$, we can also show that the integral of a difference is the difference of the integrals of the separate parts.

These ideas are demonstrated in the following examples.

Worked example 17.1

Find (a) $\int 6x^{-3}\,dx$ (b) $\int (3x^4-8x^{-\frac{4}{3}}+2)\,dx$

Add one to the power and divide by this new power

(a) $\int 6x^{-3}\,dx=\frac{6}{-3+1}x^{-3+1}+c$

Tidy up

$=\frac{6}{-2}x^{-2}+c$

$=-3x^{-2}+c$

Go through term by term adding one to the power of x and dividing by this new power

Remember the rule for integrating a constant

(b) $\int 3x^4-8x^{-\frac{4}{3}}+2dx=\frac{3}{4+1}x^{4+1}-\frac{8}{-\frac{4}{3}+1}x^{-\frac{4}{3}+1}+2x+c$

Tidy up

$=\frac{3}{5}x^5-\frac{8}{-\frac{1}{3}}x^{-\frac{1}{3}}+2x+c$

$=\frac{3}{5}x^5+24x^{-\frac{1}{3}}+2x+c$

Just as for differentiation, it may be necessary to change terms into the form kx^n before integrating.

Worked example 17.2

Find (a) $\int 5x^2\sqrt[3]{x}\,dx$ (b) $\int \frac{(x-3)^2}{\sqrt{x}}\,dx$

Write the cube root as a power and use rules of exponents

(a) $\int 5x^2\sqrt[3]{x}dx = \int 5x^2x^{1/3}dx = \int 5x^{7/3}dx$

Dividing by $\frac{10}{3}\left(\frac{7}{3}+1\right)$ is the same as multiplying by $\frac{3}{10}$

$= 5\times\frac{3}{10}\times x^{\frac{10}{3}} + c = \frac{3}{2}x^{\frac{10}{3}} + c$

Expand the brackets first, then use rules of exponents

(b) $\int\frac{(x-3)^2}{\sqrt{x}}dx = \int\frac{x^2-6x+9}{x^{\left(\frac{1}{2}\right)}}dx$

$= \int x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}\,dx$

Dividing by a fraction is the same as multiplying by its reciprocal

$= \frac{2}{5}x^{\frac{5}{2}} - 6\times\frac{2}{3}x^{\frac{3}{2}} + 9\times 2x^{\frac{1}{2}} + c$

$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$

Exercise 17C

EXAM HINT

In the integral do not forget the dx or the equivalent. We will make more use of it later! The function you are integrating is actually being multiplied by dx so you could write question 1(f)(ii) as $\int\frac{2dx}{x^3}$.

1. Find the following integrals:

(a) (i) $\int 9x^8\,dx$ (ii) $\int 12x^{11}\,dx$

(b) (i) $\int x\,dx$ (ii) $\int x^3\,dx$

(c) (i) $\int 9\,dx$ (ii) $\int \frac{1}{2}\,dx$

(d) (i) $\int 3x^5\,dx$ (ii) $\int 9x^4\,dx$

(e) (i) $\int 3\sqrt{x}\,dx$ (ii) $\int 3\sqrt[3]{x}\,dx$

(f) (i) $\int \frac{5}{x^2}dx$ (ii) $\int \frac{2}{x^3}\,dx$

2. Find the following integrals:

(a) (i) $\int 3\,dt$ (ii) $\int 7\,dz$

(b) (i) $\int q^5\,dq$ (ii) $\int r^{10}\,dr$

(c) (i) $\int 12g^{\frac{3}{5}}\,\mathrm{d}g$ (ii) $\int 5y^{\frac{7}{2}}\,\mathrm{d}y$

(d) (i) $\int 4\frac{\mathrm{d}h}{h^2}$ (ii) $\int \frac{\mathrm{d}p}{p^4}$

3. Find the following integrals:

(a) (i) $\int x^2 - x^3 + 2\,\mathrm{d}x$ (ii) $\int x^4 - 2x + 5\,\mathrm{d}x$

(b) (i) $\int \frac{1}{3t^3} + \frac{1}{4t^4}\,\mathrm{d}t$ (ii) $\int 5\times\frac{1}{v^2} - 4\times\frac{1}{v^5}\,\mathrm{d}v$

(c) (i) $\int x\sqrt{x}\,\mathrm{d}x$ (ii) $\int \frac{3\sqrt{x}}{\sqrt[3]{x}}\,\mathrm{d}x$

(d) (i) $\int (x+1)^3\,\mathrm{d}x$ (ii) $\int x(x+2)^2\,\mathrm{d}x$

4. Find $\int \frac{1+x}{\sqrt{x}}\,\mathrm{d}x$. *[4 marks]*

17D Integrating x^{-1} and e^x

When integrating $\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$, we were careful to exclude the case $n = -1$.

In Key point 16.8 we saw that $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$. Reversing this gives:

KEY POINT 17.4

$$\int x^{-1}\,\mathrm{d}x = \ln x + c$$

We will modify this rule in Section 17H.

In Key point 16.7, we saw that $\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$. We can use this to integrate the exponential function:

KEY POINT 17.5

$$\int e^x\,\mathrm{d}x = e^x + c$$

Exercise 17D

1. Find the following integrals:

(a) (i) $\int \frac{2}{x} dx$ (ii) $\int \frac{3}{x} dx$

(b) (i) $\int \frac{1}{2x} dx$ (ii) $\int \frac{1}{3x} dx$

(c) (i) $\int \frac{x^2 - 1}{x} dx$ (ii) $\int \frac{x^3 + 5}{x} dx$

(d) (i) $\int \frac{3x + 2}{x^2} dx$ (ii) $\int \frac{x - \sqrt{x}}{x^2} dx$

2. Find the following integrals:

(a) (i) $\int 5e^x \, dx$ (ii) $\int 9e^x \, dx$

(b) (i) $\int \frac{2e^x}{5} dx$ (ii) $\int \frac{7e^x}{11} dx$

(c) (i) $\int \frac{(e^x + 3x)}{2} dx$ (ii) $\int \frac{(e^x + x^3)}{5} dx$

17E Integrating trigonometric functions

We can expand the set of functions that we can integrate by continuing to refer back to work covered in chapter 16.

We saw in Key point 16.6 that $\frac{d}{dx}(\sin x) = \cos x$ which means that $\int \cos x \, dx = \sin x + c$.

Similarly, as $\frac{d}{dx}(\cos x) = -\sin x$, then $\int \sin x \, dx = -\cos x + c$.

KEY POINT 17.6

The integrals of trigonometric functions:

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

See Exercise 19B for establishing this result.

We do not have a function whose derivative is $\tan x$ and so have no way (yet) of finding $\int \tan x$. We will meet a method that enables us to establish this in chapter 19, but for completeness the result is given here:

KEY POINT 17.6a

$$\int \tan x = \ln|\sec x|$$

EXAM HINT

The integral of tan x is not given in the Formula booklet and is worth remembering

Exercise 17E

1. Find the following integrals:
   (a) (i) $\int \sin x - \cos x \, dx$ (ii) $\int 3\cos x + 4\sin x \, dx$
   (b) (i) $\int 1 + \tan x \, dx$ (ii) $\int \frac{\sin x}{2} + \frac{\tan x}{3} \, dx$
   (c) (i) $\int \frac{x + \sin x}{7} dx$ (ii) $\int \frac{\sqrt{x} + \cos x}{6} dx$
   (d) (i) $\int 1 - (\cos x + \sin x) dx$ (ii) $\int \cos x - 2(\cos x - \sin x) dx$

2. Find $\int \frac{\sin x + \cos x}{2\cos x} dx$. *[5 marks]*

3. Find $\int \frac{\cos 2x}{\cos x - \sin x} dx$. *[5 marks]*

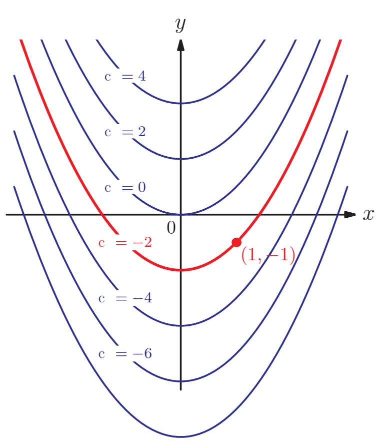
17F Finding the equation of a curve

We have seen how we can integrate the function $\frac{dy}{dx}$ to find the equation of the original curve, except for the unknown constant of integration. This is because the gradient, $\frac{dy}{dx}$, determines the shape of the curve, but not exactly where it is. However, if we are also given the coordinates of a point on the curve we can then determine the constant and hence specify the original function precisely.

Look back to Worked example 16.2 where, given the gradient, we could draw many different curves by changing the starting point.

If we again consider $\frac{dy}{dx} = 2x$ which we met at the start of this chapter, we know that the original function must have equation $y = x^2 + c$ for some constant value c.

If we are also told that the curve passes through the point (1, −1), we can find c and specify which of the family of curves our function must be.



Worked example 17.3

The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 5$ and the curve passes through the point (1, −4). Find the equation of the curve.

To find y from $\frac{dy}{dx}$ we need to integrate

Don't forget + c

$$y = \int 3x^2 - 8x + 5 dx$$

$$= x^3 - 4x^2 + 5x + c$$

The coordinates of the given point must satisfy this equation, so we can find c.

When $x = 1$, $y = -4$, so

$$-4 = (1)^3 - 4(1)^2 + 5(1) + c$$

$$\Rightarrow -4 = 1 - 4 + 5 + c \Rightarrow c = -6$$

$$\therefore y = x^3 - 4x^2 + 5x - 6$$

The above example illustrates the general procedure for finding the equation of a curve from its gradient function.

KEY POINT 17.7

To find the equation for y given the gradient $\frac{dy}{dx}$ and one point (p, q) on the curve:

1. Integrate $\frac{dy}{dx}$, remembering $+c$.
2. Find the constant of integration by substituting $x = p$, $y = q$.

Exercise 17F

1. Find the equation of the original curve if:

(a) (i) $\frac{dy}{dx} = x$ and the curve passes through (−2, 7)

(ii) $\frac{dy}{dx} = 6x^2$ and the curve passes through (0, 5)

(b) (i) $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ and the curve passes through (4, 8)

(ii) $\frac{dy}{dx} = \frac{1}{x^2}$ and the curve passes through (1, 3)

(c) (i) $\frac{dy}{dx} = 2e^x + 2$ and the curve passes through (1, 1)

(ii) $\frac{dy}{dx} = e^x$ and the curve passes through (ln5, 0)

(d) (i) $\frac{dy}{dx} = \frac{x+1}{x}$ and the curve passes through (e, e)

(ii) $\frac{dy}{dx} = \frac{1}{2x}$ and the curve passes through (e^2, 5)

(e) (i) $\frac{dy}{dx} = \cos x + \sin x$ and the curve passes through (π, 1)

(ii) $\frac{dy}{dx} = 3\tan x$ and the curve passes through (0, 4)

2. The derivative of the curve $y = f(x)$ is $\frac{1}{2x}$.

(a) Find an expression for all possible functions *f(x)*.

(b) If the curve passes through the point (2, 7), find the equation of the curve. *[5 marks]*

3. The gradient of a curve is found to be $\frac{dy}{dx} = x^2 - 4$.

(a) Find the x-coordinate of the maximum point, justifying that it is a maximum.

(b) Given that the curve passes through the point (0, 2), show that the y-coordinate of the maximum point is $-7\frac{1}{3}$. *[5 marks]*

4. The gradient of the normal to a curve at any point is equal to the x-coordinate at that point. If the curve passes through the point (e^2, 3) find the equation of the curve in the form $y = \ln(g(x))$ where $g(x)$ is a rational function, $x > 0$. *[6 marks]*

17G Definite integration

Until now we have been carrying out a process known as **indefinite integration**: indefinite in the sense that we have an unknown constant each time, for example $\int x^2\,dx = \frac{1}{3}x^3 + c$.

However, there is also a process called **definite integration** which yields a numerical answer without the involvement of the constant of integration, for example

$$\int_a^b x^2 dx = \left[\frac{1}{3}x^3\right]_a^b = \left(\frac{1}{3}b^3\right) - \left(\frac{1}{3}a^3\right).$$

Here a and b are known as the **limits** of integration: a is the lower limit and b the upper limit.

EXAM HINT

Make sure you know how to evaluate definite integrals on your calculator, as explained on Calculator skills sheet 10 on the CD-ROM. It can save you time, and you can evaluate integrals you don't know how to do algebraically. Even when you are asked to find the exact value of the integral, you can check your answer on the calculator.

The square bracket notation means that the integration has taken place but the limits have not yet been applied. To do this we simply evaluate the integrated expression at the upper limit and subtract the integrated expression evaluated at the lower limit.

You may be wondering where the constant of integration has gone. We could write it in as before but we quickly realise that this is unnecessary as it will just cancel out at the upper and lower limit each time:

$$\int_a^b x^2 \mathrm{d}x = \left[\frac{1}{3}x^3 + c\right]_a^b$$
$$= \left(\frac{1}{3}b^3 + c\right) - \left(\frac{1}{3}a^3 + c\right)$$
$$= \frac{1}{3}b^3 - \frac{1}{3}a^3$$

The value of x is a dummy variable, it does not come into the answer. But both a and b can vary and affect the result. Changing x to a different variable does not change the answer. For example:

$$\int_a^b u^2 \,\mathrm{d}u = \frac{1}{3}b^3 - \frac{1}{3}a^3 = \int_a^b x^2 \,\mathrm{d}x$$

Worked example 17.4

Find the exact value of $\int_1^{\mathrm{e}} \frac{1}{x} + 4\,\mathrm{d}x$.

Integrate and write in square brackets

$$\int_1^e \frac{1}{x} + 4dx = [\ln x + 4x]_1^e$$

Evaluate the integrated expression at the upper and lower limits and subtract the lower from the upper

$$= (\ln(e) + 4(e)) - (\ln(1) + 4(1))$$
$$= (1 + 4e) - (0 + 4) = 4e - 3$$

Exercise 17G

1. Evaluate the following definite integrals, giving exact answers.

   (a) (i) $\int_2^6 x^3 \, dx$ (ii) $\int_1^4 x^2 + x \, dx$

   (b) (i) $\int_0^{\pi/2} \cos x \, dx$ (ii) $\int_\pi^{2\pi} \sin x \, dx$

   (c) (i) $\int_0^1 e^x \, dx$ (ii) $\int_{-1}^1 3e^x \, dx$

2. Evaluate correct to three significant figures:

   (a) (i) $\int_{0.3}^{1.4} \sqrt{x} \, dx$ (ii) $\int_9^{9.1} \frac{3}{\sqrt{x}} \, dx$

   (b) (i) $\int_0^1 e^{x^2} \, dx$ (ii) $\int_1^e \ln x \, dx$

3. Find the exact value of the integral $\int_0^\pi e^x + \sin x + 1 \, dx$ *[5 marks]*

4. Show that the value of the integral $\int_k^{2k} \frac{1}{x} \, dx$ is independent of k. *[4 marks]*

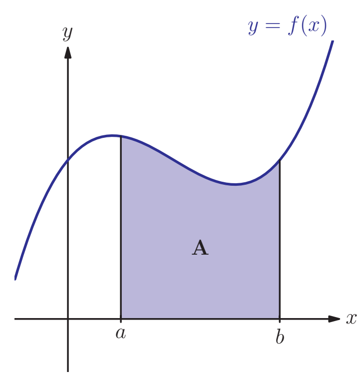
5. If $\int_3^9 f(x) \, dx = 7$, evaluate $\int_3^9 2f(x) + 1 \, dx$. *[4 marks]*

6. Solve the equation $\int_1^a \sqrt{t} \, dt = 42$. *[5 marks]*

17H Geometrical significance of definite integration

Now we have a method that gives a numerical value for an integral, the natural question to ask is: what does this number mean?

On Fill-in proof sheet 20 on the CD-ROM, The fundamental theorem of calculus, we show that, as long as $f(x)$ is positive, the definite integral of $f(x)$ between the limits a and b is the area enclosed between the curve, the x-axis and the lines $x = a$ and $x = b$.



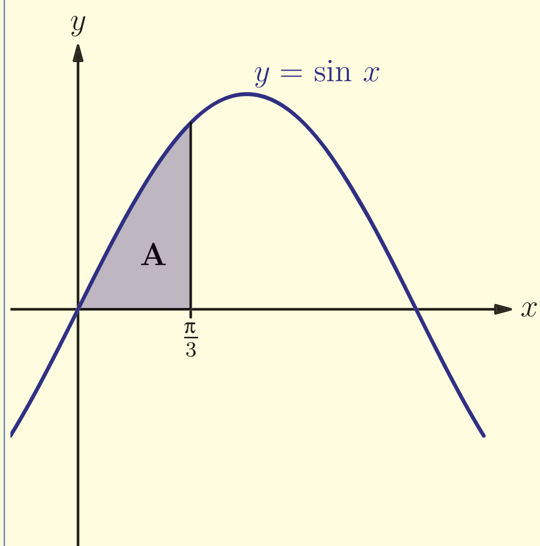
KEY POINT 17.8

$$\text{Area} = \int_a^b f(x) \, dx$$

Worked example 17.5

Find the exact area enclosed between the x-axis, the curve $y = \sin x$ and the lines $x = 0$ and $x = \frac{\pi}{3}$.

Sketch the graph and identify the area required



Integrate and write in square brackets

$$A = \int_0^{\pi/3} \sin x\,dx = [-\cos x]_0^{\pi/3}$$

Evaluate the integrated expression at the upper and lower limit and subtract the lower from the upper

$$= \left(-\cos\frac{\pi}{3}\right) - (-\cos 0)$$

$$= \left(-\frac{1}{2}\right) - (-1) \quad = \frac{1}{2}$$

If you are sketching the graph on the calculator you can get it to shade and evaluate the required area: see Calculator skills sheet 10 on the CD-ROM. You need to show the sketch as a part of your working if it is not already shown in the question.

In the 17th Century, integration was defined as the area under a curve. The area was broken down into small rectangles, each with a height $f(x)$ and a width of a small bit of x, called Δx. The total area was approximately the sum of all of these rectangles:

$$\sum_{x=a}^{x=b} f(x)\Delta x$$

Isaac Newton, one of the pioneers of calculus, was also a big fan of writing in English rather than Greek. So sigma became the English letter 'S' and delta became the English letter d so that when the limit is taken as the width of the rectangles become vanishingly small then the expression becomes:

$$\int_a^b f(x)\,dx$$

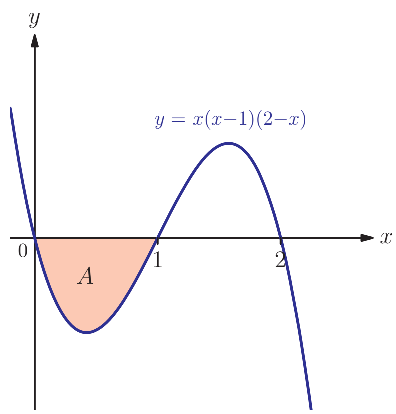
This illustrates another very important interpretation of integration – the infinite sum of infinitesimally small parts.

The Ancient Greeks had developed ideas of limiting processes similar to those used in calculus but it took nearly 2000 years for these ideas to be formalised. This was done almost simultaneously by Isaac Newton and Gottried Leibniz in the 17th Century. Is this a coincidence or is it often the case that a long period of slow progress is often necessary to get to the stage of major breakthroughs? Supplementary sheet 10 looks at some other people who can claim to have invented calculus.

When the curve is entirely below the x-axis the integral will give a negative value. The modulus of this value is the area.

Worked example 17.6

Find the area A in this graph.



Write down the integral to be evaluated, then use calculator

$\int_0^1 x(x-1)(2-x)\,dx = -0.25$ (by GDC)

The area must be positive

$\therefore A = 0.25$

Unfortunately, the relationship between integrals and areas is not so simple when there are parts of the curve above and below the axis. Those bits above the axis contribute positively to the area, but bits below the axis contribute negatively to the area. We must separate out the sections above the axis and below the axis.

Worked example 17.7

(a) Find $\int_1^4 x^2 - 4x + 3 \, dx$

(b) Find the area enclosed between the x-axis, the curve $y = x^2 - 4x + 3$ and the lines $x = 1$ and $x = 4$.

Apply standard integration

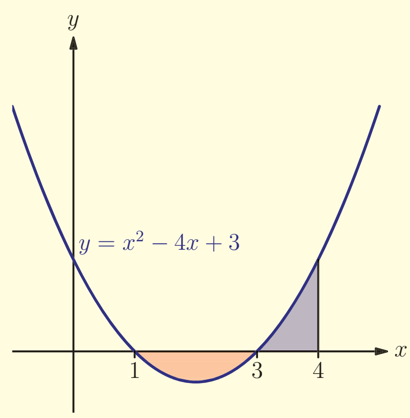
(a) $\int_1^4 x^2 - 4x + 3dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^4$

$= \left(\frac{1}{3}(4)^3 - 2(4)^2 + 3(4)\right) - \left(\frac{1}{3}(1)^3 - 2(1)^2 + 3(1)\right)$

$= \left(\frac{4}{3}\right) - \left(\frac{4}{3}\right) = 0$

The value found above can't be the correct area for (b). Sketch the curve to see exactly which area we are being asked to find

(b)



continued . . .

The area is made up of two parts, so evaluate each of them separately

$$\int_1^3 x^2 - 4x + 3dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$$

$$= (0) - \left(\frac{4}{3}\right) = -\frac{4}{3}$$

$\therefore$ Area below the axis is $\frac{4}{3}$

$$\int_3^4 x^2 - 4x + 3dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_3^4$$

$$= \left(\frac{4}{3}\right) - (0) = \frac{4}{3}$$

$\therefore$ Area above the axis is $\frac{4}{3}$

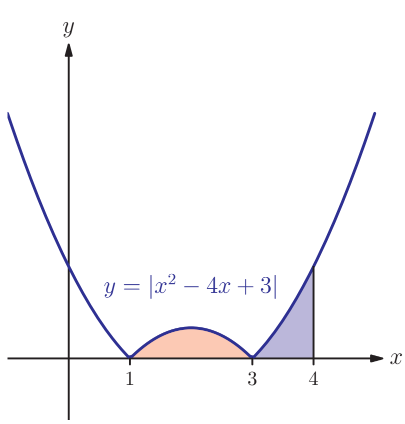
Total area $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

The fact that the integral was zero in Worked example 17.7 part (a) means that the area above the axis is exactly cancelled by the area below the axis.

Transformations of graphs using the modulus function were covered in chapter 7.

This example warns us that when asked to find an area we must always sketch the graph and identify exactly where each part of the area is. If we are evaluating the area on the calculator we can use the modulus function to ensure that the entire graph is above the x-axis. Using the function from Worked example 17.7:

$$\int_1^4 |x^2 - 4x + 3|\,dx = \frac{8}{3}$$

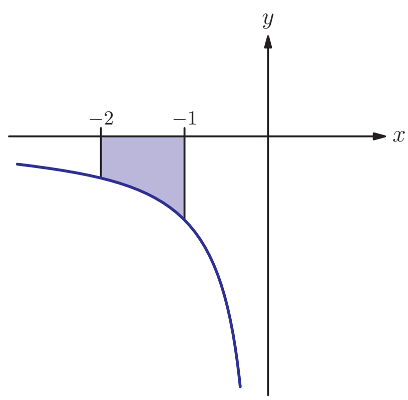


KEY POINT 17.9

The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b |f(x)|\, dx$.
When working without a calculator, if the curve crosses the x-axis between a and b we need to split the area into several parts and find each one separately.

The interpretation of integrals as areas causes one inconsistency with our previous work. Consider the integral $\int_{-2}^{-1} \frac{1}{x}\, dx$.

Graphically we can see that this area should exist.



However, if we do the integration we find that:

$$\int_{-2}^{-1} \frac{1}{x}\, dx = [\ln x]_{-2}^{-1}$$

$$= \ln(-1) - \ln(-2)$$

$$= \ln\left(\frac{-1}{-2}\right)$$

$$= \ln\left(\frac{1}{2}\right)$$

$$= -\ln 2$$

This is the correct answer (which we could have found using the symmetry of the curve) but it goes through a stage where we had to take logarithms of negative numbers, and this is something we are not allowed to do. We avoid this by redefining the integral of $\frac{1}{x}$ as:

KEY POINT 17.4 AGAIN

$$\int x^{-1}\,dx = \ln|x| + c$$

You may rightly be a little uncomfortable with inserting a modulus function 'just because it works'. In mathematics, do the ends justify the means?

With this definition we can integrate $y = \frac{1}{x}$ over negative numbers, and the integral above becomes

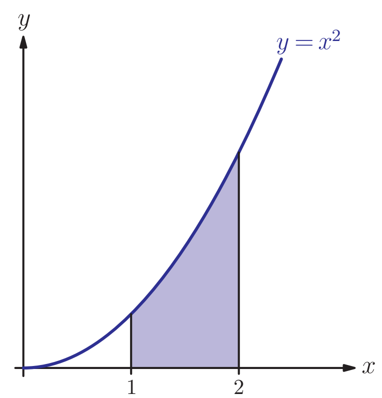
$$\begin{aligned}\int_{-2}^{-1} \frac{1}{x}\,dx &= [\ln|x|]_{-2}^{-1}\\ &= \ln 1 - \ln 2\\ &= -\ln 2 \text{ as before}\end{aligned}$$

Notice that the answer is negative, since the required area is below the x-axis. We can still not integrate $\frac{1}{x}$ with a negative lower and positive upper limit, since the graph has an asymptote at $x = 0$.

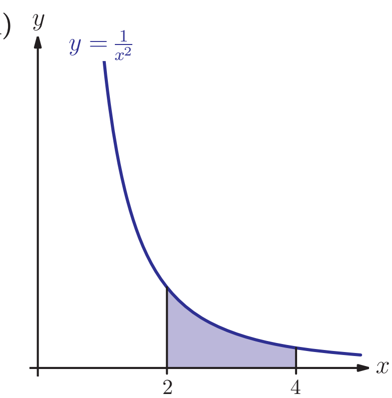
Exercise 17H

1. Find the shaded areas:

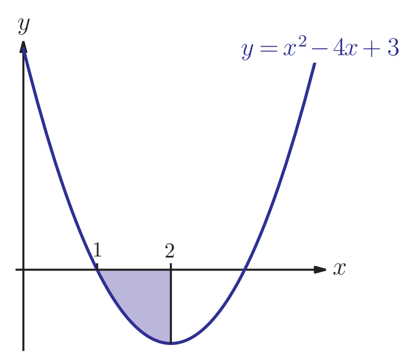
(a) (i)



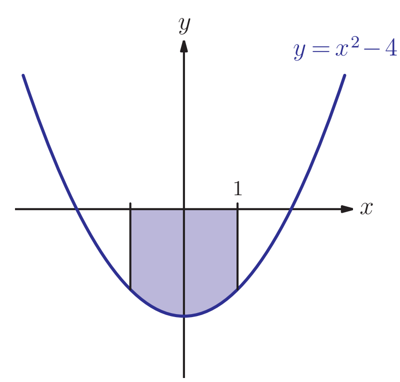
(ii)



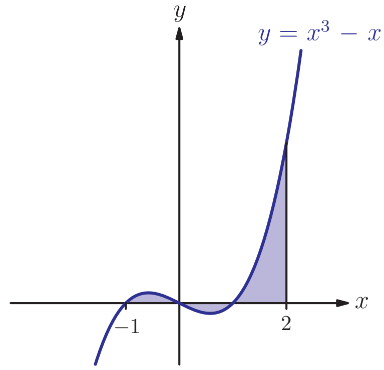
(b) (i)



(ii)



(c) (i)



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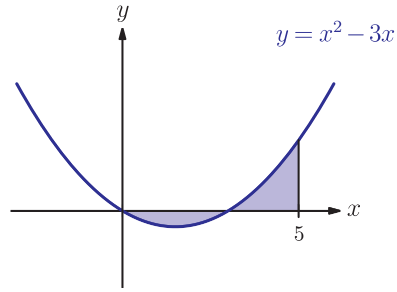


EXAM HINT

'Find the area enclosed' means first find a closed region bounded by the curves mentioned, then find its area. A sketch is a very useful tool.

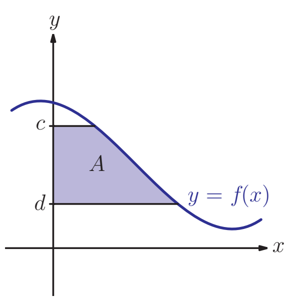
2. The area enclosed by the x-axis, the curve $y = \sqrt{x}$ and the line $x = k$ is 18. Find the value of k. *[6 marks]*

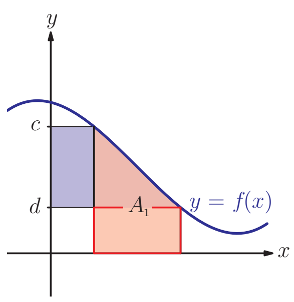
3. (a) Find $\int_0^3 x^2 - 1\,\mathrm{d}x$

(b) Find the area between the curve $y = x^2 - 1$ and the x-axis between $x = 0$ and $x = 3$. *[5 marks]*

4. Between $x = 0$ and $x = 3$, the area of the graph $y = x^2 - kx$ below the x-axis equals the area above the x-axis. Find the value of k. *[6 marks]*

5. Find the area enclosed by the curve $y = 7x - x^2 - 10$ and the x-axis. *[7 marks]*



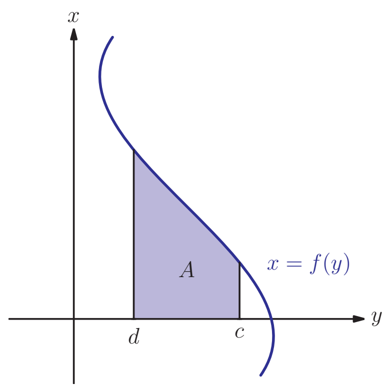


17I The area between a curve and the y-axis

Consider the diagram alongside. How can we find the shaded area A?

One possible strategy is to construct a box around the graph to divide up the regions of interest. You can integrate to find the area labelled A_1 and then, by adding and subtracting the areas of the blue and red rectangles shown, calculate A.

Happily, there is a quicker way: we can treat x as a function of y, effectively reflecting the whole diagram in the line $y = x$, and then use the same method as in the previous section.



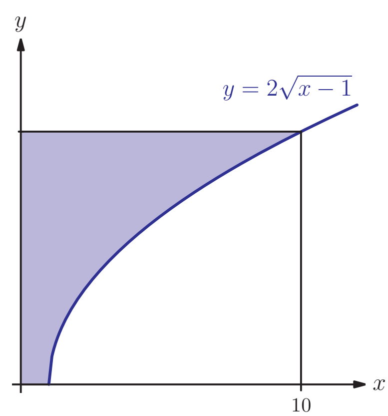
KEY POINT 17.10

The area bounded by the curve $y = f(x)$, the y-axis and the lines $y = c$ and $y = d$ is given by $\int_c^d g(y)\,dy$, where $g(y)$ is the expression for x in terms of y.

You may have realised that this is related to inverse functions from Section 5E.

Worked example 17.8

The curve shown has equation $y = 2\sqrt{x-1}$. Find the shaded area.



Express x in terms of y

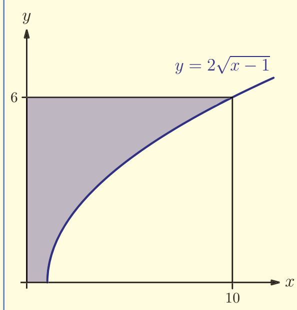
$$x - 1 = \left(\frac{y}{2}\right)^2$$

$$\Rightarrow x = \frac{y^2}{4} + 1$$

continued . . .

Find the limits on the y-axis
It may help to label them on the graph

When $x = 1$, $y = 2\sqrt{1-1} = 0$
When $x = 10$, $y = 2\sqrt{10-1} = 6$



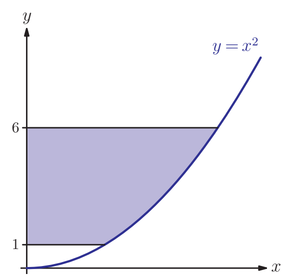
Write down the integral and evaluate using calculator

$$\text{Area} = \int_0^6 \left(\frac{y^2}{4} + 1\right) dy = 24 \text{ (from GDC)}$$

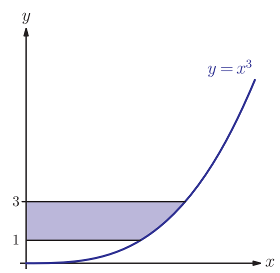
Exercise 17I

1. Find the shaded areas:

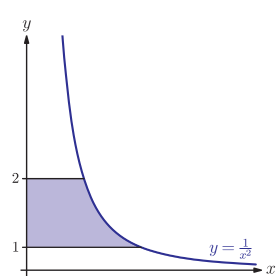
(a) (i)



(ii)



(b) (i)



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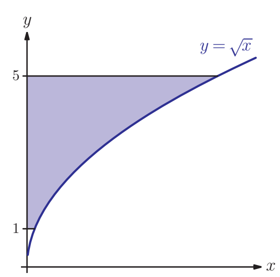
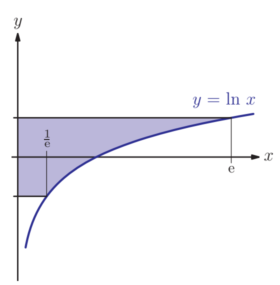
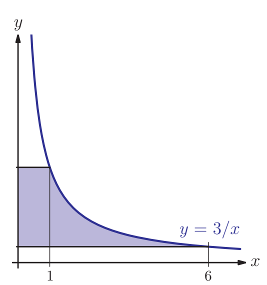


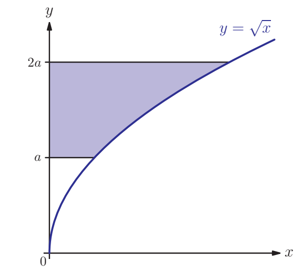
(c) (i)



(ii)

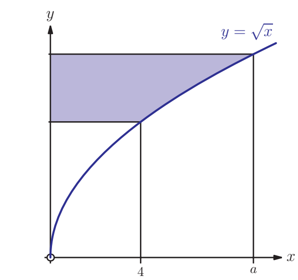


2. The diagram shows the curve $y=\sqrt{x}$. If the shaded area is 504 find the value of a. *[6 marks]*

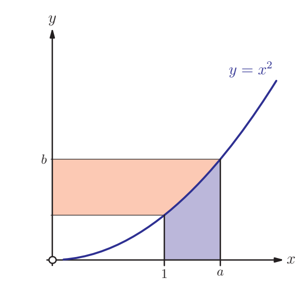


3. Find the exact value of the area enclosed by the graph of $y=\ln(x+1)$, the line $y=2$ and the y-axis. *[6 marks]*

4. The diagram shows the graph of $y=\sqrt{x}$.
The shaded area is 39 units. Find the value of a. *[7 marks]*



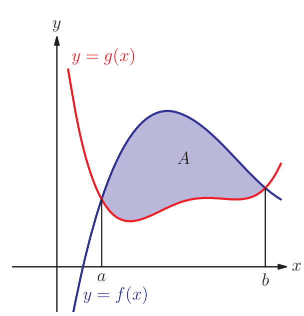
5. The diagram shows the graph of $y=x^2$, where $a\in]1,\infty[$.
The area of the pink region is equal to the area of the blue region. Give two equations for a in terms of b, and hence give a in exact form and determine the size of the blue area. *[8 marks]*



17J The area between two curves

So far we have only looked at areas bounded by a curve and one of the coordinate axes, but we can also find areas bounded by two curves.
The area A in the diagram can be found by taking the area bounded by $f(x)$ and the x-axis and subtracting the area bounded by $g(x)$ and the x-axis, that is:

$$A=\int_a^b f(x)\,\mathrm{d}x-\int_a^b g(x)\,\mathrm{d}x$$



We can do the subtraction before integrating so that we only have to integrate one expression instead of two. This gives an alternative formula for the area.

KEY POINT 17.11

The area A between two curves, $f(x)$ and $g(x)$, is:

$$A = \int_a^b \left| f(x) - g(x) \right| \mathrm{d}x$$

where a and b are the x-coordinates of the intersection points of the two curves.

Worked example 17.9

Find the area A enclosed between $y = 2x + 1$ and $y = x^2 - 3x + 5$.

First find the x-coordinates of intersection

For intersection:

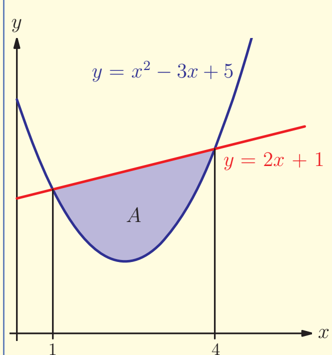
$x^2 - 3x + 5 = 2x + 1$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0$

$\Rightarrow x = 1, 4$

Make a rough sketch to see the relative positions of the two curves



Subtract the lower curve from the higher before integrating

$$A = \int_1^4 (2x+1) - (x^2 - 3x + 5)\,dx$$

$$= \int_1^4 -x^2 + 5x - 4\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x \right]_1^4$$

$$= \frac{8}{3}\left(-\frac{11}{6}\right) = \frac{9}{2}$$

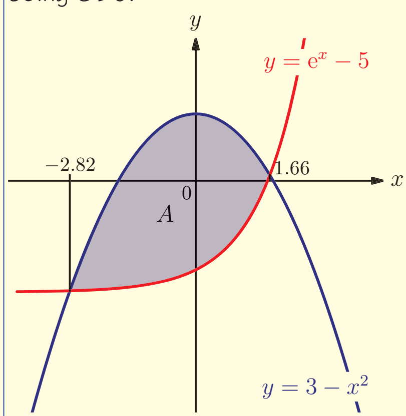
Subtracting the two equations before integrating is particularly useful when one of the curves is partly below the x-axis. If $f(x)$ is always above $g(x)$ then the expression we are integrating, $f(x)-g(x)$, is always positive, so we do not have to worry about the signs of $f(x)$ and $g(x)$ themselves.

Worked example 17.10

Find the area bounded by the curves $y = e^x - 5$ and $y = 3 - x^2$.

Sketch the graph to see the relative position of two curves

Using GDC:



Find the intersection points – use calculator

intersections: $x = -2.818$ and 1.658

Write down the integral representing the area

$$\text{Area} = \int_{-2.818}^{1.658} (3 - x^2) - (e^x - 5)\,dx$$

$$= \int_{-2.818}^{1.658} (8 - x^2 - e^x)\,dx$$

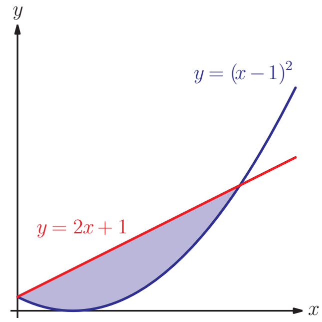
Evaluate the integral using calculator

$$= 21.6 \quad (3SF)$$

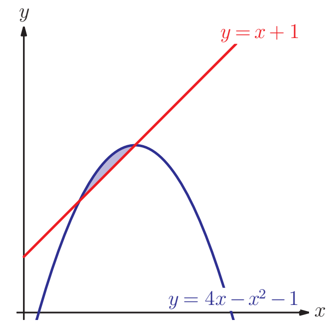
Exercise 17J

1. Find the shaded areas.

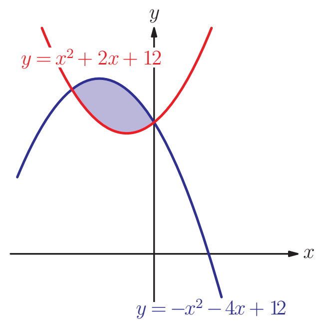
(a) (i)



(ii)



(b) (i)



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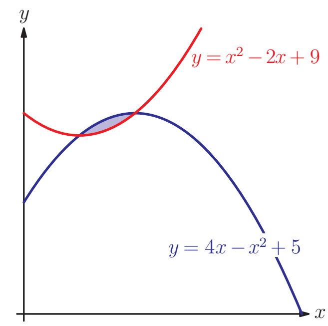
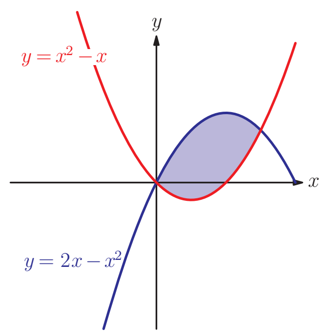


(c) (i)



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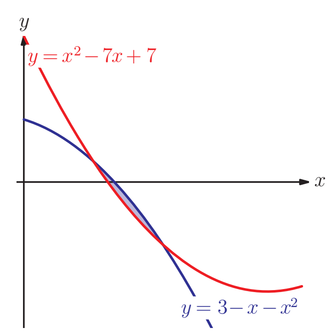
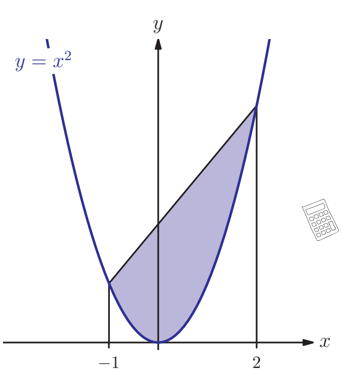




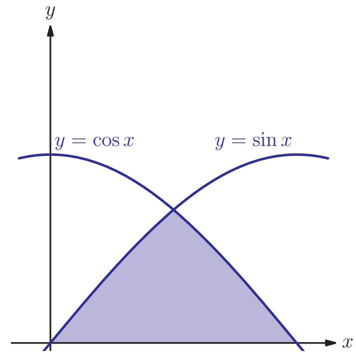
2. Find the area enclosed between the graphs of $y = x^2 + x - 2$ and $y = x + 2$. *[6 marks]*

3. Find the area enclosed by the curve $y = e^x$, $y = x^2$, the y-axis and the line $x = 2$. *[6 marks]*

4. Find the area between the curves $y = \frac{1}{x}$ and $y = \sin x$ in the region $0 < x < \pi$. *[6 marks]*

5. Show that the area of the shaded region alongside is $\frac{9}{2}$. *[6 marks]*

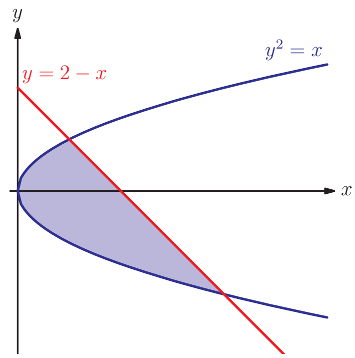
6. The diagram alongside shows the graphs of $y = \sin x$ and $y = \cos x$. Find the shaded area. *[6 marks]*



7. Find the total area enclosed between the graphs of $y = x(x-4)^2$ and $y = x^2 - 7x + 15$. *[6 marks]*

8. The area enclosed between the curve $y = x^2$ and the line $y = mx$ is $10\frac{2}{3}$. Find the value of m if $m > 0$. *[7 marks]*

9. Show that the shaded area in the diagram below is $\frac{9}{2}$. *[8 marks]*



Summary

- **Integration** is the reverse process of differentiation.
- Any integral without limits (**indefinite**) will generate a **constant of integration**.
- For all rational $n \neq -1$ $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$.
- If $n = -1$, we get the natural logarithm function: $\int x^{-1} \, dx = \ln x + c$.
- The integral of the exponential function is: $\int e^x \, dx = e^x + c$.
- The integrals of the trigonometric functions are:

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \tan x \, dx = \ln|\sec x| + c$$

- The **definite integral** has limits. $\int_a^b f(x) \, dx$ is found by evaluating the integrated expression at b and then subtracting the integrated expression evaluated at a.

- The area between the curve $y = f(x)$, the x-axis and lines $x = a$ and $x = b$ is given by:

$$A = \int_a^b f(x)\, dx$$

If the curve goes below the x-axis, the value of this integral will be negative.

- On the calculator, we can use the modulus function to ensure we are always integrating a positive function.
- The area between the curve, the y-axis and lines $y = c$ and $y = d$ is given by: $A_1 = \int_c^d g(y)\, dy$.
- The area between two curves is given by:

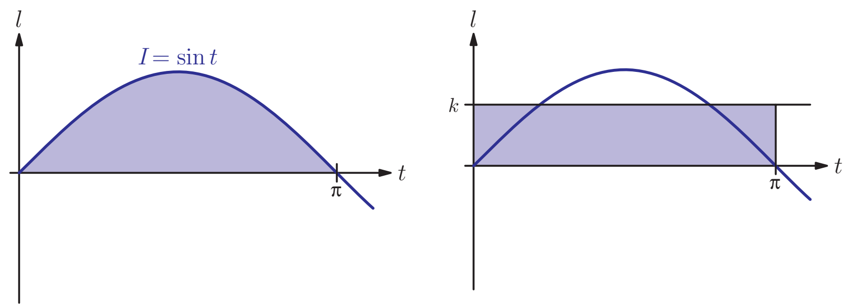
$$A = \int_a^b |f(x) - g(x)|\, dx$$

where $x = a$ and $x = b$ are the intersection points.

Introductory problem revisited

The amount of charge stored in a capacitor is given by the area under the graph of current (I) against time (t). When there is alternating current the relationship between I and t is given by $I = \sin t$. When it contains direct current the relationship between I and t is given by $I = k$. What value of k means that the amount of charge stored in the capacitor from $t = 0$ to $t = \pi$ is the same whether alternating or direct current is used?

The area under the curve of I against t is given by $\int_0^\pi \sin t\, dt = [-\cos t]_0^\pi = 2$. For a rectangle of width π to have the same area the height must be $\frac{2}{\pi}$.



You can look at integration as a quite sophisticated way of finding an average value of a function.

Mixed examination practice 17

Short questions

1. If $f'(x) = \sin x$ and $f\left(\frac{\pi}{3}\right) = 0$, find $f(x)$. *[4 marks]*

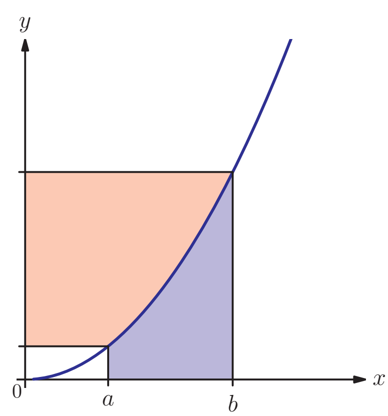
2. Calculate the area enclosed by the curves $y = \ln x$ and $y = e^x - e, x > 0$.

[6 marks]

[© IB Organization 2003]

3. Find the area enclosed between the graph of $y = k^2 - x^2$ and the x-axis, giving your answer in terms of k. *[6 marks]*

4. The diagram shows the graph of $y = x^n$ for $n > 1$.



The red area is three times larger than the blue area. Find the value of n. *[6 marks]*

5. Find the indefinite integral:

$$\int \frac{1 + x^2\sqrt{x}}{x}\,dx$$ *[5 marks]*

6. (a) Solve the equation:

$$\int_0^a x^3 - x\,dx = 0, \ a > 0.$$

(b) For this value of a, find the total area enclosed between the x-axis and the curve $y = x^3 - x$ for $0 \le x \le a$. *[6 marks]*

7. Find the area enclosed between the graphs of $y = \sin x$ and $y = 1 - \sin x$ for $0 < x < \pi$. *[3 marks]*

8. (a) The function $f(x)$ has a stationary point at $(3, 19)$ and $f''(x) = 6x + 6$.

   What kind of stationary point is $(3, 19)$? *[5 marks]*

   (b) Find $f(x)$.

Long questions

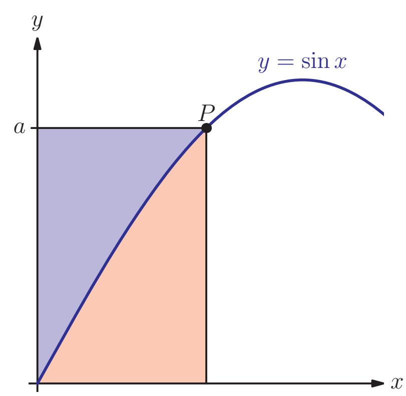
1. (a) Find the coordinates of the points of intersection of the graphs $y = 5a^2 + 4ax - x^2$ and $y = x^2 - a^2$.

   (b) Find the area enclosed between these two graphs.

   (c) Show that the fraction of this area above the axis is independent of a and state the value that this fraction takes. *[10 marks]*

2. (a) Use the identity $\cos^2 x + \sin^2 x = 1$ to show that $\cos(\arcsin x) = \sqrt{1 - x^2}$.

   (b) The diagram below shows part of the curve $y = \sin x$.



   Write down the x-coordinate of the point P in terms of a.

   (c) Find the red shaded area in terms of a, writing your answer in a form without trigonometric functions.

   (d) By considering the blue shaded area find $\int_0^a \arcsin x \, dx$ for $0 < a < 1$.

[12 marks]

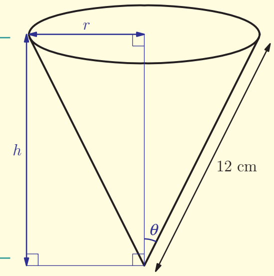
18 Further differentiation methods

In this chapter you will learn:

- how to differentiate composite functions
- how to differentiate reciprocal trigonometric functions: $\sec x, \csc x$ and $\cot x$
- how to differentiate products of functions
- how to differentiate quotients of functions
- how to differentiate functions that are not in the form $y = f(x)$
- how to differentiate exponential functions
- how to differentiate inverse trigonometric functions: $\arcsin x, \arccos x$ and $\arctan x$.

Introductory problem

Given a cone of fixed slant height 12 cm, find the maximum volume as apex angle θ varies.



In this chapter we will build on the techniques covered in chapter 16 so that we can differentiate a wider range of functions. Much of the work here will also be used in chapter 19 when we learn more integration techniques.

18A Differentiating composite functions using the chain rule

We can already differentiate functions such as $y = (3x^2 + 5x)^2$ by expanding the brackets and differentiating term by term:

$$y = (3x^2)^2 + 2(3x^2)(5x) + (5x)^2 = 9x^4 + 30x^3 + 25x^2$$

$$\therefore \frac{dy}{dx} = 36x^3 + 90x^2 + 50x = 2x(18x^2 + 45x + 25)$$

But what if the function is more complicated?

The same method would work, but it is clearly not practical to expand, for example, $y = (3x^2 + 5x + 2)^7$ and then differentiate each term. And what about functions such as $y = \sin 3x$ or $y = e^{x^2}$? While we can already differentiate $y = \sin x$ and $y = e^x$, we have no rules so far to tell us what to do when the argument is changed to $3x$ or x^2.

The functions $y=(3x^2+5x+2)^7$, $y=\sin 3x$ and $y=e^{x^2}$ may not seem related but do have something in common; they are all composite functions:

- $y=(3x^2+5x+2)^7$ is $y=u^7$ where $u(x)=3x^2+5x+2$
- $y=\sin 3x$ is $y=\sin u$ where $u(x)=3x$
- $y=e^{x^2}$ is $y=e^u$ where $u(x)=x^2$

There is a general rule for differentiating any composite function.

KEY POINT 18.1

The chain rule

If $y=g(u)$ where $u=f(x)$:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

We will accept the chain rule without proof, as it is very technical and requires differentiation from first principles. Let us apply the chain rule to the three functions above.

Worked example 18.1

Differentiate these functions:

(a) $y=(3x^2+5x+2)^7$ (b) $y=\sin(3x)$ (c) $y=e^{x^2}$

These are all composite functions so use chain rule

(a) $y=u^7$ where $u=3x^2+5x+2$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=7u^6\times(6x+5)$$

Write the answer in terms of x

$$=7(3x^2+5x+2)^6(6x+5)$$

(b) $y=\sin u$ where $u=3x$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\cos u\times(3)$$

Write the answer in terms of x

$$=3\cos(3x)$$

(c) $y=e^u$ where $u=x^2$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=e^u\times(2x)$$

Write the answer in terms of x in the conventional form

$$=2xe^{x^2}$$

Worked example 18.1(b) illustrates a special case of the chain rule when the 'inside' function is of the form $ax+b$.

KEY POINT 18.2

$$\frac{d}{dx}f(ax+b)=af'(ax+b)$$

For example,

$$\frac{d}{dx}(4x+1)^7=4\times7(4x+1)^6 \text{ and } \frac{d}{dx}(e^{3-2x})=-2e^{3-2x}$$

It is useful to remember this shortcut. In practice it is not necessary to keep specifying the function $u(x)$ each time and the chain rule calculation can be written down more directly as can be seen in the example below, i.e. imagine brackets around the inner function u and differentiate the outer function first, as if the bracketed expression were a single argument, and then multiply by the derivative of the bracketed expression.

Worked example 18.2

Differentiate these composite functions:

(a) $y=e^{x^2-3x}$ (b) $y=\frac{3}{\sqrt{x^3-5}}$

$e^{()}$ differentiates to $e^{()}$ and x^2-3x differentiates to $2x-3$

(a) $\frac{dy}{dx}=(2x-3)e^{(x^2-3x)}$

First rewrite the square root as a power

(b) $y=3(x^3-5)^{-\frac{1}{2}}$

$3()^{-\frac{1}{2}}$ differentiates to $-\frac{3}{2}()^{-\frac{3}{2}}$ and x^3-5 differentiates to $3x^2$

$$\frac{dy}{dx}=-\frac{3}{2}(x^3-5)^{-\frac{3}{2}}(3x^2)=-\frac{9x^2}{2}(x^3-5)^{-\frac{3}{2}}$$

Sometimes it is necessary to apply the chain rule more than once.

Worked example 18.3

Differentiate $y = \cos^3(\ln 2x)$.

Remember that $\cos^3 A$ means $(\cos A)^3$

$y = (\cos(\ln 2x))^3$

This is a composite of three functions, so use chain rule

$(\ )^3$ differentiates to $3(\ )^2$

$\cos(\ )$ differentiates to $-\sin(\ )$

$\ln 2x$ differentiates to $2 \times \frac{1}{2x} = \frac{1}{x}$

$$\frac{dy}{dx} = 3(\cos(\ln 2x))^2 \times (-\sin(\ln 2x)) \times \frac{1}{x}$$

$$= -\frac{3}{x}\cos^2(\ln 2x)\sin(\ln 2x)$$

Now we can use the chain rule, we can add the derivatives of $y = \sec x$, $y = \csc x$ and $y = \cot x$ (see Key point 18.3 on the next page) to those of $y = \sin x$, $y = \cos\ x$ and $y = \tan x$ already established in chapter 16.

Worked example 18.4

Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$

Express $\sec x$ in terms of $\cos x$

$y = \sec x = (\cos x)^{-1}$

This is a composite function, so apply chain rule

$(\ )^{-1}$ differentiates to $-(\ )^{-2}$

$\cos(\ )$ differentiates to $-\sin(\ )$

$$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}$$

We want the answer to contain $\tan x$, which is $\frac{\sin x}{\cos x}$

$$= \frac{1}{\cos x}\frac{\sin x}{\cos x}$$

$= \sec x \tan x$ as required

The proofs for the other two reciprocal trigonometric functions follow the same pattern, giving the following results.

KEY POINT 18.3

$$y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \tan x$$
$$y = \csc x \Rightarrow \frac{dy}{dx} = -\csc x \cot x$$
$$y = \cot x \Rightarrow \frac{dy}{dx} = -\csc^2 x$$

Exercise 18A

1. Differentiate the following using the chain rule:
   (a) (i) $(x^2 - 3x + 1)^7$ (ii) $(x^3 + 1)^5$
   (b) (i) $e^{x^2 - 2x}$ (ii) $e^{4 - x^3}$
   (c) (i) $(2e^x + 1)^{-3}$ (ii) $(2 - 5e^x)^{-4}$
   (d) (i) $\sin(3x^2 + 1)$ (ii) $\cos(x^2 + 2x)$
   (e) (i) $\cos^3 x$ (ii) $\sin^4 x$
   (f) (i) $\ln(2x - 5x^3)$ (ii) $\ln(4x^2 - 1)$
   (g) (i) $(4\ln x - 1)^4$ (ii) $(\ln x + 3)^{-5}$

2. Differentiate the following using the short cut from Key point 18.2:
   (a) (i) $(2x + 3)^5$ (ii) $(4x - 1)^8$
   (b) (i) $(5 - x)^{-4}$ (ii) $(1 - x)^{-7}$
   (c) (i) $\cos(1 - 4x)$ (ii) $\cos(2 - x)$
   (d) (i) $\ln(5x + 2)$ (ii) $\ln(x - 4)$
   (e) (i) $\cot(3x)$ (ii) $\csc(5x)$
   (f) (i) $\sec(2x + 1)$ (ii) $\tan(1 - x)$

3. Differentiate the following using the chain rule twice:
   (a) (i) $\sec^2 3x$ (ii) $\tan^2 2x$
   (b) (i) $e^{\sin^2 3x}$ (ii) $e^{(\ln 2x)^2}$
   (c) (i) $(1 - 2\sin^2 2x)^2$ (ii) $(4\cos 3x + 1)^2$
   (d) (i) $\ln(1 - 3\cos 2x)$ (ii) $\ln(2 - \cos 5x)$

4. Find the equation of the normal to the curve $y = \frac{1}{\sqrt{4x^2 + 1}}$ at the point where $x = \sqrt{2}$.

5. Find the exact coordinates of stationary points on the curve $y = e^{\sin x}$ for $x \in [0, 2\pi]$. *[5 marks]*

6. Given that $f(x) = \csc^2 x$:

(a) Find $f'(x)$.

(b) Solve the equation $f'(x) = 2f(x)$ for $-\pi < x < \pi$. *[7 marks]*

7. For what values of x does the function $f : x \mapsto \ln(x^2 - 35)$ have a gradient of 1? *[5 marks]*

8. (a) If a, b, p and q are positive with $a < b$ find the x-coordinate of the stationary point of the curve $y = (x-a)^p (x-b)^q$ in the domain $a < x < b$.

(b) Sketch the graph in the case when $p = 2$ and $q = 3$.

(c) By considering the graph or otherwise, determine a condition involving p and/or q to determine when this stationary point is a maximum. *[10 marks]*

9. A non-uniform chain hangs from two posts. Its height (h) satisfies the equation

$$h = e^x + \frac{1}{e^{2x}} \text{ for } -1 \le x \le 2.$$

The left post is positioned at $x = -1$. The right post is positioned at $x = 2$.

(a) State, with reasons, which post is taller.

(b) Show that the minimum height occurs when $x = \frac{1}{3}\ln 2$.

(c) Find the exact value of the minimum height of the chain. *[8 marks]*

Many people think that a chain hangs as a parabola but it can be proved that it actually hangs in the shape of the curve in question 9, called a *catenary*. To prove this requires a topic called *differential geometry*.

10. (a) Solve the equation $\sin 2x = \sin x$ for $0 \le x \le 2\pi$.

(b) Find the coordinates of the stationary points of the curve $y = \sin 2x - \sin x$ for $0 \le x \le 2\pi$.

(c) Hence sketch the curve $y = \sin 2x - \sin x$. *[8 marks]*

18B Differentiating products using the product rule

We now look at products of two functions. We can already differentiate some products, such as $y = x^4(3x^2 - 5)$, by expanding and differentiating term by term. However, like composite functions, this is tricky when the function becomes more complicated, for example $y = x^4(3x^2 - 5)^9$, and expanding is no help at all with functions such as $y = x^2 \cos x$ or $y = x \ln x$.

Just as there is a rule for differentiating composite functions, there is a rule for differentiating products.

KEY POINT 18.4

The product rule

If $y = u(x)v(x)$ then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

If you are interested in the proof, see Fill-in proof 21 on the CD-ROM.

Let us apply the product rule to the first function in the previous paragraph.

Worked example 18.5

Differentiate $y = x^4(3x^2 - 5)$.

This is a product so use the product rule. It doesn't make any difference which function is $u(x)$ and which is $v(x)$

Let $u = x^4$ and $v = 3x^2 - 5$

$\frac{du}{dx} = 4x^3$, $\frac{dv}{dx} = 6x$

Apply the product rule

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= (3x^2 - 5)4x^3 + x^4 \times 6x$$

$$= 12x^5 - 20x^3 + 6x^5$$

$$= 18x^5 - 20x^3$$

EXAM HINT

After applying the product rule you do not need to simplify the resulting expression unless the question clearly tells you to do so.

With a more complicated function, we may need the chain rule as well as the product rule.

Worked example 18.6

Differentiate $y = x^4(3x^2-5)^5$ and factorise your answer.

This is a product so use the product rule. It doesn't make any difference which function is $u(x)$ and which is $v(x)$

Let $u = x^4$ and $v = (3x^2-5)^5$

$$\frac{du}{dx} = 4x^3$$

$v(x)$ is a composite function, so use chain rule

$$\frac{dv}{dx} = 5(3x^2-5)^4(6x)$$
$$= 30x(3x^2-5)^4$$

Now apply the product rule

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= (3x^2-5)^5\,4x^3 + (x^4)\times 30x(3x^2-5)^4$$

We are asked to factorise the answer, so look for common factors

$$= 2x^3(3x^2-5)^4\left[2(3x^2-5)+15x^2\right]$$
$$= 2x^3(3x^2-5)^4(6x^2-10+15x^2)$$
$$= 2x^3(3x^2-5)^4(21x^2-10)$$

Exercise 18B

1. Use the product rule to differentiate the following:

   (a) (i) $y = x^2\cos x$ (ii) $y = x^{-1}\sin x$

   (b) (i) $y = x^{-2}\ln x$ (ii) $y = x\ln x$

   (c) (i) $y = x^3\sqrt{2x+1}$ (ii) $y = x^{-1}\sqrt{4x}$

   (d) (i) $y = e^{2x}\tan x$ (ii) $y = e^{x+1}\sec 3x$

2. Find $f'(x)$ and fully factorise your answer:

   (a) (i) $f(x) = (x+1)^4(x-2)^5$ (ii) $f(x) = (x-3)^7(x+5)^4$

   (b) (i) $f(x) = (2x-1)^4(1-3x)^3$ (ii) $f(x) = (1-x)^5(4x+1)^2$

3. Differentiate $y = (3x^2-x+2)e^{2x}$ giving your answer in the form $P(x)e^{2x}$ where $P(x)$ is a polynomial. *[4 marks]*

4. Given that $f(x) = x^2e^{3x}$, find $f''(x)$ in the form $(ax^2+bx+c)e^{3x}$. *[4 marks]*

5. Find the x-coordinates of the stationary points on the curve $y=(2x+1)^5 e^{-2x}$. *[5 marks]*

6. Find the exact values of the x-coordinates of the stationary points on the curve $y=(3x+1)^5(3-x)^3$. *[6 marks]*

7. Given that $y = x\sin 2x$ for $x \in [0, 2\pi]$:

(a) show that the x-coordinates of the points of inflexion satisfy $\cos 2x = x\sin 2x$

(b) hence find the coordinates of the points of inflexion. *[6 marks]*

8. Find the derivative of $\sin(xe^x)$ with respect to x. *[5 marks]*

9. (a) If $f(x) = x\ln x$, find $f'(x)$.

(b) Hence find $\int \ln x \, dx$. *[5 marks]*

10. Find the exact coordinates of the minimum point of the curve $y = e^{-x}\cos x,\ 0 \le x \le \pi$. *[6 marks]*

11. Given that $f(x) = x^2\sqrt{1+x}$, show that $f'(x) = \dfrac{x(a+bx)}{2\sqrt{1+x}}$

where a and b are constants to be found. *[6 marks]*

12. (a) Write $y = x^x$ in the form $y = e^{f(x)}$.

(b) Hence or otherwise find $\dfrac{dy}{dx}$.

(c) Find the exact coordinates of the stationary points of the curve $y = x^x$. *[8 marks]*

18C Differentiating quotients using the quotient rule

A combination of the product rule and chain rule provides us with a method for differentiating quotients such as:

$$y = \frac{x^2 - 4x + 12}{(x-3)^2}$$

We can express it as $y = (x^2 - 4x + 12)(x-3)^{-2}$ then using the product rule and taking

$$u = (x^2 - 4x + 12) \text{ and } v = (x-3)^{-2}$$

$$\Rightarrow \frac{du}{dx} = 2x - 4 \text{ and } \frac{dv}{dx} = (-2)(x-3)^{-3}$$

we have:

$$\frac{dy}{dx} = (x-3)^{-2}(2x-4) + (x^2-4x+12)(-2)(x-3)^{-3}$$

After tidying up the negative powers and fractions, this simplifies to $\frac{dy}{dx} = \frac{-2x-12}{(x-3)^3}$.

This process is laborious, but it can be applied to a general function of the form $\frac{u(x)}{v(x)}$ to produce a new rule for differentiating quotients.

The details are given in the Fill-in proof 22 on the CD-ROM, but you only need to know how to use the result.

KEY POINT 18.5

The quotient rule

If $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Worked example 18.7

Use the quotient rule to differentiate $y = \frac{x^2-4x+12}{(x-3)^2}$. Simplify your answer as far as possible.

This is a quotient. Make sure to get u and v the right way round

$$y = \frac{u}{v},\quad u = x^2-4x+12,\quad v = (x-3)^2$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Use chain rule to differentiate v then substitute the appropriate values into the quotient rule

$$= \frac{(x-3)^2(2x-4) - (x^2-4x+12)2(x-3)}{\left[(x-3)^2\right]^2}$$

Cancel a factor of $(x - 3)$

$$= \frac{(2x-4)(x-3) - (x^2-4x+12)2}{(x-3)^3}$$

$$= \frac{2x^2-10x+12-2x^2+8x-24}{(x-3)^3} = \frac{-2x-12}{(x-3)^3}$$

In Section 16E we stated the result that the derivative of tan x is $\sec^2 x$. We can now use the quotient rule, together with the derivatives of $\sin x$ and $\cos x$, to prove this result.

Worked example 18.8

Prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

We know how to differentiate $\sin x$ and $\cos x$, so use them to express $\tan x$

$\tan x = \frac{\sin x}{\cos x}$, $u = \sin x$, $v = \cos x$

Use quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

The quotient rule, like the product rule, often leads to a long expression. You do not need to simplify this expression unless asked to do so. However, sometimes product and quotient rule questions are also used to test your skill with fractions and exponents, as in the following example.

Worked example 18.9

Differentiate $\frac{x}{\sqrt{x+1}}$, giving your answer in the form $\frac{x+c}{k\sqrt{(x+1)^p}}$ where $c, k, p \in \mathbb{N}$.

This is a quotient

$y = \frac{x}{\sqrt{x+1}}$, $u = x$, $v = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

continued . . .

Use quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(x+1)^{-\frac{1}{2}}}{\left((x+1)^{\frac{1}{2}}\right)^2}$$

As we want a square root in the answer, turn the fractional powers back into roots

$$= \frac{\sqrt{x+1} - \frac{x}{2\sqrt{x+1}}}{x+1}$$

Remove 'fractions within fractions' by multiplying top and bottom by $2\sqrt{x+1}$

$$= \frac{2(x+1) - x}{2(x+1)\sqrt{x+1}}$$

Notice that $x\sqrt{x} = x^{\frac{3}{2}} = \sqrt{x^3}$

$$= \frac{x+2}{2\sqrt{(x+1)^3}}$$

Exercise 18C

1. Differentiate using the quotient rule:

(a) (i) $y = \dfrac{x-1}{x+1}$ (ii) $y = \dfrac{x+2}{x-3}$

(b) (i) $y = \dfrac{\sqrt{2x+1}}{x}$ (ii) $y = \dfrac{x^2}{\sqrt{x-1}}$

(c) (i) $y = \dfrac{1-2x}{x^2+2}$ (ii) $y = \dfrac{4-x^2}{1+x}$

(d) (i) $y = \dfrac{\ln 3x}{x}$ (ii) $y = \dfrac{\ln 2x}{x^2}$

2. Find the equation of the normal to the curve $y = \dfrac{\sin x}{x}$ at the point where $x = \dfrac{\pi}{2}$, giving your answer in the form $y = mx + c$ where m and c are exact. *[7 marks]*

3. Find the coordinates of the stationary points on the graph of $y=\frac{x^2}{2x-1}$. *[5 marks]*

4. The graph of $y=\frac{x-a}{x+2}$ has gradient 1 at the point $(a, 0)$ and $a \neq -2$. Find the value of a. *[5 marks]*

5. Find the exact coordinates of the stationary point on the curve $y=\frac{\ln x}{x}$ and determine its nature. *[6 marks]*

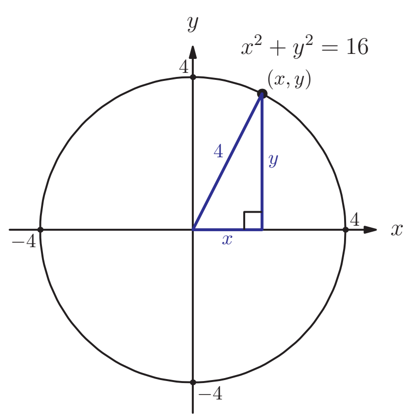
6. Find the range of values of x for which the function $f(x)=\frac{x^2}{1-x}$ is increasing. *[6 marks]*

7. Given that $y=\frac{x^2}{\sqrt{x+1}}$ show that $\frac{dy}{dx}=\frac{x(ax+b)}{2(x+1)^p}$, stating clearly the value of the constants a, b and p. *[6 marks]*

8. Show that if the curve $y=f(x)$ has a maximum stationary point at $x=a$ then the curve $y=\frac{1}{f(x)}$ has a minimum stationary point at $x=a$ as long as $f(a)\neq 0$. *[7 marks]*

18D Implicit differentiation

The functions we have differentiated so far have been of the form $y=f(x)$, but we will also meet functions that are not expressed in this form. For example, the coordinates of a point on the circle shown in the diagram satisfy the equation $x^2+y^2=16$. Such functions are said to be **implicit** (and those in the form $y=f(x)$ are said to be explicit).



Rather than trying to rearrange the equation, we can just differentiate the equation term by term with respect to x:

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(16)$$

Note that care is needed when differentiating y^2 as it is a composite function. We will need the chain rule:

$$\frac{d(y^2)}{dx}=\frac{d(y^2)}{dy}\times\frac{dy}{dx}=2y\frac{dy}{dx}$$

The chain rule will be needed when differentiating any terms involving y.

KEY POINT 18.6

When differentiating implicitly, we need to use:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

We can now find $\frac{dy}{dx}$ for the equation of the circle above.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Notice that the expression for $\frac{dy}{dx}$ will often be in terms of both x and y. Sometimes implicit differentiation may also need the product rule.

Worked example 18.10

Find an expression for $\frac{dy}{dx}$ if $e^x + x \sin y = \cos 2y$.

Differentiate term by term, using chain rule on all y terms

$$\frac{d}{dx}(e^x) + \frac{d}{dx}(x \sin y) = \frac{d}{dx}(\cos 2y)$$

$x \sin y$ is a product, so use the product rule and the chain rule on all y terms

$$\Rightarrow e^x + \left(x \times \cos y \frac{dy}{dx} + \sin y \times 1\right) = -2\sin 2y \frac{dy}{dx}$$

Group the terms involving $\frac{dy}{dx}$

$$\Rightarrow x\cos y \frac{dy}{dx} + 2\sin 2y \frac{dy}{dx} = -e^x - \sin y$$

$$\Rightarrow (x\cos y + 2\sin 2y)\frac{dy}{dx} = -e^x - \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^x - \sin y}{x\cos y + 2\sin 2y}$$

If we are only interested in the gradient at a particular point, or we are given the gradient and need to find the x- and y-coordinates, we can substitute given values into the differentiated equation without rearranging it.

Worked example 18.11

Find the coordinates of the turning points on the curve $y^3 + 3xy^2 - x^3 = 27$.

Differentiate each term with respect to x but notice that the term $3xy^2$ will need the product rule

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(3xy^2) - \frac{d}{dx}(x^3) = \frac{d}{dx}(27)$$

Use the chain rule on all y terms

$$\Rightarrow 3y^2\frac{dy}{dx} + \left(3x \times 2y\frac{dy}{dx} + y^2 \times 3\right) - 3x^2 = 0$$

$$\Rightarrow 3y^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2 - 3x^2 = 0$$

We know the value of $\frac{dy}{dx}$

For stationary points, $\frac{dy}{dx} = 0$

$\Rightarrow 3y^2 - 3x^2 = 0$

$\Rightarrow (y - x)(y + x) = 0$

$\Rightarrow y = x$ or $y = -x$

We have found a relationship between x and y at the stationary points, to actually find the points substitute back into the original function

When $x = y$:

$x^3 + 3xx^2 - x^3 = 27$

$\Rightarrow 3x^3 = 27$

$\Rightarrow x^3 = 9$

$\Rightarrow x = \sqrt[3]{9}$

$\therefore \left(\sqrt[3]{9}, \sqrt[3]{9}\right)$ is a stationary point

When $x = -y$:

$(-x)^3 + 3x(-x)^2 - x^3 = 27$

$\Rightarrow -x^3 + 3x^3 - x^3 = 27$

$\Rightarrow x^3 = 27$

$\Rightarrow x = 3$

$\therefore (3, -3)$ is a stationary point

One application of implicit differentiation is to differentiate exponential functions with a base other than e.

Worked example 18.12

Show that $\frac{\mathrm{d}}{\mathrm{d}x}(5^x) = 5^x \ln 5$.

Take ln of both sides to 'remove' the power

Let $y = 5^x$

Then $\ln y = x \ln 5$

We can differentiate implicitly

$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln 5)$

Remember that ln*a* is a constant

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln 5$

$\Rightarrow \frac{dy}{dx} = y \ln 5 = 5^x \ln 5$

EXAM HINT

Although these results are given in the Formula booklet, you could be asked to prove them.

We can use this procedure, and a similar one for $y = \log_a x$ (using the change of base rule), to derive the following general results:

KEY POINT 18.7

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}$$

Exercise 18D

1. Find the gradient of each curve at the given point:

(a) (i) $x^2 + 3y^2 = 7$ at $(2, -1)$ (ii) $2x^3 - y^3 = -6$ at $(1, 2)$

(b) (i) $\cos x + \sin y = 0$ at $(0, \pi)$

(ii) $\tan x + \tan y = 2$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

(c) (i) $x^2 + 3xy + y^2 = 20$ at $(2, 2)$

(ii) $3x^2 - xy^2 + 3y = 21$ at $(-1, 3)$

(d) (i) $xe^y + ye^x = 2e$ at $(1, 1)$ (ii) $x \ln y - \frac{x}{y} = 2$ at $(-1, 1)$

2. Find $\frac{dy}{dx}$ in terms of x and y:

(a) (i) $3x^2 - y^3 = 15$ (ii) $x^4 + 3y^2 = 20$

(b) (i) $xy^2 - 4x^2y = 6$ (ii) $y^2 - xy = 7$

(c) (i) $\frac{x+y}{x-y} = 2y$ (ii) $\frac{y^2}{xy+1} = 1$

(d) (i) $xe^y - 4\ln y = x^2$ (ii) $3x\sin y + 2\cos y = \sin x$

3. Find the coordinates of stationary points on the curves given by these implicit equations:

(i) $-x^2 + 3xy + y^2 = 13$ (ii) $2x^2 - xy + y^2 = 28$

4. Find the exact value of the gradient at the given point:

(a) (i) $y = 3^x$ at $(1, 3)$ (ii) $y = 5^x$ at $(2, 25)$

(b) (i) $y = \left(\frac{1}{2}\right)^x$ when $x = -2$ (ii) $y = \left(\frac{1}{3}\right)^x$ when $x = -1$

(c) (i) $y = 2^{3x}$ when $x = -1$ (ii) $y = 4^{2x}$ when $x = \frac{1}{4}$

(d) (i) $y = 3^{3-x}$ when $x = 2$ (ii) $y = 5^{1-x}$ when $x = 2$

5. (a) On Fill-in proof 18 'Differentiating logarithmic functions graphically' on the CD-ROM we constructed an argument which suggested that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Use the fact that $\ln x$ is the inverse function of e^x and implicit differentiation to prove this result.

(b) Show that $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$.

(c) Differentiate $\ln kx$ and $\ln x^n$ using chain rule. What do you notice? Why is this the case? *[6 marks]*

6. Find the gradient of the curve with equation $x^2 - 3xy + y^2 + 1 = 0$ at the point (1, 2). *[6 marks]*

7. Find the equation of the tangent to the curve with equation $4x^2 - 3xy - y^2 = 25$ at the point (2, –3). *[6 marks]*

8. A curve has implicit equation $x2^y = \ln y$. Find an expression for $\frac{dy}{dx}$ in terms of x and y. *[6 marks]*

9. Find the coordinates of the stationary point on the curve given by $e^x + ye^{-x} = 2e^2$. *[6 marks]*

10. The line L is tangent to the curve C which has the equation $y^2 = x^3$ when $x = 4$ and $y > 0$.

(a) By rearranging the curve into the form $y = \pm f(x)$ or otherwise, sketch C.

(b) Find the equation of L.

(c) Show that L meets C again at the point P with an x-coordinate which satisfies the equation $x^3 - 9x^2 + 24x - 16 = 0$.

(d) Find the coordinates of the point P. *[10 marks]*

18E Differentiating inverse trigonometric functions

Implicit differentiation can also be used to find the derivatives of the inverse trigonometric functions $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$.

Worked example 18.13

If $y = \arcsin x$, find $\frac{dy}{dx}$ in terms of x.

We know how to differentiate sin, so express x in terms of y

$y = \arcsin x \quad \Rightarrow \sin y = x$

Differentiate each term with respect to x, remembering the chain rule

$\Rightarrow \cos y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

We want the answer in terms of x, so we need to change cos to sin

$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

We can establish the results for the inverse cos and tan functions similarly giving:

KEY POINT 18.8

$$y = \arcsin x \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$y = \arccos x \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan x \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

Worked example 18.14

Differentiate:

(a) $y = \arctan 4x$ (b) $y = \arccos\sqrt{x-3}$

Multiply the standard result by 4, the derivative of 4x (using chain rule)

(a) $\frac{dy}{dx} = \frac{1}{1+(4x)^2} \times 4$

$= \frac{4}{1+16x^2}$

Again using the chain rule multiply by $\frac{1}{2}(x-3)^{-\frac{1}{2}}$, the derivative of $\sqrt{x-3}$

(b) $\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\sqrt{x-3}\right)^2}} \times \frac{1}{2}(x-3)^{-\frac{1}{2}}$

$= \frac{-1}{\sqrt{1-(x-3)}} \times \frac{1}{2\sqrt{x-3}}$

$= \frac{-1}{2\sqrt{(4-x)(x-3)}}$

Exercise 18E

1. Find $\frac{dy}{dx}$ for each of the following:

(a) (i) $y = \arccos(3x)$ (ii) $y = \arccos(2x)$

(b) (i) $y = \arctan\left(\frac{x}{2}\right)$ (ii) $y = \arctan\left(\frac{2x}{5}\right)$

(c) (i) $y = x\arcsin x$ (ii) $y = x^2 \arccos x$

(d) (i) $y = \arctan(x^2+1)$ (ii) $y = \arcsin(1-x^2)$

2. Find the exact value of the gradient of the graph of $y = \arccos\left(\frac{x}{2}\right)$ at the point where $x = \frac{1}{3}$. *[5 marks]*

3. Given that $y = \arcsin\left(\frac{3x}{2}\right)$, show that $\frac{dy}{dx} = \frac{3}{\sqrt{4-9x^2}}$. *[5 marks]*

4. Given that $x \arctan y = 1$, find an expression for $\frac{dy}{dx}$. *[5 marks]*

5. (a) Find $\frac{d}{dx}(x \arcsin x)$. *[5 marks]*

(b) Hence find $\int \arcsin x \, dx$. *[6 marks]*

6. Show that the graph of $y = \arcsin(x^2)$ has no points of inflexion. *[6 marks]*

Summary

- The **chain rule** is used to differentiate composite functions.

$$\text{If } y = f(u) \text{ where } u = g(x), \text{ then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

- The **product rule** is used to differentiate two functions multiplied together.

$$\text{If } y = u(x)v(x), \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

- The **quotient rule** is used to differentiate one function divided by another.

$$\text{If } y = \frac{u(x)}{v(x)}, \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

- The derivatives of the reciprocal trigonometric functions are:

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

- The derivative of an exponential function is:

$$\frac{d}{dx}(a^x) = a^x \ln a$$

- The derivative of a log function is:

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

- The derivatives of the inverse trigonometric functions are:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

- In an **implicit equation**, differentiate each term separately noting that for functions of y the chain rule needs to be used:

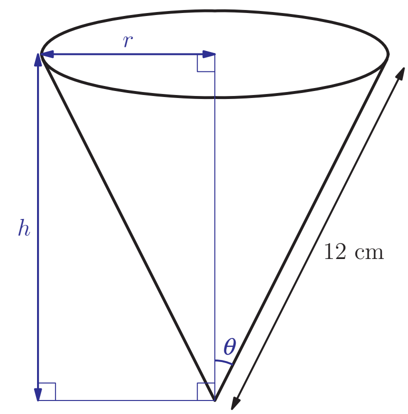
$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

Introductory problem revisited

> Given a cone of fixed slant height 12 cm, find the maximum volume as apex angle θ varies.

First we need to write an expression for the volume of the cone. Then we can differentiate with respect to θ and solve $\frac{dV}{d\theta} = 0$ to find the value of θ at which the maximum occurs.

$V = \frac{1}{3}\pi r^2 h$



Using the right-angled triangle highlighted in the diagram:

$r = 12\sin\theta$

$h = 12\cos\theta$

Therefore, substituting into the formula for V we have:

$$V = \frac{1}{3}\pi(12\sin\theta)^2(12\cos\theta) = \frac{12^3}{3}\pi\sin^2\theta\cos\theta$$

For stationary points, $\frac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = \frac{12^3}{3}\pi\left[(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta(-\sin\theta)\right]$$

$$= \frac{12^3}{3}\pi\left[2\sin\theta\cos^2\theta - \sin^3\theta\right] = 0$$

$\Rightarrow 2\sin\theta\cos^2\theta - \sin^3\theta = 0$

$\Rightarrow \sin\theta = 0 \quad \text{or} \quad 2\cos^2\theta - \sin^2\theta = 0$

$\sin\theta = 0$ has no valid solutions, since for a cone, $0 < \theta < 90°$.

$2\cos^2\theta - \sin^2\theta = 0 \Rightarrow 2\tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$ ($\tan\theta = -\sqrt{2}$ has no solutions $0 < \theta < 90°$)

Therefore the maximum volume occurs when $\tan\theta = \sqrt{2}$, which means $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$.

Therefore, substituting into $V = \frac{12^3}{3}\pi\sin^2\theta\cos\theta$:

$$V_{\max} = \frac{12^3}{3}\pi\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) = \frac{12^3\,2\sqrt{3}\pi}{3^3} = 4^3\,2\sqrt{3}\pi = 128\sqrt{3}\pi$$

Mixed examination practice 18

Short questions

1. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = x^2 \arcsin x$

(b) $xe^y = 4y^2$ *[7 marks]*

2. Differentiate $f(x) = \arccos(1 - x^2)$. *[4 marks]*

3. Find the exact value of the gradient of the curve with equation $y = \frac{1}{4 - x^2}$ when $x = \frac{1}{2}$. *[5 marks]*

4. Find the equation of the normal to the curve with equation $4x^2 + xy^2 - 3y^3 = 56$ at the point $(-5, 2)$. *[7 marks]*

5. Given that $y = \arctan(x^2)$ find $\frac{d^2y}{dx^2}$. *[5 marks]*

6. Find the gradient of the curve with equation $4\sin x \cos y + \sec^2 y = 5$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. *[6 marks]*

7. The graph of $y = xe^{-kx}$ has a stationary point when $x = \frac{2}{5}$. Find the value of k. *[4 marks]*

8. A curve has equation $f(x) = \frac{a}{b + e^{-cx}}$, $a \neq 0, b, c > 0$.

(a) Show that $f''(x) = \frac{ac^2 e^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3}$.

(b) Find the coordinates of the point on the curve where $f''(x) = 0$.

(c) Show that this is a point of inflexion. *[8 marks]*

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9. Find the coordinates of stationary points on the curve with equation $(y - 2)^2 e^x = 4x$. *[7 marks]*

Long questions

1. A curve has equation $y = \frac{x^2}{1-2x}$.

(a) Write down the equation of the vertical asymptote of the curve.

(b) Use differentiation to find the coordinates of stationary points on the curve.

(c) Determine the nature of the stationary points.

(d) Sketch the graph of $y = \frac{x^2}{1-2x}$. *[15 marks]*

2. The function f is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that $f'(x) = \frac{2x - x^2 \ln 2}{2^x}$.

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(b) (i) Find the exact value of x satisfying the equation $f'(x) = 0$.

(ii) Show that this value gives a maximum value for $f(x)$.

(c) Find the x-coordinates of the two points of inflexion on the graph of f.

[12 marks]

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3. Let $f(x) = \arccos(\sqrt{1-9x^2})$ for $0 < x < \frac{1}{3}$.

(a) Show that $f'(x) = \frac{3}{\sqrt{1-9x^2}}$.

(b) Show that $f''(x) > 0$ for all $x \in]0, \frac{1}{3}[$.

(c) Let $g(x) = \arccos(kx)$. If $g'(x) = -pf'(x)$ for $0 < x < \frac{1}{3}$, find the values of p and k. *[12 marks]*

4. A curve is given by the implicit equation $x^2 - xy + y^2 = 12$.

(a) Find the coordinates of the stationary points on the curve.

(b) Show that at the stationary points, $(x-2y)\frac{d^2y}{dx^2} = 2$.

(c) Hence determine the nature of the stationary points. *[16 marks]*

5. If $f(x) = \sec x$, $0 \le x \le \pi$ the inverse function is $f^{-1}(x) = \text{arcsec}\, x$.

(a) Write down the domain of $\text{arcsec}\, x$.

(b) Sketch the graph of $y = \text{arcsec}\, x$.

(c) Show that the derivative of $\sec x$ is $\sec x \tan x$.

(d) Find the derivative of $\text{arcsec}\, x$ with respect to x, justifying carefully the sign of your answer. *[12 marks]*

19 Further integration methods

In this chapter you will learn:

- to integrate using known derivatives
- to use the chain rule in reverse
- to integrate using trigonometric identities
- to integrate using inverse trigonometric functions
- to use the product rule in reverse (integration by parts)
- to integrate using a change of variable (substitution)
- to integrate using the separation of a fraction into two fractions.

Introductory problem

Use integration to prove that the area of a circle of radius r is equal to πr^2.

Having extended the range of functions we can differentiate, we now need to do the same for integration. Sometimes we will be able to use results from the previous chapter, but in other cases we will require new techniques. In this chapter we look at each of these in turn and then face the challenge of selecting the appropriate technique from the list of options we have.

19A Reversing standard derivatives

In chapter 17 we reversed a number of **standard derivatives** that had been established in chapter 16 to give us this list of functions we could integrate.

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \; n \neq -1$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

EXAM HINT

These are all given in the Formula booklet.

In chapter 18 (Key point 18.3) we differentiated sec x, csc x and cot x. We can now reverse these standard derivatives too and add them to our list:

$$\int \sec^2 x \, dx = \tan x + c$$
$$\int \sec x \tan x \, dx = \sec x + c$$
$$\int \csc x \cot x \, dx = -\csc x + c$$
$$\int \csc^2 x \, dx = -\cot x + c$$

EXAM HINT

These are not given in the list of **standard integrals** in the Formula booklet, but can be deduced from the list of standard derivatives.

The chain rule for differentiation (chapter 18A) allows us to go further and deal with integrals such as $\int 2\cos(2x)\,dx$. Here we think about integrating cos to sin and then consider what the chain rule would give us if we differentiated back. In this case the chain rule would give the 2 anyway as $\frac{d}{dx}(\sin 2x) = 2\cos 2x$ (2 is the derivative of $2x$) so we have the correct integration straight away:

$$\int 2\cos(2x)\,dx = \sin 2x + c$$

We may have a similar question in which we do not have the exact derivative and then we need to compensate by cancelling out any unwanted constant generated by the chain rule.

For example, in finding $\int (2x-3)^4\,dx$ we proceed as before integrating $(\ )^4$ to $\frac{1}{5}(\ )^5$ but now when we differentiate back the chain rule gives us an unwanted 2:

$$\frac{d}{dx}\left(\frac{1}{5}(2x-3)^5\right) = 2(2x-3)^4$$

so we divide by 2 to remove it:

$$\int (2x-3)^4\,dx = \frac{1}{2} \times \frac{1}{5}(2x-3)^5 + c = \frac{1}{10}(2x-3) + c.$$

You may notice a pattern here, we always divide by the coefficient of x. This is indeed a general rule, which follows simply by reversing the special case of the chain rule from Key point 18.2.

KEY POINT 19.1

The reverse chain rule

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$$

where $F(x)$ is the integral of $f(x)$.

EXAM HINT

This rule only applies when the 'inside' function is of the form $(ax + b)$!

With this shortcut we do not need to work through the chain rule every time.

Worked example 19.1

Find the following:

(a) $\int \frac{1}{2}e^{4x}dx$ (b) $\int \frac{2}{5-x}dx$

Integrate $e^{(\ )}$ to $e^{(\ )}$ and divide by the coefficient of x

(a) $\int \frac{1}{2}e^{4x}dx = \frac{1}{2} \times \frac{1}{4}e^{4x} + c$

$= \frac{1}{8}e^{4x} + c$

Integrate $\frac{1}{(\ )}$ to $\ln|\ \ |$ and divide by the coefficient of x

(b) $\int \frac{2}{5-x}dx = 2\left(\frac{1}{-1}\right)\ln|5-x| + c$

$= -2\ln|5-x| + c$

Exercise 19A

1. Find:

(a) (i) $\int 5(x+3)^4\,dx$ (ii) $\int (x-2)^5\,dx$

(b) (i) $\int (4x-5)^7\,dx$ (ii) $\int \left(\frac{1}{8}x+1\right)^3 dx$

(c) (i) $\int 4\left(3-\frac{1}{2}x\right)^6 dx$ (ii) $\int (4-x)^8\,dx$

(d) (i) $\int \sqrt{2x-1}\,dx$ (ii) $\int 7(2-5x)^{3/4}\,dx$

(e) (i) $\int \frac{1}{\sqrt[4]{2+\frac{x}{3}}}\,dx$ (ii) $\int \frac{6}{(4-3x)^2}\,dx$

2. Find these integrals:

(a) (i) $\int 3e^{3x}\,dx$ (ii) $\int e^{2x+5}\,dx$

(b) (i) $\int 4e^{\frac{2x-1}{3}}\,dx$ (ii) $\int e^{\frac{1}{2}x}\,dx$

(c) (i) $\int -6e^{-3x}\,dx$ (ii) $\int \frac{1}{e^{4x}}\,dx$

(d) (i) $\int \frac{-2}{e^{x/4}}\,dx$ (ii) $\int e^{-\frac{2}{3}x}\,dx$

3. Find:

(a) (i) $\int \frac{1}{x+4}\,dx$ (ii) $\int \frac{5}{5x-2}\,dx$

(b) (i) $\int \frac{2}{3x+4}\,dx$ (ii) $\int \frac{-8}{2x-5}\,dx$

(c) (i) $\int \frac{-3}{1-4x}\,dx$ (ii) $\int \frac{1}{7-2x}\,dx$

(d) (i) $\int 1-\frac{3}{5-x}\,dx$ (ii) $\int 3+\frac{1}{3-x}\,dx$

4. Integrate the following:

(a) $\int -\csc x \cot x\,dx$

(b) $\int 3\sec^2 3x\,dx$

(c) $\int \sin(2-3x)\,dx$

(d) $\int \csc^2\left(\frac{1}{4}x\right)dx$

(e) $\int 2\cos 4x\,dx$

(f) $\int \sec\frac{x}{2}\tan\frac{x}{2}\,dx$

5. Two students integrate $\int \frac{1}{3x}\,dx$ in two different ways.
Marina writes:

$$\int \frac{1}{3x}\,dx = \frac{1}{3}\int \frac{1}{x}\,dx = \frac{1}{3}\ln|x| + c$$

Jack uses the special case of the reverse chain rule and divides by the coefficient of x:

$$\int \frac{1}{3x}\,dx = \frac{1}{3}\ln|3x| + c$$

Who has the right answer?

6. Given that $0 < a < 1$ and the area between the x-axis, the lines $x = a^2$, $x = a$ and the graph of $y = \frac{1}{1-x}$ is 0.4, find the value of a correct to 3 significant figures. *[5 marks]*

19B Integration by substitution

The shortcut for reversing the chain rule works only when the derivative of the 'inside' function is a constant. This is because a constant factor can 'move through the integral sign', for example:

$$\int \cos 2x \, dx = \int \frac{1}{2} \times 2\cos 2x \, dx = \frac{1}{2}\int 2\cos 2x \, dx = \frac{1}{2}\sin 2x + c$$

Another method for integrating products is integration by parts, the reverse of the product rule. We will meet this in Section 19F.

This cannot be done with a variable: $\int x \sin x \, dx$ is not the same as $x\int \sin x \, dx$. So we need a different rule for integrating a product of two functions. In some cases this can be done by extending the principle of reversing the chain rule, leading to the method of **integration by substitution**.

When using the chain rule to differentiate a composite function, we differentiate the outer function and multiply this by the derivative of the inner function; for example

$$\frac{d}{dx}\left(\sin\left(x^2+2\right)\right) = \cos\left(x^2+2\right) \times 2x$$

We can think of this as using a substitution $u = x^2 + 2$, and then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Let us now look at $\int x\cos(x^2+2)\, dx$.

As $\cos(x^2+2)$ is a composite function we can write it as $\cos u$, where $u = x^2 + 2$. So our integral becomes $\int x\cos u \, dx$. We know how to integrate $\cos u$, so we want to change our variable to u. But then we need to be integrating with respect to u, so we should have du instead of dx. Those two are not the same thing, but they are related because $u = x^2 + 2 \Rightarrow \frac{du}{dx} = 2x$.

We can now 'rearrange' this to get $dx = \frac{1}{2x} du$.

Substituting all this into our integral we now have

$$\begin{aligned}\int x\cos\left(x^2+2\right)dx &= \int x\cos u\left(\frac{1}{2x}\right)du \\ &= \int \frac{1}{2}\cos u \, du \\ &= \frac{1}{2}\sin u + c\end{aligned}$$

This answer is in terms of u so we need to write it in terms of x.

$$\int x\cos(x^2+2)\, dx = \frac{1}{2}\sin(x^2+2) + c$$

A word of warning here: $\frac{du}{dx}$ is not really a fraction, so it is not clear that the above 'rearrangement' is valid. However, it can be shown that it follows from the chain rule that it is valid to replace dx by $\frac{1}{f'(x)}du$.

Worked example 19.2

Find the following:

(a) $\int \sin^5 x \cos x \, dx$ (b) $\int x^2 e^{x^3+4} \, dx$

It is helpful here to think of $\sin^5 x$ as $(\sin x)^5$. Therefore the inner function is $\sin x$

(a) Let $u = \sin x$

Then $\frac{du}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} du$

Make the substitution

$$\int (\sin x)^5 \cos x \, dx = \int u^5 \cos x \frac{1}{\cos x} du$$

$$= \int u^5 du$$

$$= \frac{1}{6} u^6 + c$$

Write the answer in terms of x

$$= \frac{1}{6} \sin^6 x + c$$

e^{x^3+4} is a composite function with inner function $x^3 + 4$

(b) Let $u = x^3 + 4$

Then $\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{1}{3x^2} du$

Make the substitution

$$\int x^2 e^{x^3+4} dx = \int x^2 e^u \frac{1}{3x^2} du$$

$$= \int \frac{1}{3} e^u du$$

$$= \frac{1}{3} e^u + c$$

Write the answer in terms of x

$$= \frac{1}{3} e^{x^3+4} + c$$

You may have noticed in all of the above examples that, after making the substitution, the part of the integrand which was still in terms of x cancelled with a similar term coming from $\frac{du}{dx}$.

For example, in (b), $\int x^2 e^{x^3+4} dx = \int x^2 e^u \frac{1}{3x^2} du = \int \frac{1}{3} e^u du$.

This will always happen when one part of the integrand is an exact multiple of the derivative of the inner function, and can be explained by looking at the chain rule.

For example, consider the integral $\int (2x+3)(x^2+3x-5)^4 \, dx$.

To find this integral, think about what we would need to differentiate to get $(2x+3)(x^2+3x-5)^4$. As $2x+3$ is the derivative of x^2+3x-5 we know that we would get $2x+3$ 'for free' when differentiating some power of x^2+3x-5 using the chain rule. In this case to end up with $(x^2+3x-5)^4$ we would want to be differentiating $\frac{1}{5}(x^2+3x-5)^5$, that is

$$\frac{d}{dx}\left(\frac{1}{5}(x^2+3x-5)^5\right) = (2x+3)(x^2+3x-5)^4$$

and therefore:

$$\int (2x+3)(x^2+3x-5)^4 \, dx = \frac{1}{5}(x^2+3x-5)^5 + c$$

This is the same answer we would get by using the substitution $u = x^2+3x-5$. If you notice that you can integrate an expression by reversing the chain rule, you can just write down the answer without any working. However, if you are not sure, it is safer to go through the whole process of substitution.

In some cases this cancelling of the remaining x-terms will not happen and you will have to express x in terms of u. The full method of substitution will then be as follows:

EXAM HINT

You will nearly always be told which substitution to use. If you are not, look for a composite function and take $u =$ 'inner' function.

KEY POINT 19.2

Integration by substitution

1. Select a substitution (if not already given).
2. Differentiate the substitution and write dx in terms of du.
3. Replace dx by the above expression, and replace any obvious occurrences of u.
4. Simplify as far as possible.
5. If any terms with x remain, write them in terms of u.
6. Work out the new integral in terms of u.
7. Write the answer in terms of x.

For the integral in the next example there are two possible substitutions. As there is a composite function $\sqrt{4x-1}$, we could use the 'inner' function: $u = 4x-1$. However, we must always use the substitution we are given.

Worked example 19.3

Find $\int x\sqrt{4x-1}\,dx$ using the substitution $u=\sqrt{4x-1}$.

Differentiate the substitution

$$u=\sqrt{4x-1}$$

$$\Rightarrow \frac{du}{dx}=\frac{4}{2\sqrt{4x-1}}$$

$$=\frac{2}{u}$$

... and write dx in terms of du (Key point 19.2 Step 2)

$$\therefore dx=\frac{1}{2}u\,du$$

Replace those parts that we already have expressions for, and simplify if possible (Steps 3 and 4)

$$\int x\sqrt{4x-1}\,dx=\int xu\frac{1}{2}u\,du$$

$$=\int\frac{1}{2}xu^2du$$

There is still an x remaining, so replace it by using $u=\sqrt{4x-1}\Rightarrow x=\frac{u^2+1}{4}$ (Step 5)

$$=\int\frac{1}{2}\frac{u^2+1}{4}u^2\,du$$

Now everything is in terms of u so we can integrate (Step 6)

$$=\frac{1}{8}\int u^4+u^2\,du=\frac{1}{8}\left(\frac{1}{5}u^5+\frac{1}{3}u^3\right)+c$$

Write the answer in terms of x using $u=\sqrt{4x-1}$ (Step 7)

$$=\frac{1}{8}\left(\frac{1}{5}\left(\sqrt{4x-1}\right)^5+\frac{1}{3}\left(\sqrt{4x-1}\right)^3\right)+c$$

When limits are given, we must change them too. Then there is no need to change back to the original variable at the end.

KEY POINT 19.3

When evaluating a definite integral using substitution, add the following step to the process in Key point 19.2:

Step 3a. Write the limits in terms of u.

The next example shows one of the most common uses of substitution; integrating a quotient where the numerator is a multiple of the derivative of the denominator.

Worked example 19.4

Evaluate $\int_0^1 \frac{x-3}{x^2-6x+7}\,dx$ giving your answer in the form $a \ln p$.

This is of the form 'something' $\times \frac{1}{(\ )}$ so the 'inner' function is x^2-6x+7	Let $u = x^2-6x+7$. Then $\frac{du}{dx} = 2x-6 \Rightarrow dx = \frac{1}{2x-6}du$
Write limits in terms of u	Limits: $x=0 \Rightarrow u=7$, $x=1 \Rightarrow u=2$
Make the substitution	$\int_0^1 \frac{x-3}{x^2-6x+7}dx = \int_7^2 \frac{x-3}{u}\frac{1}{2x-6}du$
Simplify $2x-6=2(x-3)$	$= \int_7^2 \frac{1}{2u}du$ $= \left[\frac{1}{2}\ln\lvert u \rvert\right]_7^2$ $= \frac{1}{2}(\ln 2 - \ln 7)$ $= \frac{1}{2}\ln\left(\frac{2}{7}\right)$

In the above example it is possible to write down the result of the integration without using the full substitution method, if we notice that $x-3$ is half of the derivative of x^2-6x+7, and so $(x-3)\times\frac{1}{x^2-6x+7}$ comes from differentiating $\frac{1}{2}\ln\lvert x^2-6x+7\rvert$.

This particular case of substitution, where the top of the fraction is the derivative of the bottom, is definitely worth remembering:

KEY POINT 19.4

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

The next example shows that the substitution can also be given as x in terms of u (or θ in this case). It also illustrates that substitutions can lead to integrals where the use of trigonometric identities is required.

We will see more examples of using trigonometric identities in the next section.

Worked example 19.5

Use the substitution $x = \sec\theta$ to find the exact value of $\int_{\sqrt{2}}^{2} (x^2 - 1)^{-(3/2)}\,dx$.

Differentiate the substitution and express dx in terms of dθ

$x = \sec\theta$

$\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta$

$dx = \sec\theta\tan\theta\,d\theta$

Change the limits

When $x = \sqrt{2}$:

$\sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}$

When $x = 2$:

$\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

→

continued . . .

Replace those parts that we already have expressions for. Remember that $\sec^2\theta - 1 = \tan\theta$

$$\int_{\sqrt{2}}^{2}(x^2-1)^{-\frac{3}{2}}\,dx = \int_{\pi/4}^{\pi/3}(\sec^2\theta-1)^{-\frac{3}{2}}\sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3}(\tan^2\theta)^{-\frac{3}{2}}\sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3}(\tan\theta)^{-3}\sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3}(\tan\theta)^{-2}\sec\theta\,d\theta$$

We seem to be stuck; writing everything in terms of sin and cos often helps

$$= \int_{\pi/4}^{\pi/3}\frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3}\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Writing this as $\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$, we now have a standard derivative (Section 19A)

$$= \int_{\pi/4}^{\pi/3}\csc\theta\cot\theta\,d\theta$$

$$= [-\csc\theta]_{\pi/4}^{\pi/3} = \left(-\frac{2}{\sqrt{3}}\right)-(-\sqrt{2}) = \sqrt{2}-\frac{2}{\sqrt{3}}$$

Exercise 19B

1. Either by using a suitable substitution, or by considering the chain rule, find these integrals:

(a) (i) $\int x(x^2+3)^3\,dx$ (ii) $\int 3x(x^2-1)^5\,dx$

(b) (i) $\int (2x-5)(3x^2-15x+4)^4\,dx$

(ii) $\int (x^2+2x)(x^3+3x^2-5)^3\,dx$

(c) (i) $\int \frac{2x}{x^2+3}\,dx$ (ii) $\int \frac{6x^2-12}{x^3-6x+1}\,dx$

(d) (i) $\int 4\cos^5 3x\sin 3x\,dx$ (ii) $\int \cos 2x\sin^3 2x\,dx$

(e) (i) $\int 3xe^{3x^2-1}\,dx$ (ii) $\int 3xe^{x^2}\,dx$

(f) (i) $\int \frac{e^{2x+3}}{e^{2x+3}+4}\,dx$ (ii) $\int \frac{\cos x}{3+4\sin x}\,dx$

(g) (i) $\int 32\sec^2 2x\tan^3 2x\,\mathrm{d}x$

(ii) $\int 6\sec^6\left(\frac{x}{4}\right)\tan\left(\frac{x}{4}\right)\mathrm{d}x$

(h) (i) $\int \csc^4 x\cot x\,\mathrm{d}x$

(ii) $\int \frac{\csc^2 2x}{(3+\cot 2x)}\mathrm{d}x$

(i) (i) $\int \frac{x}{\sqrt{3-x^2}}\mathrm{d}x$

(ii) $\int 2\mathrm{e}^{-4x+1}\sqrt{\mathrm{e}^{-4x+1}}\,\mathrm{d}x$

2. Use a suitable substitution to show that $\int \tan x\,\mathrm{d}x = \ln|\sec x| + c$.

EXAM HINT

The integral of tan is not given in the Formula booklet, and is worth remembering.

3. Find the following integrals using the given substitution:

(a) (i) $\int x\sqrt{x+1}\,\mathrm{d}x,\ u = x+1$

(ii) $\int x^2\sqrt{x-2}\,\mathrm{d}x,\ u = x-2$

(b) (i) $\int 2x(x-5)^7\,\mathrm{d}x,\ u = x-5$

(ii) $\int x(x+3)^5\,\mathrm{d}x,\ u = x+3$

4. Find the following using an appropriate substitution:

(a) (i) $\int x(2x-1)^4\,\mathrm{d}x$ (ii) $\int 9x(3x+2)^5\,\mathrm{d}x$

(b) (i) $\int x\sqrt{x-3}\,\mathrm{d}x$ (ii) $\int (x+1)\sqrt{5x-6}\,\mathrm{d}x$

(c) (i) $\int \frac{x^2}{\sqrt{x-5}}\mathrm{d}x$ (ii) $\int \frac{4(x+5)}{(2x-3)^3}\mathrm{d}x$

5. Use the given substitution to evaluate these definite integrals:

(a) (i) $\int_2^3 \left(\frac{x}{4-x}\right)^2 \mathrm{d}x,\ u = 4-x$

(ii) $\int_1^3 \frac{x^3}{(x+2)^2}\mathrm{d}x,\ u = x+2$

(b) (i) $\int_0^{\pi/6} \frac{\cos\theta}{1+\sin\theta}\mathrm{d}\theta,\ u = \sin\theta$

(ii) $\int_0^{\pi/2} \frac{\sin\theta}{1+\cos\theta}\mathrm{d}\theta,\ u = 1+\cos\theta$

(c) (i) $\int_0^{1/3} \frac{1}{\sqrt{4-9x^2}}\mathrm{d}x,\ x = \frac{2}{3}\sin\theta$

(ii) $\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\mathrm{d}x,\ x = \frac{1}{2}\cos\theta$

EXAM HINT

When an unusual substitution is required it will always be given in the question.

6. Find the exact value of $\int_0^2 (2x+1)e^{x^2+x-1}\,dx$. *[6 marks]*

7. Evaluate $\int_2^5 \frac{2x}{x^2-1}\,dx$, giving your answer in the form $\ln k$. *[4 marks]*

8. Use the substitution $u = x-2$ to find $\int \frac{x}{\sqrt{x-2}}\,dx$. *[6 marks]*

9. (a) Show that $(x-1)$ is a factor of x^3-1.

(b) Find $\int \frac{2x^2-x-1}{x^3-1}\,dx$. *[4 marks]*

10. Use the substitution $u = \ln x$ to find $\int \frac{\sec^2(\ln(x^2))}{2x}\,dx$. *[6 marks]*

11. Find $\int \frac{\cos x}{\sin^5 x}\,dx$. *[3 marks]*

12. Evaluate $\int_1^3 \frac{(2x-3)\sqrt{x^2-3x+3}}{x^2-3x+3}\,dx$. *[6 marks]*

19C Using trigonometric identities in integration

Sometimes it is necessary to rearrange the expression before reversing a standard derivative or using a substitution. In this section we will take a more systematic look at using trigonometric identities in order to integrate a wide range of functions.

As seen in the previous section, the presence of the $\cos x$ in $\int \sin^3 x \cos x\,dx$ makes it possible to apply the reverse chain rule (or a substitution) but how do we cope with just $\int \sin^3 x\,dx$? As a mixture of sin and cos helps us in the use of the reverse chain rule, we aim to introduce cos by using $\sin^2 x + \cos^2 x = 1$.

EXAM HINT

If you are asked to do an integral like this in the exam you will be given a hint, as in the example below.

Worked example 19.6

(a) Show that $\sin^3 x = \sin x - \cos^2 x \sin x$.

(b) Hence find $\int \sin^3 x \, dx$.

Introduce $\cos^2 x$ by using $\sin^2 x + \cos^2 x = 1$; to do this we need to 'split' $\sin^3 x$

(a) $\sin^3 x = \sin^2 x \sin x$
$= (1 - \cos^2 x)\sin x$
$= \sin x - \cos^2 x \sin x$

We can use the result from part (a)

(b) $\int \sin^3 x \, dx = \int \sin x - \cos^2 x \sin x \, dx$

Use a substitution $u = \cos x$

$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$\int \sin^3 x \, dx = -\cos x - \int u^2 \sin x \left(-\frac{1}{\sin x}\right) du$

$= -\cos x + \int u^2 du$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

The same trick does not work for $\int \sin^2 x \, dx$, as we can only rewrite it as $\int 1 - \cos^2 x \, dx$ which we also cannot integrate. Instead we notice that $\sin^2 x$ appears in one of the versions of the double-angle formulae for $\cos 2x$: $\cos 2x = 1 - 2\sin^2 x$, and we know how to integrate $\cos 2x$.

Double angle identities were covered in Section 12A.

Worked example 19.7

Find $\sin^2 x \, dx$.

Write an alternative expression for $\sin^2 x$ by using a double angle identity

$\cos 2x = 1 - 2\sin^2 x$

$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\therefore \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

Remember to divide by the coefficient of x when integrating $\cos 2x$

$= \frac{1}{2}x - \frac{1}{2}\frac{1}{2}\sin 2x + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

A similar method is used to integrate $\cos^2 x$ and should be learnt.

EXAM HINT
You will be expected to recall this method without hints.

KEY POINT 19.5

> To integrate $\sin^2 x$, use $\cos 2x = 1 - 2\sin^2 x$.
>
> To integrate $\cos^2 x$, use $\cos 2x = 2\cos^2 x - 1$.

The methods from Worked examples 19.6 and 19.7 can be extended to deal with any powers of $\sin x$ and $\cos x$. The method from Worked example 19.6 can be applied to any odd power, for example:

$$\sin^5 x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x = (1 - 2\cos^2 x + \cos^4 x)\sin x$$

which can be integrated using the reverse chain rule.

For even powers, we can use the identity from Key point 19.5, for example

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

and then the double angle identity has to be used again to relate $\cos^2 2x$ to $\cos 4x$.

See Worked example 15.25 in Section 15H to remind you of this method.

This becomes increasingly complicated for larger powers. Luckily, there is a ready-made alternative from the unlikely source of complex numbers. We saw in chapter 15 that we could use De Moivre's Theorem to generate expressions for powers of sin and cos in terms of multiple angles.

Worked example 19.8

(a) Show that $\cos^6 x = \frac{1}{16}\cos 6x + \frac{3}{8}\cos 4x + \frac{15}{16}\cos 2x + \frac{5}{8}$.

(b) Hence find $\int \cos^6 x \, dx$.

We derived similar identities in Section 15H, so we will not repeat it here

(a) See Section 15H for how to do this.

Use the result from part (a)

(b) $\int \cos^6 x \, dx = \int \frac{1}{16}\cos 6x + \frac{3}{8}\cos 4x + \frac{15}{16}\cos 2x + \frac{5}{8} \, dx$

Don't forget to divide by the coefficient of x

$= \frac{1}{96}\sin 6x + \frac{3}{32}\sin 4x + \frac{15}{32}\sin 2x + \frac{5}{8}x + c$

Don't worry if this seems complicated – a question like this will always be split into several parts, as in questions 10 and 11 in Worked exercise 19C at the end of this section.

We shall now integrate tan x and its powers. We have already integrated tan x as an application of the reverse chain rule in Exercise 19B, question 2: $\int \tan x \, dx = \ln|\sec x| + c$. However, this does not help when trying to integrate more complicated functions, for example $\tan^2 x$. We do, however, have an identity relating $\tan^2 x$ to something we know how to integrate: $1 + \tan^2 x = \sec^2 x$.

This identity was derived in chapter 12.

Worked example 19.9

Find $\int \tan^2 2x \, dx$.

We have an identity relating $\tan^2(\ )$ to $\sec^2(\ )$, which we know how to integrate

$$\int \tan^2 2x \, dx = \int \sec^2 2x - 1 \, dx$$

Use the standard result for integrating $\sec^2(\ )$, remembering to divide by the coefficient of x

$$= \frac{1}{2}\tan 2x - x + c$$

The same identity is used in integrating any power of tan x.

KEY POINT 19.6

To integrate $\tan^n x$ use the identity $1 + \tan^2 x = \sec^2 x$ and the fact that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Worked example 19.10

Find $\int \tan^3 x \, dx$.

Introduce $\sec^2 x$ by using $\tan^2 x = \sec^2 x - 1$

$$\int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx$$
$$= \int (\sec^2 x - 1) \tan x \, dx$$

We integrate the two terms separately

$$= \int \sec^2 x \tan x \, dx - \int \tan x \, dx$$

We can apply the reverse chain rule (or a substitution $u = \tan x$) to $\sec^2 x \tan x$ because $\frac{d}{dx}(\tan x) = \sec^2 x$

$(\ )$ integrates to $\frac{1}{2}(\ )^2$

First integral:

$$\int \sec^2 x \tan x = \frac{1}{2}(\tan x)^2 + c$$

We found $\int \tan x$ in the previous section

Second integral:

$$\int \tan x \, dx = \ln|\sec x| + c$$

$$\therefore \int \tan^3 x \, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + c$$

The above examples illustrate standard methods used to integrate powers of trigonometric functions. Many other trigonometric integrals can be rearranged into a form where we can simply reverse a standard derivative. Here we give one example of using trigonometric identities to do this.

Worked example 19.11

Find $\int \frac{\sin 4x}{\sin^3 2x} \, dx$.

As we have $4x$ and $2x$, apply the double angle identity for sine

$$\int \frac{\sin 4x}{\sin^3 2x} dx = \int \frac{2\sin 2x \cos 2x}{\sin^3 2x} dx$$
$$= \int \frac{2\cos 2x}{\sin^2 2x} dx$$

continued . . .

Check if this can be manipulated into the form of one of our standard derivatives. If we cannot see it immediately, it is a good idea to try and split the expression into a product of two trigonometric functions

$$= 2\int \frac{1}{\sin 2x}\frac{\cos 2x}{\sin 2x}dx$$

$$= 2\int \csc 2x \cot 2x \, dx$$

Remember to divide by the coefficient of x

$$= -\csc 2x + c$$

Exercise 19C

1. Simplify to get standard integrals, and then integrate:

(a) $\int \frac{\tan 3x}{\cos 3x} \, dx$

(b) $\int \frac{1}{\sin^2 x} \, dx$

(c) $\int \sin 5x \cos x - \cos 5x \sin x \, dx$

(d) $\int \frac{3 - \cos 2x}{\sin^2 2x} \, dx$

(e) $\int \frac{\cos 2x}{\cos x + \sin x} \, dx$

2. Use trigonometric identities before using a substitution (or reversing the chain rule) to integrate:

(a) $\int \cos^3 x \sin^2 x \, dx$

(b) $\int \frac{\cos^3 x}{\sin^2 x} \, dx$

(c) $\int \sin x \cos x e^{\cos 2x} \, dx$

(d) $\int \tan^4 3x + \tan^6 3x \, dx$

(e) $\int \frac{\sin 2x \cos 2x}{\sqrt{1 + \cos 4x}} \, dx$

3. Find the following integrals:

(a) (i) $\int 2\cos^2 x \, dx$ (ii) $\int \cos^2 3x \, dx$

(b) (i) $\int 2\tan^2\left(\frac{x}{2}\right) dx$ (ii) $\int \tan^2 3x \, dx$

4. Find the exact value of the following:

(a) (i) $\int_0^{\pi} \sin^2 2x \, dx$ (ii) $\int_0^{2\pi} \tan^2\left(\frac{x}{6}\right) dx$

(b) (i) $\int_0^{\pi/4} (\tan x - 1)^2 \, dx$ (ii) $\int_{\pi/4}^{\pi} (1 + \cos 2x)^2 \, dx$

5. Three students integrate $\cos x \sin x$ in three different ways:

Amara uses reverse chain rule with $u = \sin x$:

$$\frac{du}{dx} = \cos x, so$$

$$\int \cos x \sin x \, dx = \int u \, du$$

$$= \frac{1}{2}\sin^2 x + c$$

Ben uses reverse chain rule with $u = \cos x$:

$$\frac{du}{dx} = -\sin x, \text{ so}$$

$$\int \cos x \sin x \, dx = \int -u \, du$$

$$= -\frac{1}{2}\cos^2 x + c$$

Carlos uses a double angle formula:

$$\int \cos x \sin x \, dx = \int \frac{1}{2}\sin 2x \, dx$$

$$= -\frac{1}{4}\cos 2x + c$$

Who is right?

6. Find $\int \sin^2\left(\frac{x}{3}\right) dx$. *[5 marks]*

7. (a) Show that $\tan^3 x = \tan x \sec^2 x - \tan x$.

(b) Hence find $\int \tan^3 x \, dx$. *[6 marks]*

8. Given that $\int_0^{\pi/12} \tan^2(kx) \, dx = \frac{4 - \pi}{12}$ find the value of k. *[6 marks]*

9. (a) Use the formula for $\cos(A + B)$ to show that $\cos 2x = 2\cos^2 x - 1$

(b) Hence find $\int \cos 2x \sin x \, dx$. *[7 marks]*

10. (a) Show that $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$.

(b) Hence find the exact value of $\int_0^{3\pi} \sin^3\left(\frac{x}{3}\right) dx$. *[7 marks]*

11. A complex number is defined by $z = \cos\theta + i\sin\theta$.

(a) (i) Show that $\frac{1}{z} = \cos\theta - i\sin\theta$.

(ii) Use De Moivre's Theorem to deduce that:

$z^n - \frac{1}{z^n} = 2i\sin n\theta$.

(b) (i) Expand $\left(z - \frac{1}{z}\right)^5$.

(ii) Hence find integers a, b and c such that:

$16\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$.

(c) Find $\int \sin^5 2x \, dx$. *[14 marks]*

19D Integration using inverse trigonometric functions

In the last section we saw examples of similar-looking integrals that required very different methods: $\int \sin^2 x \cos x \, dx$ could be done by reversing the chain rule. But $\int \sin^2 x \, dx$ required the more complex method of substituting with a trigonometric identity; without the derivative of $\sin x$ the integration was more difficult.

Similarly, consider $\int \frac{x}{\sqrt{1-x^2}} dx$ and $\int \frac{1}{\sqrt{1-x^2}} dx$.

The first integral features a function $(1-x^2)$ and a multiple of its derivative $(-2x)$, so we can apply the reverse chain rule: $\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + c$. However, the absence of the derivative of $(1-x^2)$ in the second integral means that we need another method.

Fortunately, we have already met the expression $\frac{1}{\sqrt{1-x^2}}$ in chapter 18 as the derivative of $\arcsin x$. In the same chapter we saw that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$. This means that:

These results are given in Key point 18.8.

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \text{ and } \int \frac{1}{1+x^2} dx = \arctan x.$$

We can extend these results slightly to include expressions of the form $\frac{1}{\sqrt{a^2-x^2}}$ and $\frac{1}{a^2+x^2}$. For example, to integrate $\frac{1}{16+x^2}$ we can take out a factor of $\frac{1}{16}$ to turn the denominator into the form $1+Y^2$ and then use a substitution:

$$\begin{aligned}\int \frac{1}{16+x^2}\,\mathrm{d}x &= \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,\mathrm{d}x \\ &= \frac{1}{16}\int \frac{1}{1+Y^2}\times 4\mathrm{d}Y \text{ where } Y=\frac{x}{4} \\ &= \frac{1}{4}\int \frac{1}{1+Y^2}\,\mathrm{d}Y \\ &= \frac{1}{4}\arctan Y + c \\ &= \frac{1}{4}\arctan\left(\frac{x}{4}\right)+c\end{aligned}$$

We can use this method to obtain the general result for the two integrals:

KEY POINT 19.7

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right)+c \quad (|x|<a)$$

It is worth noting several things about these; first of all, the arctan result has a factor of $\frac{1}{a}$ and the arcsin one does not. Secondly, the arcsin result only applies when $|x|<a$ because of the presence of the square root. Finally, you may be wondering why there is no corresponding result with arccos; if you look back at Key point 18.7 you will see that the derivative of $\arccos x$ is $-\frac{1}{\sqrt{1-x^2}}$, which, according to the Key point above, would integrate to $-\arcsin x$. This is because the graphs of the arcin and the arccos functions are related through a reflection and a translation.

You may wonder whether there are rules for integrating $\frac{1}{\sqrt{x^2-a^2}}$ when $|x|>a$ or $\frac{1}{\sqrt{x^2+a^2}}$. These require the study of hyperbolic functions, which are in many ways similar to trigonometric functions, and can be used to describe some important curves, such as the shape of a hanging chain.

Worked example 19.12

Find the value of: $\int \frac{1}{1+9x^2}\,dx$

This is similar to the derivative of arctan x but x^2 has been replaced by $9x^2=(3x)^2$. So reverse the standard derivative, remembering to divide by the coefficient of x

(a) $\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$

$= \frac{1}{3}\arctan(3x)+c$

When working through an integration, we may first need to put the expression into the correct form. As there is an x^2 term in the derivatives of both arcsin x and arctan x, this often involves **completing the square**. (For a reminder of this term, see glossary on CD-ROM.)

Worked example 19.13

Find $\int \frac{3}{\sqrt{-4x^2-4x+8}}\,dx$.

This is not a reverse chain rule integral and there is a square root in the denominator, so perhaps arcsin? The only way of producing $\sqrt{1-X^2}$ in the denominator is to start by completing the square to get $\sqrt{C-X^2}$

$\int \frac{3}{\sqrt{-4x^2-4x+8}}\,dx = \int \frac{3}{\sqrt{-(4x^2+4x-8)}}\,dx$

$= \int \frac{3}{\sqrt{-\left[(2x+1)^2-1-8\right]}}\,dx$

$= \int \frac{3}{\sqrt{9-(2x+1)^2}}\,dx$

Now reverse the standard derivative, remembering to divide by the coefficient of x which is 2

$= 3\int \frac{1}{\sqrt{3^2-(2x+1)^2}}\,dx$

$= \frac{3}{2}\arcsin\left(\frac{2x+1}{3}\right)+c$

Exercise 19D

1. Find the following:

(a) (i) $\int \frac{1}{1+2x^2}\,dx$ (ii) $\int \frac{1}{1+5x^2}\,dx$

(b) (i) $\int \frac{1}{\sqrt{1-3x^2}}\,dx$ (ii) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

(c) (i) $\int \frac{9}{x^2+9}\,dx$ (ii) $\int \frac{10}{x^2+10}\,dx$

(d) (i) $\int \frac{2}{\sqrt{25-x^2}}\,dx$ (ii) $\int \frac{5}{\sqrt{4-x^2}}\,dx$

2. By first completing the square, find the following:

(a) (i) $\int \frac{1}{x^2+4x+5}\,dx$ (ii) $\int \frac{1}{x^2-6x+10}\,dx$

(b) (i) $\int \frac{1}{\sqrt{8x-x^2-15}}\,dx$ (ii) $\int \frac{1}{\sqrt{2x-x^2}}\,dx$

(c) (i) $\int \frac{6}{x^2+10x+27}\,dx$ (ii) $\int \frac{5}{\sqrt{-4x^2-12x}}\,dx$

3. Find the exact value of $\int_0^{\sqrt{3}/2} \frac{3}{1+4x^2}\,dx$. *[4 marks]*

4. (a) Write $2x^2+4x+11$ in the form $2(x+p)^2+q$.
(b) Hence find $\int \frac{3}{2x^2+4x+11}\,dx$. *[5 marks]*

5. (a) Write $1+6x-3x^2$ in the form $a^2-3(x-b)^2$.

(b) Hence find the exact value of $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx$. *[5 marks]*

6. Show that $\int_3^{5.5} \frac{10}{4x^2-24x+61}\,dx = \frac{\pi}{4}$. *[6 marks]*

7. By using the substitution $u = e^x$, find the exact value of

$$\int_0^{\frac{1}{2}\ln 3} \frac{1}{e^x+e^{-x}}\,dx.$$ *[6 marks]*

19E Other strategies for integrating quotients

We now have a number of techniques to use when integrating a quotient of functions; we can reverse a standard derivative, apply the reverse chain rule or use an inverse trigonometric function. However, there are occasions when none of these methods seem to be useful. In such cases our only remaining option is to split the fraction into two separate fractions, each of which we *can* solve.

The easiest way to do this is to split the numerator.

Worked example 19.14

Find $\int \frac{x+1}{\sqrt{1-x^2}}\,dx$.

Split the fraction into two

$$\int \frac{x+1}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{1}{\sqrt{1-x^2}}dx$$

The first part can be integrated using the reverse chain rule or the substitution $u = 1 - x$

First integral:

Let $u = 1 - x^2$, so $\frac{du}{dx} = -2x \Rightarrow dx = -\frac{1}{2x}du$

$$\therefore \int \frac{x}{\sqrt{1-x^2}}dx = \int -\frac{x}{\sqrt{u}}\frac{1}{2x}du$$

$$= \int -\frac{1}{2}u^{-\frac{1}{2}}du$$

$$= -\sqrt{u} + c = -\sqrt{1-x^2} + c$$

We recognise the second part as the derivative of arcsin

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + c$$

$$\therefore \int \frac{x+1}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + \arcsin x + c$$

Sometimes it is not obvious how to split the numerator.

Worked example 19.15

Find $\int \frac{4x+19}{x^2+12x+41}\,dx$.

No obvious options, so split the fraction into two
However, simply writing this as:

$$\frac{4x}{x^2+12x+41} + \frac{19}{x^2+12x+41}$$

is no use as we still can't integrate

Instead, make the numerator of the first fraction a multiple of the derivative of the denominator in order to apply the reverse chain rule and then hope to be able to deal with the resulting second fraction

$$\int \frac{4x+19}{x^2+12x+41}dx$$

$$= \int \frac{4x+24}{x^2+12x+41}dx - \int \frac{5}{x^2+12x+41}dx$$

continued . . .

This works; second fraction can be integrated with the arctan function (after completing the square)

$$= \int \frac{4x+24}{x^2+12x+41}\,dx - \int \frac{5}{(x+6)^2+5}\,dx$$

$$= 2\int \frac{2x+12}{x^2+12x+41}\,dx - \int \frac{1}{\left(\frac{x+6}{\sqrt{5}}\right)^2+1}\,dx$$

Apply reverse chain rule to the first fraction (it is of the form $\frac{u'}{u}$) and arctan to the second

$$= 2\ln(x^2+12x+41) - \sqrt{5}\arctan\left(\frac{x+6}{\sqrt{5}}\right) + c$$

We should always check whether the fraction can be simplified before trying to split the numerator.

Worked example 19.16

Integrate $\int \frac{x+4}{12-5x-2x^2}\,dx$.

Check whether the polynomial factorises

$$\int \frac{x+4}{12-5x-2x^2}\,dx = \int \frac{x+4}{(3-2x)(x+4)}\,dx$$

$$= \int \frac{1}{3-2x}\,dx$$

We now have a standard integral, just remember to divide by the coefficient of x

$$= -\frac{1}{2}\ln|3-2x| + c$$

You will need the method of comparing coefficients from Section 3A.

The final type of functions to consider are improper fractions. These can be integrated by splitting them into a polynomial plus a proper fraction.

Worked example 19.17

(a) Find constants A, B and C such that:

$$\frac{x^2+5}{x+2} = Ax + B + \frac{c}{x+2}$$

(b) Hence find $\int \frac{x^2+5}{x+2}\,dx$.

We can multiply both sides by $x + 2$ to get rid of fractions

(a) $x^2+5=(Ax+B)(x+2)+C$

Setting $x = -2$ will eliminate the first term on the right, so we can find C

When $x=-2$:

$(-2^2)+5=(-2A+B)(0)$

$\therefore C=9$

To find A and B we need to expand the brackets and compare coefficients

$x^2+5=Ax^2+2Ax+Bx+2B+9$

x^2 terms: $1=A$

x terms: $0=2A+B=2+B \therefore B=-2$

So

$$\frac{x^2+5}{x+2}=x-2+\frac{9}{x+2}$$

The result from part (a) allows us to use standard integrals

(b) $\int \frac{x^2+5}{x+2}\,dx = \int x-2+\frac{9}{x+2}\,dx$

$=\frac{1}{2}x^2-2x+9\ln|x+2|+C$

Exercise 19E

1. By first simplifying, find:

(a) $\int \frac{(4x^2-9)^2}{(2x+3)^2}\,dx$ (b) $\int \frac{x+3}{6-13x-5x^2}\,dx$

2. Find the following by splitting the numerator:

(a) $\int \frac{5x+1}{x^2+6}\,dx$ (b) $\int \frac{x-3}{\sqrt{4-x^2}}\,dx$

(c) $\int \frac{8x+23}{x^2+8x+25}\,dx$ (d) $\int \frac{x-5}{\sqrt{-x^2+6x-7}}\,dx$

3. Find the following by splitting into a polynomial and a proper fraction:

(a) (i) $\int \frac{x+1}{x+2}\,dx$ (ii) $\int \frac{2x+3}{x-1}\,dx$

(b) (i) $\int \frac{x^2+2}{x-3}\,dx$ (ii) $\int \frac{x^2+2x-1}{x+5}\,dx$

(c) $\int \frac{x^2+5x+1}{x^2+3}\,dx$

4. (a) Show that $\frac{1}{x-2} = \frac{1}{x+3} = \frac{5}{x^2+x-6}$.

(b) Hence find $\int \frac{5}{x^2+x-6}\,dx$ giving your answer in the form $\ln(f(x))+c$. *[5 marks]*

5. Find the exact value of $\int_0^2 \frac{4}{x^2+4}\,dx$. *[4 marks]*

6. (a) Show that $\frac{5-x}{2+x-x^2} = \frac{1}{2-x} + \frac{2}{1+x}$.

(b) Given that $\int_0^1 \frac{5-x}{2+x-x^2}\,dx = \ln k$, find the value of k. *[6 marks]*

7. Find $\int \frac{4x+5}{\sqrt{1-x^2}}\,dx$. *[5 marks]*

8. (a) Write $2x^2-8x+17$ in the form $a(x-p)^2+q$.

(b) Hence find $\int \frac{2x+8}{2x^2-8x+17}\,dx$. *[7 marks]*

19F Integration by parts

In Section B we saw cases where we could integrate products of functions by using the reverse chain rule or a substitution, but we cannot yet solve integrals such as $\int x \sin x \, dx$ or $\int x^2 e^x \, dx$.

In order to deal with these we return to the product rule for differentiation (Key point 18.4).

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating with respect to x we get:

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\Rightarrow \int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

KEY POINT 19.8

The **integration by parts** formula.

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

When using integration by parts, the challenge is deciding which of the functions is to be u and which $\frac{dv}{dx}$. The aim is to select them so that the product $v\frac{du}{dx}$ is easier to integrate than the original product. This often (but not always) means that you choose u to be a polynomial.

Worked example 19.18

Find $\int x \sin x \, dx$.

This is a product to which we can't apply the reverse chain rule, so try integration by parts
Choose u to be the polynomial part

$u = x$ and $\frac{dv}{dx} = \sin x$

$\Rightarrow \frac{du}{dx} = 1$ and $v = -\cos x$

Apply the formula

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int x\sin x\,dx = x(-\cos x) - \int(-\cos x)1\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

It may be necessary to use integration by parts more than once. As long as the integrals are becoming simpler each time, you are on the right track! The next example also shows you how to deal with the limits.

Worked example 19.19

Find the exact value of $\int_0^{\ln 2} x^2 e^x \, dx$.

This is a product to which we can't apply the reverse chain rule, so try integration by parts. Choose the polynomial as u

$u = x^2$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 2x$ and $v = e^x$

Apply the formula. Put in the limits on the uv part straight away

$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int_0^{\ln 2} x^2 e^x \, dx = \left[x^2 e^x\right]_0^{\ln 2} - \int_0^{\ln 2} 2x e^x \, dx$

We have to integrate a product again, so use integration by parts again. Choose u to be the polynomial again

(If we used $u = e^x$ and $\frac{dv}{dx} = 2x$ we would end up back where we started!)

$u = 2x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 2$ and $v = e^x$

Apply the formula again and use the limits

So,

$\int_0^{\ln 2} 2x e^x \, dx = \left[2x e^x\right]_0^{\ln 2} - \int_0^{\ln 2} 2e^x \, dx = \left[2x e^x\right]_0^{\ln 2} - \left[2e^x\right]_0^{\ln 2}$

Put both integrals together, making sure to keep track of negative signs by using brackets appropriately

Therefore,

$$\int_0^{\ln 2} x^2 e^x \, dx = \left[x^2 e^x\right]_0^{\ln 2} - \left\{\left[2x e^x\right]_0^{\ln 2} - \left[2e^x\right]_0^{\ln 2}\right\}$$

$$= \left((\ln 2)^2 e^{\ln 2} - 0\right) - \left(2\ln 2 e^{\ln 2} - 0\right) + \left(2e^{\ln 2} - 2\right)$$

$$= 2(\ln 2)^2 - 4\ln 2 + 2$$

Sometimes it seems that we are getting nowhere, as the new integral resulting from integration by parts is no easier than the original. However, as long as things are not getting worse, they will eventually get better as the following example shows.

Worked example 19.20

Use integration by parts to find $\int e^x \cos x \, dx$.

This is one of the rare occasions when it makes no difference which way round we choose u and $\frac{dv}{dx}$

$u = \cos x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = -\sin x$ and $v = e^x$

Applying the formula, the new integral is no better but also no worse than the original

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

$$\int e^x \cos x \, dx = \cos x \, e^x - \int (-\sin x) e^x \, dx$$

$$= \cos x e^x + \int e^x \sin x \, dx$$

With no other option, we proceed with a second integration by parts

We need to be consistent and choose u and $\frac{dv}{dx}$ in the same way as before to avoid undoing what we have just done

$u = \sin x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = \cos x$ and $v = e^x$

It seems that no progress has been made as we have ended up with the integral we started with (except with a different sign)

So,

$$\int e^x \sin x \, dx = \sin x \, e^x - \int \cos x e^x \, dx$$

However, when we put everything together it becomes apparent that the different sign allows us to rearrange and find an expression for $\int e^x \cos x \, dx$

Therefore,

$$\int e^x \cos x \, dx = \cos x \, e^x + \left\{ \sin x \, e^x - \int \cos x e^x \, dx \right\}$$

$$\Rightarrow 2\int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{e^x}{2}(\cos x + \sin x) + c$$

We are able to differentiate and integrate e^x but so far we have only been able to *differentiate* ln x.

In order to be able to integrate ln x, we can use integration by parts. This might not seem an obvious method at first because there is no product of functions here, but with a little creativity we can proceed.

Worked example 19.21

Find $\int \ln x \, dx$.

The seemingly trivial step of writing ln x as the product of 1 and ln x sets up integration by parts

$$\int \ln x \, dx = \int 1 \times \ln x \, dx$$

Cannot integrate $\ln x$ so let $u = \ln x$

$$u = \ln x \quad \text{and} \quad \frac{dv}{dx} = 1$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad v = x$$

Apply the parts formula

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

$$\int 1 \times \ln x \, dx = (\ln x)x - \int \frac{1}{x} x \, d$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + c$$

Although this example only shows how to integrate ln x, in most other cases of integration by parts involving ln x we would still let ln $x = u$. The choices for u and v in the common cases are summarised below.

KEY POINT 19.9

When integrating $\int x^n f(x) \, dx$ by parts, choose $u = x^n$ in all cases except when $f(x) = \ln x$.

Exercise 19F

1. Use integration by parts to find the following:

(a) (i) $\int x\cos 2x \, dx$ (ii) $\int x\sin\left(\frac{x}{2}\right) dx$

(b) (i) $\int 4xe^{-2x} \, dx$ (ii) $\int xe^{4x} \, dx$

(c) (i) $\int 2x\ln 5x \, dx$ (ii) $\int x\ln x \, dx$

(d) (i) $\int x^2 \cos 3x \, dx$ (ii) $\int x^2 \sin x \, dx$

(e) $\int \frac{1}{4}x^2 e^{\frac{x}{4}} \, dx$

(f) $\int \frac{\ln x}{x^3} \, dx$

(g) $\int (\ln x)^2 \, dx$

2. Use integration by parts to find the following:

(a) $\int \arctan x \, dx$ (b) $\int \ln(2x+1) \, dx$

3. Evaluate the following exactly:

(a) $\int_0^{\pi/2} x\cos x \, dx$

(b) $\int_1^2 \frac{\ln x}{x^2} \, dx$

(c) $\int_0^{\pi/3} \sin x \ln(\sec x) \, dx$

4. When using the integration by parts formula, we start with $\frac{dv}{dx}$ and find v. Why do we not include a constant of integration when we do this? Try a few examples adding $+C$ to v and see what happens.

5. Find $\int 2xe^{-3x} \, dx$. *[5 marks]*

6. Evaluate $\int_1^e x^5 \ln x \, dx$. *[6 marks]*

7. (a) Show that $\int \tan x \, dx = \ln|\sec x| + c$.

(b) Hence find $\int \frac{x}{\cos^2 x} \, dx$. *[8 marks]*

8. Find the value of k such that $\int_0^k \arccos x \, dx = 0.5$ *[7 marks]*

9. Use the substitution $\sqrt{x+1} = u$ to find the exact value of $\int_{-1}^{3} \frac{1}{2} e^{\sqrt{x+1}}\, dx$. *[8 marks]*

Summary

- Look for **standard derivatives** before attempting any more complicated methods. They are given in the Formula booklet, but you may need to divide by the coefficient of x.
- If the expression contains both a function and its derivative and the 'inside' function is of the form $(ax + b)$ it is highly likely to be susceptible to the **reverse chain rule** or a substitution.
- **Integration by substitution** can also work in other situations, and you need to be able to use any given substitution. The steps of integration by substitution are given in Key point 19.2.
- When evaluating a definite integral using substitution, see Key point 19.3.
- A particular case of substitution, where the top of the fraction is the derivative of the bottom, is worth remembering:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

- Many integrals involving trigonometric functions can be simplified using identities.

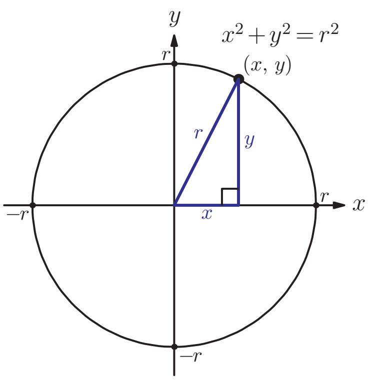
  Particularly useful identities are: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$; $\sin^2 x = \frac{1}{2}(1-\cos 2x)$; $\tan^2 x = \sec^2 x - 1$.

  For example, to integrate $\sin^2 x$, use $\cos 2x = 1 - 2\sin^2 x$; to integrate $\cos^2 x$, use $\cos 2x = 2\cos^2 x - 1$; to integrate $\tan^n x$, use $1 + \tan^2 x = \sec^2 x$ and that $\frac{d}{dx}(\tan x) = \sec^2 x$.
- The integral of tan is worth knowing: $\int \tan x\, dx = \ln|\sec x| + c$.
- Higher powers of sine and cosine can be integrated using De Moivre's Theorem.
- Integration can be done using inverse trigonometric functions: $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$;

  $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right) + c \ (|x| < a)$.
- It may be necessary to split a fraction that contains two (or more) terms in the numerator into two separate functions before integrating each with the above methods. If the degree of the numerator is at least as large as the degree of the denominator, then write it as a polynomial plus a proper fraction and compare coefficients.
- It may be possible to integrate a product of functions using the **integration by parts** formula:
- $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$
- The challenge with integration questions is often not in carrying out any of the above methods, but actually in selecting the correct method to use in the first place. In the exam you will often, but not always, be told which method to use. For extra practice see the Extension worksheet 17 'Basic integration'.

Introductory problem revisited

Use integration to prove that the area of a circle of radius r is equal to πr^2.



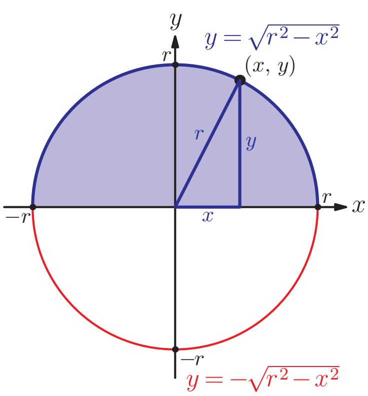
In order to use integration, we need to think of a circle as a graph of a function. We saw at the beginning of chapter 18 that the coordinates of a point on the circle with radius r satisfy the equation $x^2 + y^2 = r^2$. (You are not required to know this equation for the exam!) In chapter 18 we used implicit differentiation to find the gradient; in order to integrate we need an explicit expression for y in terms of x.

There is a small problem: the equation of the circle above is a relation, not a function – the graph of a circle does not pass the vertical line test.

We can only integrate functions, but we can avoid the problem by considering only the top half of the circle and then doubling the answer we get for the area.

You might find Key point 5.1 from chapter 5 useful here.

For the top half of the circle $y > 0$, so $y = \sqrt{r^2 - x^2}$. Now that you have done lots of integration practice, you may suspect that this one needs a substitution and it turns out that a useful substitution is $x = r\cos\theta$. This makes some sense, as we know that trigonometric functions are closely related to circles.



EXAM HINT

This is not one of the standard integrals, so if you have to do it in the exam you should be given a hint.

Now that we have decided on the strategy we can carry out the integration.

Write down the integral to be evaluated	The area of the top half of the circle is given by: $\frac{A}{2} = \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$
State the method to be used	Substitution: $x = r\cos\theta$

→

continued . . .

Differentiate

$$\frac{dx}{d\theta} = -r\sin\theta$$

$$\Rightarrow dx = -r\sin\theta \, d\theta$$

Express the integrand in terms of θ

$$r^2 - (r\cos\theta)^2 = r^2(1-\cos^2\theta) = r^2\sin^2\theta$$

Notice that $\sin\theta$ is positive on the top half of the circle

$$\Rightarrow \sqrt{r^2 - x^2} = r\sin\theta$$

Change the limits

Limits:

when $x = -r$, $\cos\theta = -1$ so $\theta = \pi$

when $x = r$, $\cos\theta = 1$ so $\theta = 0$

Put everything together

$$\frac{A}{2} = \int_{\pi}^{0} (r\sin\theta)(-r\sin\theta)\, d\theta$$

Remove the minus sign by swapping limits and take the constant outside the integral

$$= r^2 \int_0^{\pi} \sin^2\theta \, d\theta$$

Use the double angle formula to integrate $\sin^2\theta$

$$= r^2 \int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= r^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= r^2 \left\{\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right] - [0]\right\}$$

$$= \frac{\pi r^2}{2}$$

Hence the area of the whole circle is $2 \times \frac{\pi r^2}{2} = \pi r^2$, as required.

Mixed examination practice 19

Short questions

1. Find the exact value of $\int_0^{\pi} \cos^2(3x)\,dx$. *[6 marks]*

2. Use integration by parts to find $\int x\cos 2x\,dx$. *[6 marks]*

3. Given that $\int_0^m \frac{dx}{3x+1}\,dx = 1$ calculate, to 3 significant figures, the value of m. *[6 marks]*

4. Find the exact value of $\int_0^{\pi/12} \frac{1}{\cos^2 4x}\,dx$. *[5 marks]*

5. Find the following integrals:

(a) $\int \frac{1}{1-3x}\,dx$ (b) $\int \frac{1}{(2x+3)^2}\,dx$ *[6 marks]*

6. Find $\int \ln x\,dx$. *[5 marks]*

7. (a) Simplify $\frac{e^{-4x}+3e^{-2x}}{e^{-4x}-9}$ (b) Hence find $\int \frac{e^{-4x}+3e^{-2x}}{e^{-4x}-9}\,dx$. *[6 marks]*

8. Find $\int \frac{6x+4}{x^2+4}\,dx$. *[5 marks]*

9. (a) Show that $\frac{x+5}{(x+1)(x+2)}$ can be written as $\frac{2}{x-1}-\frac{1}{x+2}$.

(b) Hence find, in the form $\ln k$, the exact value of $\int_5^7 \frac{x+5}{(x-1)(x+2)}\,dx$. *[8 marks]*

10. Find $\int \frac{1}{x\ln x}\,dx$. *[6 marks]*

11. Using the substitution $u = \frac{1}{2}x - 1$, or otherwise, find $\int \frac{x}{\sqrt{\frac{1}{2}x-1}}\,dx$. *[5 marks]*

12. Find the exact value of $\int_2^5 \frac{x-1}{x+2}\,dx$. *[6 marks]*

13. Use integration by parts to find $\int \arctan x\,dx$. *[7 marks]*

14. Given that $\int_{-a}^{a} \frac{2}{1-x^2}\,dx = 1$, find the exact value of a. *[7 marks]*

Long questions

1. (a) Show that $\dfrac{4-3x}{(x+2)(x^2+1)}$ can be written in the form $\dfrac{A}{x+2}+\dfrac{1-Bx}{x^2+1}$ finding the constants A and B.

   (b) Hence find $\displaystyle\int \frac{4-3x}{(x+2)(x^2+1)}\,\mathrm{d}x$.

   (c) Find the exact value of $\displaystyle\int_0^{\sqrt{3}/2} \frac{4-3x}{\sqrt{1-x^2}}\,\mathrm{d}x$. *[15 marks]*

2. Let $I=\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$ and $J=\displaystyle\int \frac{\cos x}{\sin x+\cos x}\,\mathrm{d}x$.

   (a) Find $I+J$.

   (b) By using the substitution $u=\sin x+\cos x$, find $J-I$.

   (c) Hence find $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$. *[9 marks]*

3. Let $t=\tan\left(\dfrac{x}{2}\right)\mathrm{d}x$.

   (a) Find $\dfrac{\mathrm{d}t}{\mathrm{d}x}$ in terms of t.

   (b) (i) Show that $\sin 2\theta=\dfrac{2\tan\theta}{\sec^2\theta}$.

   (ii) Hence show that $\sin x=\dfrac{2t}{1+t^2}$.

   (c) Use the substitution $t=\tan\left(\dfrac{x}{2}\right)$ to evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin x}\,\mathrm{d}x$. *[14 marks]*

4. Consider the complex number $z=\cos\theta+\mathrm{i}\sin\theta$.

   (a) Using De Moivre's Theorem show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$.

   (b) By expanding $\left(z+\dfrac{1}{z}\right)^4$, show that $\cos^4\theta=\dfrac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$.

   (c) Let $g(a)=\displaystyle\int_0^a \cos^4\theta\,\mathrm{d}\theta$.

   (i) Find $g(a)$.

   (ii) Solve $g(a)=1$. *[11 marks]*

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20 Further applications of calculus

In this chapter you will learn:

- to write real world problems as equations involving variables and their derivatives
- how to relate different rates of change
- to apply calculus to problems involving motion (kinematics)
- to find volumes of shapes rotated around an axis
- to maximise or minimise functions with constraints.

Introductory problem

A forest fire spreads in a circle at the speed of 12 km/h. How fast is the area affected by the fire increasing when its radius is 68 km?

Did you know that if you are in a sealed box you cannot measure your velocity but you can measure your acceleration? Or that Newton's second law says that force is the rate of change of momentum? These are two examples where a rate of change is easier to find than the underlying variable. To get from this rate of change to the underlying variable requires the use of integration. This chapter will look at various applications of the calculus you have met in the previous four chapters, with a particular emphasis on real-world applications of rates of change.

20A Related rates of change

When blowing up a balloon we can control the amount of gas in the balloon (V), but we may want to know how fast the radius (r) is increasing. These are two different rates of change, but they are linked – the faster the gas fills the balloon the faster the radius will increase. We need to link two derivatives: $\frac{dV}{dt}$ and $\frac{dr}{dt}$. This is done by using the chain rule and the geometric context.

Worked example 20.1

A spherical balloon is being inflated with air at a rate of 200 cm³ per minute. At what rate is the radius increasing when the radius is 8 cm?

Define variables

V = volume of air in balloon in cm³
r = radius of balloon in cm
t = time in minutes

Write the given rate of change and the required rate of change

$$\frac{dV}{dt} = 200$$

$$\frac{dr}{dt} = ?$$

Relate these rates of change using the chain rule

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

So we need to find $\frac{dV}{dr}$

Use geometric context

Since the balloon is spherical, $V = \frac{4}{3}\pi r^3$,

$$\Rightarrow \quad \frac{dV}{dr} = 4\pi r^2$$

Put into the chain rule

$$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 200, \ r = 8$$

$$\therefore 200 = 256\pi \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = 0.249 \text{ (3SF)}$$

So radius is increasing at about 0.249 cm/minute

EXAM HINT

Don't use units in the working, as long as the units in the information are consistent. Always give units with your final answer.

The rate required may be linked to several other variables.

Worked example 20.2

As a conical icicle melts the rate of decrease of height h is 1 cm^{-1} and the rate of decrease of the radius of the base, r, is 0.1 cm h^{-1}. At what rate is the volume (V) of the icicle decreasing when the height is 30 cm and the base radius is 4 cm?

Write the given rates of change and the required rates of change
Remember that decrease means negative derivative

$$\frac{dh}{dt} = -1$$

$$\frac{dr}{dt} = -0.1$$

$$\frac{dV}{dt} = ?$$

Use geometry to link the variables

$$V = \frac{1}{3}\pi r^2 h$$

Differentiate both sides with respect to t, requiring the product rule and the chain rule

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2\right)h + \left(\frac{1}{3}\pi r^2\right)\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r\frac{dr}{dt}h + \frac{1}{3}\pi r^2\frac{dh}{dt}$$

$$r = 30,\ h = 4,\ \frac{dr}{dt} = -0.1,\ \frac{dh}{dt} = -1$$

Put in given values

$$\therefore \frac{dV}{dt} = \frac{2}{3}\pi \times 4 \times (-0.1) \times 30 + \frac{1}{3}\pi \times 4^2 \times (-1)$$

$$= -41.9 \text{ cm}^3\text{h}^{-1}$$

The volume is decreasing at 41.9 cm^3 per hour

Exercise 20A

1. In each case, find an expression for $\frac{dz}{dx}$ in terms of x.

(a) (i) $z = 4y^2,\ y = 3x^2$ (ii) $z = y^2, y = x^3 + 1$

(b) (i) $z = \cos y,\ y = 3x^2$ (ii) $z = \tan y,\ y = x^2 + 1$

2. (a) (i) Given that $z = y^2 + 1$ and $\frac{dy}{dx} = 5$, find $\frac{dz}{dx}$ when $y = 5$.

(ii) Given that $z = 2y^3$ and $\frac{dy}{dx} = -2$, find $\frac{dz}{dx}$ when $y = 1$.

(b) (i) If $w = \sin x$ and $\frac{dw}{dt} = -3$, find $\frac{dx}{dt}$ when $x = \frac{\pi}{3}$.

(ii) If $P = \tan h$ and $\frac{dP}{dx} = 2$, find $\frac{dh}{dx}$ when $h = \frac{\pi}{4}$.

(c) (i) Given that $V = 12r^3$, $\frac{dr}{dt} = 1$ and $\frac{dV}{dt} = 4$, find the possible values of r.

(ii) Given that $H = 3S^{-2}$, find the value of S for which

$\frac{dH}{dx} = 3$ and $\frac{dS}{dx} = 4$.

3. (a) (i) Given that $V = 3r^2h$, find $\frac{dV}{dt}$ when $r = 3$, $h = 2$, $\frac{dr}{dt} = 2$

and $\frac{dh}{dt} = -1$.

(ii) Given that $N = kx^4$, find $\frac{dN}{dt}$ when

$x = 2$, $k = 5$, $\frac{dk}{dt} = 1$ and $\frac{dx}{dt} = 1$.

(b) (i) Given that $m = \frac{S}{n}$ and that

$S = 100$, $\frac{dS}{dt} = 20$, $n = 50$ and $\frac{dn}{dt} = 4$, find $\frac{dm}{dt}$.

(ii) Given that $\rho = \frac{m}{V}$ and that

$m = 24$, $\frac{dm}{dt} = 2$, $V = 120$ and $\frac{dV}{dt} = 6$, find $\frac{d\rho}{dt}$.

4. A circular stain is spreading so that the radius is increasing at the constant rate of 1.5 cm s^{-1}. Find the rate of increase of the area when the radius is 12 cm. *[5 marks]*

5. The area of a square is increasing at the constant rate of 50 $cm^2\,s^{-1}$. Find the rate of increase of the side of the square when the length of the side is 12.5 cm. *[5 marks]*

6. The surface area of a closed cylinder is given by $A = 2\pi r^2 + 2\pi rh$, where h is the height and r is the radius of the base. At the time when the surface area is increasing at the rate of 20π cm^2 s^{-1} the radius is 4 cm, the height is 1 cm and is decreasing at the rate of 2 cm s^{-1}. Find the rate of change of radius at this time. *[6 marks]*

7. A spherical balloon is being inflated at a constant rate of 500 cm^3 s^{-1}. The radius at time t seconds is r cm.

Find the radius of the balloon at the time when it is increasing at the rate of 0.5 cm s^{-1}. *[6 marks]*

8. A ship is 5 km east and 7 km North of a lighthouse. It is moving North at a rate of 12 kmh^{-1} and East at a rate of 16 kmh^{-1}. At what rate is its distance from the lighthouse changing? *[7 marks]*

20B Kinematics

Kinematics is the study of movement – especially position, speed and acceleration. We first need to define some terms carefully:

Time is normally given the symbol t. We can normally define $t = 0$ at any convenient time.

In a 400 m race athletes run a single lap so, despite running 400 m they have returned to where they started. This distance is how much ground someone has covered, whilst the **displacement** is how far away they are from a particular position. The symbol s is normally used to represent displacement.

The rate of change of displacement with respect to time is called **velocity**, and it is normally given the symbol v.

In the IB you will only have to deal with motion in one dimension. However, motion is often in two or three dimensions. To deal with this requires a combination of vectors and calculus called (unsurprisingly) vector calculus.

KEY POINT 20.1

Velocity is given by: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$.

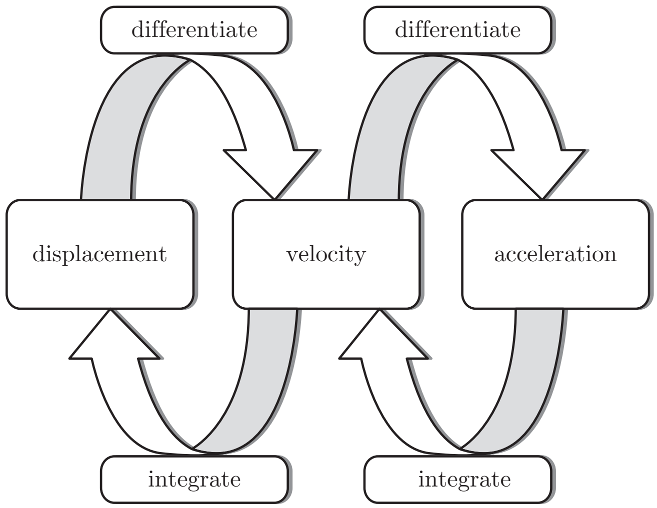
Speed is the magnitude of the velocity: $|v|$.

The rate of change of velocity with respect to time is called **acceleration**, and it is given the symbol a.

KEY POINT 20.2

Acceleration is given by: $a = \frac{\mathrm{d}v}{\mathrm{d}t}$.

To reverse the process – going from acceleration to velocity to displacement – is done by integration.



There is an important difference between finding distance and displacement between times a and b.

See Section 17H for more on the differences between areas and integrals.

KEY POINT 20.3

Displacement is the integral: $\int_a^b v\,\mathrm{d}t$

Distance travelled is the area: $\int_a^b |v|\,\mathrm{d}t$

Worked example 20.3

The velocity (ms^{-1}) of a car at time t seconds after passing a flag is modelled by $v = 17 - 4t$, for $0 \le t < 5$.

(a) What is the initial speed of the car?

(b) Find the acceleration of the car.

(c) What is the maximum displacement of the car from the flag?

continued . . .

(d) Find the distance the car travels.

(a) When $t = 0$, $v = 17\ \text{ms}^{-1}$

(b) $a = \frac{dv}{dt} = -4\ \text{ms}^{-2}$

Maximum displacement occurs when $\frac{ds}{dt} = 0$, which is the same as $v = 0$

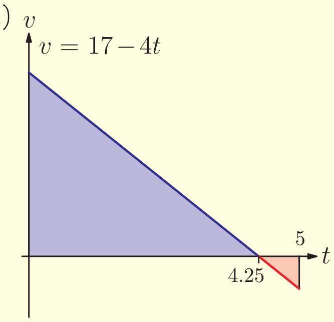
(c) When $v = 0$, $t = 4.25$

$s = \int_0^{4.25} v\,dt = \int_0^{4.25} 17 - 4t\,dt$

$= 36.125m$ (from GDC)

Distance is actual area between graph and x-axis of v-t graph

(d)



$\int_0^5 |v|\,dt = 37.25$ (from GDC)

So total distance is 37.25 m

In all the examples so far, velocity and acceleration were given as functions of time. But there are many practical situations where it is easier to see how velocity depends on the displacement. For example, a speed camera records a car's speed as it passes over certain marks along the road. From this data we may produce an equation for the speed of the car, such as $v = 60 - \sqrt{20000s}$ (v is km and s in km/h). Is it possible to deduce the acceleration of the car from this equation? We know that $a = \frac{dv}{dt}$, but as v depends on s we cannot differentiate it with respect to t. Using related rates of change allows us to get around this problem:

KEY POINT 20.4

$$a = \frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}$$

Worked example 20.4

A car is braking from the initial speed of 60 km/h. The speed of the car depends on the distance travelled since the brakes were applied, and is given by the equation $v = 60 - \sqrt{20000s}$. Find the acceleration of the car after it has been braking for 50 m.

We have v in terms of s and want to find a, so use $a = v\frac{dv}{ds}$

$$a = v\frac{dv}{ds}$$

$$= \left(60 - \sqrt{20\,000s}\right) \times \frac{-1}{2}(20\,000s)^{-\frac{1}{2}} \times 20\,000$$

$$= \frac{-10\,000\left(60 - \sqrt{20\,000s}\right)}{\sqrt{20\,000s}}$$

Remember that s should be in kilometres!

When $s = 0.05$:

$a = -8970 \text{ km/h}^2$

The car is decelerating at 8970 km/h^2.

This is about 0.7 m/s^2

Exercise 20B

1. Find the expressions for the velocity and acceleration in terms of time if the displacement is given by the equation:

(a) (i) $s = 4e^{-2t}$ (ii) $s = 5 - 2e^{3t}$

(b) (i) $s = 5\sin\left(\frac{t}{2}\right)$ (ii) $s = 2 - 3\cos(2t)$

2. A particle moves with the given velocity. The particle is at the origin at $t = 0$. Find the displacement in terms of t:

(a) (i) $v = 3t^2 - 1$ (ii) $v = \frac{1}{2}(1 - t^3)$

(b) (i) $v = 2e^{-t}$ (ii) $v = 1 + e^{2t}$

(c) (i) $v = \frac{3}{t+2}$ (ii) $v = 3 - \frac{1}{t+1}$

3. For the given velocity function, find the distance travelled between the given times:

(a) (i) $v = 2e^{-t}$ between $t = 0$ and $t = 2$
(ii) $v = 4(\ln t)^3$ between $t = 2$ and $t = 3$

(b) (i) $v = 1 - 5\cos t$ between $t = 0.2$ and $t = 0.9$
(ii) $v = 2\cos(3t)$ between $t = 1$ and $t = 1.5$

(c) (i) $v = t^2 - 2$ between $t = 0$ and $t = 2.3$
(ii) $v = 5\sin(2t)$ between $t = 0.5$ and $t = 2.5$

4. An object moves in a straight line so that the velocity is a function of the displacement. Find the acceleration of the object for the given value of the displacement or the velocity.

(a) (i) $v = e^{-2s}$, $s = \ln 3$ (ii) $v = 3\sin 2s$, $s = \frac{\pi}{24}$

(b) (i) $v = \frac{s-1}{s+2}$, $v = \frac{2}{5}$ (ii) $v = 3\ln(2s)$, $v = 10$

5. Use integration to derive these constant acceleration formulae for an object moving with constant acceleration a, and initial velocity u, where s is the displacement from the initial position.

(a) $v = u + at$
(b) $s = ut + \frac{1}{2}at^2$
(c) $v^2 = u^2 + 2as$

6. An object moves in a straight line so that its velocity at time t is given by $v = \frac{t}{t^2+1}$.

(a) Find an expression for the acceleration of the object at time t.

(b) Given that the object is initially at the origin, find its displacement from the origin when $t = 5$. *[6 marks]*

7. A ball is projected vertically upwards so that its velocity v ms^{-1} at time t s is given by $v = 12 - 9.8t$. Find the distance travelled by the ball in the first 2 seconds of motion. *[5 marks]*

8. The velocity of an object, in ms^{-1}, is given by $v = 5\cos\left(\frac{t}{3}\right)$.

(a) Find the displacement of the object from the starting point when $t = 6$.

(b) Find the total distance travelled by the object in the first 6 seconds. *[6 marks]*

9. The displacement of an object varies with time as

$$s = -\frac{1}{3}t^3 + \frac{3}{2}t^2 + 4t, \text{ for } 0 \le t \le 5. \text{ Find the maximum}$$

velocity of the object. *[5 marks]*

10. An object moves in a straight line so that its velocity, v, is a function of the displacement, s, given by $v = \ln(s+2)$. Find the acceleration of the object when $v = 4$. *[5 marks]*

11. The velocity of an object, in ms^{-1}, is given by $v = \dfrac{10(s-2)}{s^2+4}$, where s is the displacement in metres.

(a) Find the maximum velocity of the object.

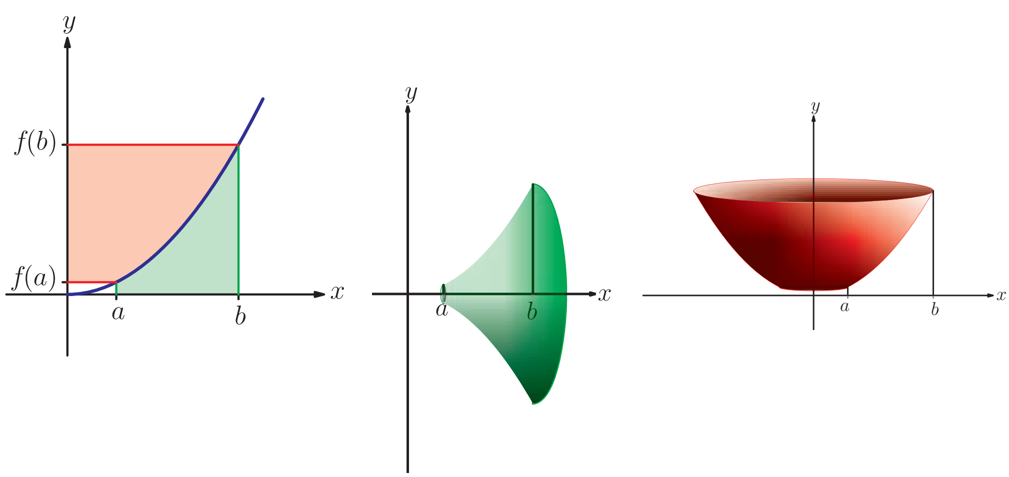
(b) Find the acceleration of the object when $s = 3$. *[6 marks]*

20C Volumes of revolution

You might find Key point 17.8 and Key point 17.9 useful here.

In chapter 17 you saw that the area between a curve and the x-axis from $x = a$ to $x = b$ is given by $\int_a^b y \, dx$ as long as $y > 0$, and also that the area between a curve and the y-axis from $y = c$ to $y = d$ is given by $\int_c^d x \, dy$. In this section we will use a similar formula to find the volume of a shape formed by rotating the curve around either the x-axis or the y-axis.

If a curve is rotated fully around the x-axis or the y-axis the resulting shape is called a **volume of revolution**.



KEY POINT 20.5

The volume of revolution around the x-axis is given by:

$$\int_{x=a}^{x=b} \pi y^2 \, dx$$

The volume of revolution around the y-axis is given by:

$$\int_{y=c}^{y=d} \pi x^2 \, dy$$

EXAM HINT

Notice that the limits use the variable you are integrating with respect to.

The formulae are derived on the Fill-in proof sheet 23 'Volumes of revolution' on the CD-ROM.

Worked example 20.5

The graph of $y = \sin 2x$, $0 \le x \le \frac{\pi}{2}$, is rotated 360° around the x-axis. Find the volume of the solid generated, in terms of π.

Use the formula for the volume of revolution

$$V = \int_0^{\pi/2} \pi (\sin 2x)^2 \, dx$$

$$= \pi \int_0^{\pi/2} \sin^2 2x \, dx$$

Integrate $\sin^2 2x$ using the double-angle formula

$$= \pi \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x) \, dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{8}\sin 4x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left\{ \left(\frac{\pi}{4} - \frac{1}{8}\sin 2\pi \right) - \left(0 - \frac{1}{8}\sin 0 \right) \right\}$$

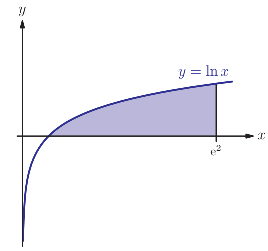
$$= \pi \left(\frac{\pi}{4} - 0 - 0 \right)$$

$$= \frac{\pi^2}{4}$$

The formulae in Key point 20.5 apply when the curve is rotated through a full turn (2π radians) around an axis. You can also form a solid by rotating the curve through a part of the full turn, most commonly π radians (half a turn).

Worked example 20.6

Find the volume of revolution generated when the shaded region is rotated π radians around the y-axis.



First establish limits in terms of y

When $x = 1$, $y = 0$
When $x = e^2$, $y = 2$

The volume when rotated by π is half the volume when rotated by 2π

$$V = \frac{1}{2}\int_0^2 \pi x^2 \, dy$$

Rearrange equation of line to get x^2 in terms of y

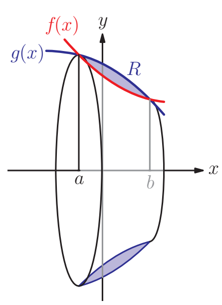
$$x = e^y$$

$$x^2 = (e^y)^2 = e^{2y}$$

$$V = \frac{\pi}{2}\int_0^2 e^{2y} \, dy$$

$$= \frac{\pi}{2} \times 53.6 \quad \text{(from GDC)}$$

$$= 84.2 \quad \text{(3SF)}$$



You might also be asked to find a volume of revolution of an area between two curves. We can apply a similar argument to the one we used for areas in Section 17J.

From the diagram we can see that the volume formed when the region R is rotated around the x-axis is given by the volume of revolution of $g(x)$ minus the volume of revolution of $f(x)$.

EXAM HINT

Do not fall into the trap of saying that the volume is:

$$\int_a^b \pi[g(x) - f(x)]^2 \, dx$$

KEY POINT 20.6

The volume of revolution of the region between curves $g(x)$ and $f(x)$ is

$$\int_a^b \pi\left(g(x)^2 - f(x)^2\right)dx$$

where $g(x)$ is above $f(x)$ and the curves intersect at $x = a$ and $x = b$.

Worked example 20.7

The region bounded by the curves $y = x^2 + 6$ and $y = 8x - x^2$ is rotated $360°$ about the x-axis.

(a) Show that the volume of revolution is given by $4\pi\int_1^3 13x^2 - 4x^3 - 9\,dx$.

(b) Evaluate this volume, correct to 3 significant figures.

The limits of integration are the intersection points

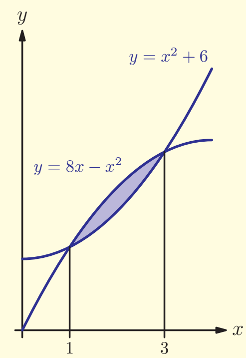
(a) Intersections:

$x^2 + 6 = 8x - x^2$

$2x^2 - 8x + 6 = 0$

$2(x-1)(x-2) = 0$

$x = 1$ or 3



Use $V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$

Draw a sketch to see which curve is above

$V = \pi\int_1^3 (8x - x^2)^2 - (x^2+6)^2\,dx$

$= \pi\int_1^3 (64x^2 - 16x^3 + x^4) - (x^4 + 12x^3 + 36)\,dx$

$= \pi\int_1^3 52x^2 - 16x^3 - 36\,dx$

$= 4\pi\int_1^3 13x^2 - 4x^3 - 9\,dx$

We can evaluate the integral using GDC

(b) Using GDC,

$V = 184$ (3SF)

There are also formulae for finding the surface area of a solid formed by rotating a region around an axis. Some particularly interesting examples arise if we allow one end of the region to tend to infinity. For example, rotating the region formed by the lines $y = \frac{1}{x}$, $x = 1$ and the x-axis results in a solid called the Gabriel's Horn, or Torricelli's trumpet. Areas and volumes can still be calculated using something called improper integrals, and it turns out that it is possible to have a solid of finite volume but infinite surface area!

EXAM HINT

Notice that the calculation for $\int [f(x)]^2 - [g(x)]^2\,dx$ is easier than doing $\int [f(x)]^2\,dx - \int [g(x)]^2\,dx$

Exercise 20C

1. Find the volume of revolution formed when the curve $y = f(x)$, with $a \le x \le b$, is rotated through 2π radians about the x-axis.

(a) (i) $f(x) = x^2 + 6, \quad a = -1, \quad b = 3$
(ii) $f(x) = 2x^3 + 1, \quad a = 0, \quad b = 1$

(b) (i) $f(x) = e^{2x} + 1, \quad a = 0, \quad b = 1$
(ii) $f(x) = e^{-x} + 2, \quad a = 0, \quad b = 2$

(c) (i) $f(x) = \sqrt{\sin x}, \quad a = 0, \quad b = \pi$
(ii) $f(x) = \sec x, \quad a = -\frac{\pi}{4}, \quad b = \frac{\pi}{4}$

2. The part of the curve $y = g(x)$ with $a \le x \le b$, is rotated 360° around the y-axis. Find the volume of revolution generated, correct to 3 significant figures:

(a) (i) $g(x) = 4x^2 + 1, \quad a = 0, \quad b = 2$
(ii) $g(x) = \frac{x^2 - 1}{3}, \quad a = 1, \quad b = 4$

(b) (i) $g(x) = \ln x + 1, \quad a = 1, \quad b = 3$
(ii) $g(x) = \ln(2x - 1), \quad a = 1, \quad b = 5$

(c) (i) $g(x) = \cos x, \quad a = 0, \quad b = \frac{\pi}{2}$
(ii) $g(x) = \tan x, \quad a = 0, \quad b = \frac{\pi}{4}$

3. The part of the graph of $y = \ln x$ between $x = 1$ and $x = 2e$ is rotated 360° around the x-axis. Find the volume generated. *[4 marks]*

4. The part of the curve $y^2 = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated 2π radians around the x-axis. Find the exact volume of the solid generated. *[4 marks]*

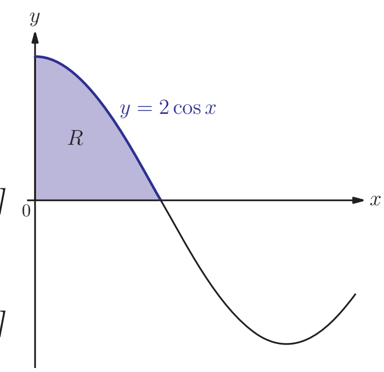
5. The part of the curve $y = \ln(x^2)$ between $x = 1$ and $x = e^2$ is rotated 360° around the y-axis. Find the exact value of the resulting volume or revolution. *[6 marks]*

6. (a) (i) Find an equation of the straight line passing through points $(0, h)$ and $(r, 0)$.

(ii) By finding the volume of revolution formed when the line is rotated around the y-axis, show that the volume of a cone is $\frac{1}{3}\pi r^2 h$.

(b) A circle of radius r and the centre at the origin has equation $x^2 + y^2 = r^2$, where $-r \le x, y \le r$. By rotating the circle around the x-axis prove that the volume of a sphere is $\frac{4}{3}\pi r^3$. *[9 marks]*

7. Region R is bounded by the curve $y = 2\cos x$ and the coordinate axes, as shown in the diagram.



Find the volume generated when R is rotated 2π radians about the y-axis.

[5 marks]

8. Find the exact volume generated when the region between the graph of $y = \sqrt{x}$, the y-axis and the line $y = 3$ is rotated π radians about the the y-axis. *[7 marks]*

9. The part of the curve $y = \frac{3}{\sqrt{x}}$ between $x = 1$ and $x = a$ rotated 2π radians around the x-axis. The volume of the resulting solid is $\pi \ln\left(\frac{64}{27}\right)$. Find the exact value of a. *[7 marks]*

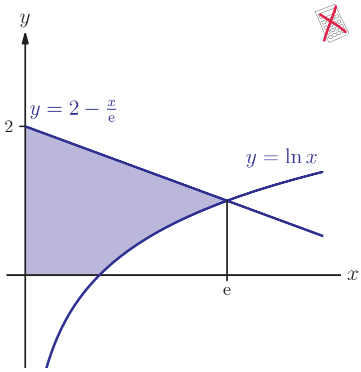
10. The region bounded by the curve $y = e^{2x} - 1$, the y-axis and the line $y = 3$ is rotated π radians around the y-axis. Find the volume of the solid generated. *[5 marks]*

11. (a) Find the coordinates of the points of intersection of the curves $y = 4\sqrt{x}$ and $y = x + 3$.

(b) The region between the curves $y = 4\sqrt{x}$ and $y = x + 3$ is rotated 2π radians around the y-axis. Find the volume of the solid generated. *[7 marks]*

12. (a) Find the coordinates of the points of intersection of curves $y = x^2 + 3$ and $y = 4x + 3$.

(b) Find the volume of revolution generated when the region between the curves $y = x^2 + 3$ and $y = 4x + 3$ is rotated 360° around the x-axis. *[7 marks]*



13. The diagram shows the curve $y = \ln x$ and the line $y = -\frac{1}{e}x + 2$. The two graphs intersect at (e, 1). The shaded region is rotated 360° around the y-axis.

Find the exact value of the volume of revolution. *[8 marks]*

In many real world situations we have information about the rate of change of a quantity. These are called differential equations, and they are introduced in Fill-in proof 20 'Fundamental theorem of calculus'.

20D Optimisation with constraints

In this section we shall look at how to maximise or minimise a function that appears to depend upon two different variables. However, these two variables will always be related by a constraint which will allow one of them to be eliminated. We can then follow the normal procedure for finding maxima or minima.

See Section 16J for the procedure for finding maxima and minima.

Worked example 20.8

Find the maximum value of $F = xy - y$ given that $x + 3y = 7$.

Define variables

We wish to maximise $F = xy - y$

Write F in terms of only one variable

$x = 7 - 3y$

$\therefore F = (7 - 3y)y - y$

$= 6y - 3y^2$

Find stationary points

$\Rightarrow \frac{dF}{dy} = 6 - 6y$

But $\frac{dF}{dy} = 0$ at a maximum point

$\therefore 6 - 6y = 0,\ y = 1$

$\Rightarrow x = 7 - 3y = 4$

$\therefore F = 4 \times 1 - 1 = 3$

continued . . .

Classify stationary points

$\frac{d^2F}{dx^2} = -6 < 0$

so $F = 3$ is a local maximum

Check endpoints and asymptotes

There are no asymptotes and when $|y|$ is large F becomes negative so 3 is the global maximum

Sometimes the constraint is not explicitly given, and needs to be deduced from the context. The two common types of constraints are:

- A shape has a fixed perimeter, area or volume – this gives an equation relating different variables (height, length, radius...)
- A point lies on a given curve – this gives a relationship between x and y.

Worked example 20.9

A rectangle has perimeter 100 cm. What is the largest its area can be?

The area of the rectangle is *length* × *width*. Introduce those as variables so we can write equations

Let $x = length$ and $y = width$.
Then $Area = xy$

It is impossible to see from this equation alone what the maximum possible value of the area is. But x and y are related: We can write an equation to express the fact that we know the perimeter

$Perimeter = 2x + 2y = 100$

continued . . .

This means that we can express the area in terms of only one of the variables

$2y = 100 - 2x$

$\Rightarrow y = 50 - x$

$\therefore \text{Area} = x(50 - x)$

$= 50x - x^2$

We can now use differentiation to find the maximum point

$A = 50x - x^2$

$\Rightarrow \frac{dA}{dx} = 50 - 2x$

For stationary points:

$\frac{dA}{dx} = 0$

$\Rightarrow 50 - 2x = 0$

$\Rightarrow x = 25$

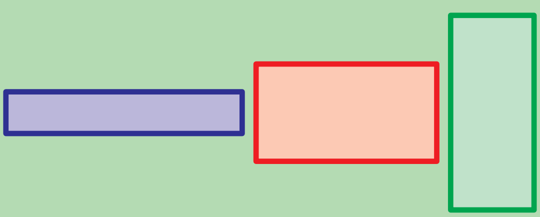
We need to check whether this is a minimum or a maximum by using the second derivative

$\frac{d^2A}{dx^2} = -2 < 0$, so this is a maximum point.

The maximum area is

$A = 50 \times 25 - 25^2 = 625 \text{cm}^2$.

It is intuitively clear that a long and thin or a short and wide rectangle will have a very small area, so the largest area should be somewhere in between.



A related problem is finding the minimum possible surface area for an object of a fixed volume. Examples of this can be seen in nature: Snakes have evolved to be long and thin in order to maximise their surface area for heat absorption, while polar bears avoid losing too much heat by adopting a rounder shape which minimises the surface area for their volume.

You may have noticed in the above example that the rectangle with the largest area is actually a square ($x = y = 25$). It turns out that out of all plane shapes with a fixed perimeter, the circle has the largest possible area. This is called 'the isoperimetric problem', and has several intriguing proofs and many applications.

Worked example 20.10

Find the point on the curve $y = x^3$ closest to the point (2, 0).

Define variables

L is the length from the point (2, 0) to the point $P(x, y)$

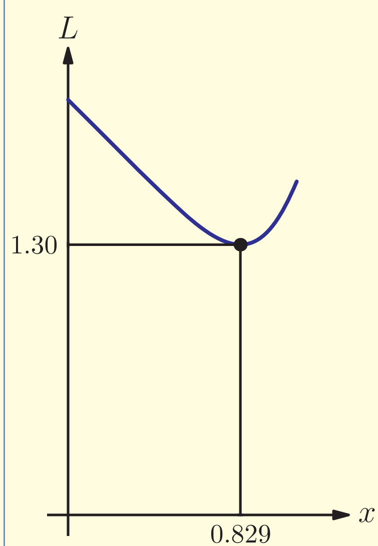
So $L = \sqrt{(x-2)^2 + y^2}$

Write L in terms of only one variable

If P lies on the curve then $y = x^3$

$\therefore L = \sqrt{(x-2)^2 + x^6}$

Find stationary points. This looks complicated and there is no requirement for exact answers so use GDC



From GDC, the minimum is when $x = 0.829$ (3SF) and $y = 0.569$ (3SF).

Exercise 20D

1. (a) (i) Find the maximum value of xy given that $x + 2y = 4$.

(ii) Find the maximum possible value of xy given that $3x + y = 7$.

(b) (i) Find the minimum possible value of $a + b$ given that $ab = 3$ and $a, b > 0$.

(ii) Find the minimum possible value of $2x + y$ given that $ab = 4$ and $a, b > 0$.

(c) (i) Find the maximum possible value of $4r^2h$ if $2r^2 + rh = 3$ and $r, h > 0$.

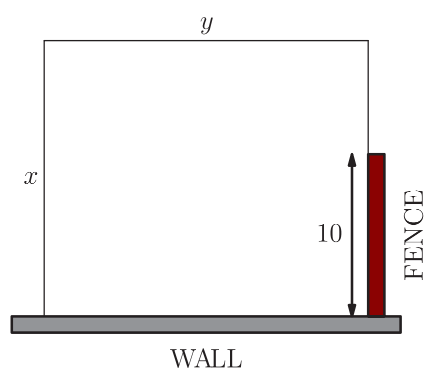
(ii) Find the maximum possible value of rh^2 if $4r^2 + 3h^2 = 12$ and $r, h > 0$.

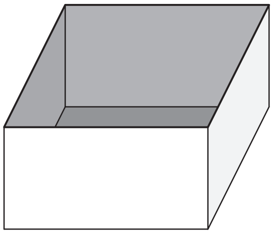
2. A farmer wishes to fence off a rectangular area adjacent to a wall. There is an existing piece of fence, 10 m in length, and perpendicular to the wall, as shown in the diagram.

Let x and y be the dimensions of the enclosure. Given that the length of the new fencing is to be 200 m:

(a) Write down an expression for the area of the enclosure in terms of x only.

(b) Hence find the values of x and y to create the maximum possible area.





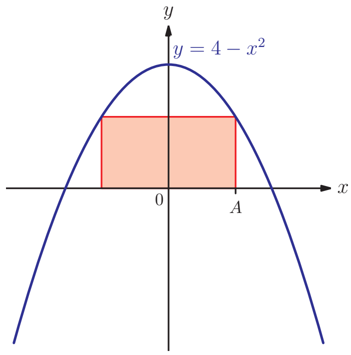
3. A square sheet of card of side 12 cm has four squares of side x cm cut from the corners. The sides are then folded to make a small open box.

(a) Find an expression for the volume of the box in terms of x.

(b) Find the value of x for which the volume is maximum possible, and prove that it is a maximum. *[6 marks]*

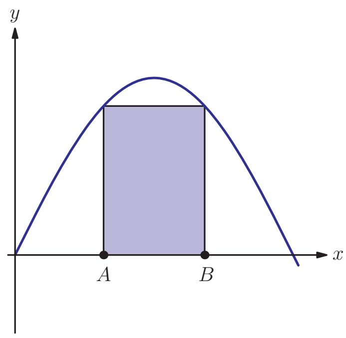
4. An open box in the shape of a square-based prism has volume 32 cm². Find the minimum possible surface area of the box. *[6 marks]*

5. A rectangle is drawn inside the region bounded by the curve $y = 4 - x^2$ and the x-axis, so that two of the vertices lie on the axis and the other two on the curve.



Find the coordinates of vertex A so that the area of the rectangle is a maximum. *[6 marks]*

6. A rectangle is drawn inside the region bounded by the curve $y = \sin x$ and the x-axis, as shown in the diagram. The vertex A has coordinates $(x, 0)$.



(a) (i) Write down the coordinates of point B.

(ii) Find an expression for the area of the rectangle in terms of x.

(b) Show that the rectangle has maximum area when $2\tan x = \pi - 2x$.

(c) Find the maximum possible area of the rectangle.

[8 marks]

7. What is the largest possible capacity of a closed cylindrical cuboid with surface area 450 cm? *[6 marks]*

8. What is the largest possible capacity of a closed square based cuboid with surface area 450 cm? *[6 marks]*

9. The sum of two numbers, x and y, is 6, and $x, y \geq 0$. Find the two numbers if the sum of their squares is:

(a) the minimum possible

(b) the maximum possible. *[7 marks]*

10. A cone of radius r and height h has volume 81π.

(a) Show that the curved surface area of the cone is given by $S = \frac{\pi}{r}\sqrt{r^6 + 243^2}$.

(b) It is required to make the cone so that the curved surface area is the minimum possible. Find the radius and the height of the cone. *[7 marks]*

11. A 20 cm piece of wire is bent to form an isosceles triangle with base b.

(a) Show that the area of the triangle is given by:

$A = \frac{b}{2}\sqrt{100 - 10b}$.

(b) Show that the area of the triangle is the largest possible when the triangle is equilateral. *[6 marks]*

12. The sum of the square of the two positive numbers is a. Prove that their product is the maximum possible when the two numbers are equal. *[6 marks]*

13. Find the coordinates of the point on the curve $y = x^2$, $x \geq 0$, closest to the point (0, 4). *[7 marks]*

Summary

- If there are more than two variables involved in a question, you may need to relate their rates of change using the chain rule, e.g. $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$
- Do not confuse distance (how much ground has been covered) and **displacement** (how far away from a particular position), or **velocity** (rate of change of displacement with respect to time: $v = \frac{ds}{dt}$) and **speed** (magnitude of velocity: $|v|$).
- **Acceleration** is the rate of change of velocity with respect to time: $a = \frac{dv}{dt}$.
- In **kinematics**, differentiate to go from displacement to velocity to acceleration. Integrate to go from acceleration to velocity to displacement.
- The displacement between times a and b is $\int_a^b v\, dt$.
- The distance between times a and b is $\int_a^b |v|\, dt$.
- If the velocity depends on displacement we need to use $a = v\frac{dv}{ds}$.

- If a curve is rotated fully around the x- or y-axis, the resulting shape is called a **volume of revolution**.
- The volume of revolution is given by

$$V = \int_{x=a}^{x=b} \pi y^2 \mathrm{d}x \text{ for rotation around the } x\text{-axis}$$

$$V = \int_{y=c}^{y=d} \pi x^2 \mathrm{d}y \text{ for rotation around the } y\text{-axis}$$

- The volume formed by rotating the region between two curves $g(x)$ and $f(x)$, where $g(x)$ is above $f(x)$ and the curves intersect at $x = a$ and $x = b$, is:

$$\int_a^b \pi\left[g(x)^2 - f(x)^2\right]\mathrm{d}x$$

- When solving optimisation problems that involve a function which depends on two variables, the variables will be related by a constraint that will allow one variable to be eliminated before differentiating to find stationary points. Two common types of constraint are:
  - a shape has a fixed perimeter, area or volume (this gives an equation relating different variables)
  - a point lies on a given curve (this gives a relationship between x and y).

Introductory problem revisited

A forest fire spreads in a circle at the speed of 12 km/h. How fast is the area affected by the fire increasing when its radius is 68 km?

Let r be the radius of the region affected by the fire and let A be its area. We are told that $\frac{\mathrm{d}r}{\mathrm{d}t} = 12$, where t is the time since the start of the fire, measured in hours. We need to find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 68$. To do this, we need to relate the rate of change of A to the rate of change of r.
Using the chain rule:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

Since the region is a circle, we know that $A = \pi r^2$, so $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$. Hence,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \times 12 = 24\pi r.$$

When $r = 68$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 5127$, so the area affected by the fire is increasing at the rate of about 5130 km^2/h.

Mixed examination practice 20

Short questions

1. The region bounded by the curve $y = ax - x^2$ is rotated $360°$ around the x-axis. Find, in terms of a, the resulting volume of revolution. *[7 marks]*

2. An object moves in a straight line so that its velocity, in ms^{-1} is given by $v = t^3 - 6t^2 + 8t$, where t is measured in seconds.

   (a) Find the displacement from the initial position when $t = 5$.

   (b) Find the total distance travelled in the first 5 seconds. *[6 marks]*

3. The sum of the squares of two positive numbers is 32. Find the two numbers so that their sum is the maximum possible. *[6 marks]*

4. A circular stain is spreading so that the rate of increase of radius is inversely proportional to the square root of the radius. Initially, the radius of the stain is 4 cm and it is increasing at the rate of 2 cms^{-1}. Find the radius of the stain at the time when its area is increasing at the rate of 115 cm^2s^{-1}. *[6 marks]*

5. An object moves in a straight line so that its displacement, s, is given by the equation $s = 3e^{-t}\sin t$, where t is time.

   (a) Calculate the velocity of the object when $t = 3$.

   (b) Sketch the graph of $v(t)$ for $0 \le t \le 3$. *[6 marks]*

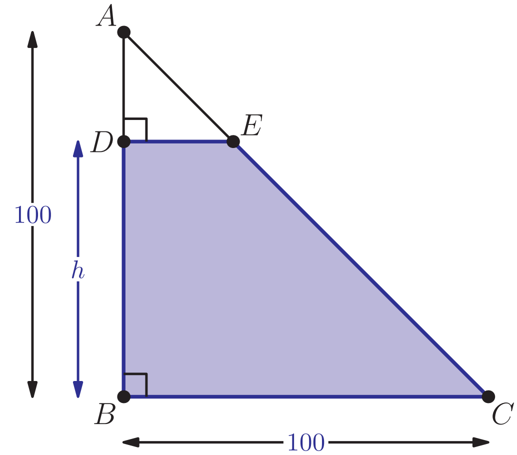
6. The diagram shows an isosceles right-angled triangle of side 100 cm. Point D is moving along the side AB towards point B so that the area of the trapezium $DBCE$ is decreasing at the constant rate of 18 cm^3s^{-1}. Let $BD = h$.

(a) Write down an expression for the area of the trapezium $DBCE$ in terms of h.

(b) Show that $\frac{dh}{dt} = \frac{18}{h-100}$.

Initially point D is at vertex A.

(c) Given that $h = 100 - k\sqrt{t}$, find the value k. *[8 marks]*



7. An aeroplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level.

At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in kilometres per hour, at which the aeroplane is moving towards the observer. *[6 marks]*

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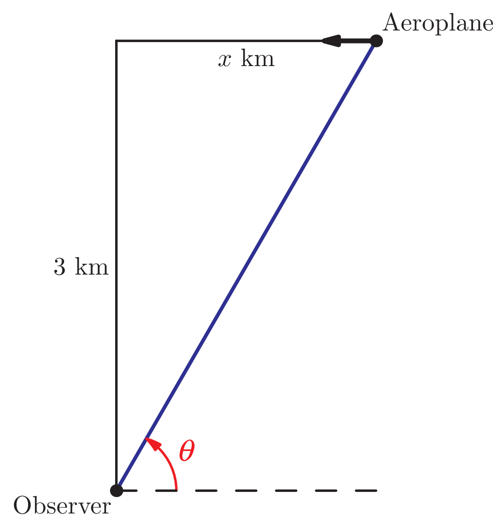
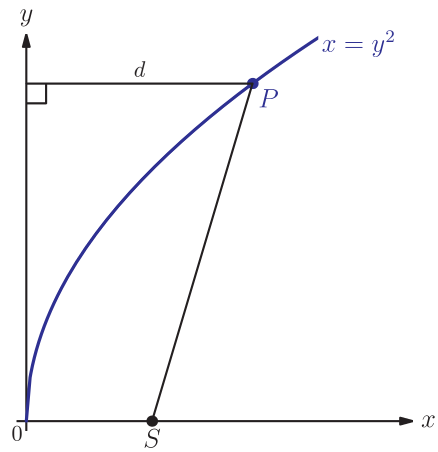


8. The diagram shows a part of the curve with equation $x = y^2$ and a fixed point $S(1, 0)$. Point P lies on the curve and has y-coordinate k ($k \geq 0$).

Let d denote the distance of P from the y-axis, and let r denote the ratio $\dfrac{d}{SP}$

(a) Show that $r = \dfrac{k^2}{\sqrt{k^4 - k^2 + 1}}$. *[7 marks]*

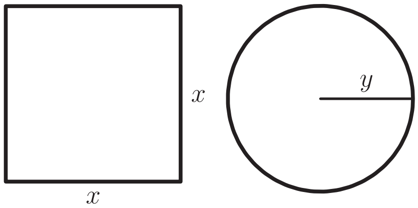
(b) Find the maximum possible value of r.



9. The acceleration of an object depends on its velocity as $a = \dfrac{v^2 + 4}{2v}$. The initial velocity is 3. Show that $v^2 = 13e^t - 4$. *[6 marks]*

Long questions

1. The diagram shows a square with side x cm and a circle with radius y cm.



(a) Write down an expression for the perimeter:

(i) of the square (ii) of the circle

The two shapes are made out of a piece of wire of total length 8 cm.

(b) Find an expression for x in terms of y.

(c) Show that the total area of the two shapes is given by:

$$A = \frac{\pi}{4}(\pi + 4)y^2 - 2\pi y + 4$$

(d) If the total area of the two shapes is the minimum possible, what percentage of the wire is used for the circle? *[10 marks]*

2. Consider two curves with equations $y = x^2 - 8x + 12$ and $y = 12 + x - x^2$ for $0 \le x \le 5$.

(a) Write down the coordinates of the points of intersection of the two curves.

(b) Find the greatest vertical distance between the two curves.

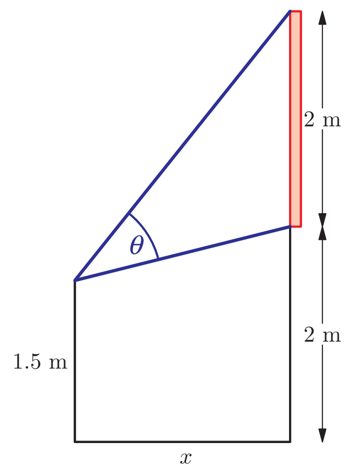
(c) The region between the curves is rotated 360° around the x-axis.

(i) Write down an expression for the volume of the solid generated.

(ii) Evaluate the volume, giving your answer to the nearest integer.

[10 marks]

3. A painting of height 2 m is hanging on the wall of an art gallery so that the bottom of the painting is 2 m above the floor. A visitor is sitting on a stool so that his eyes are at the height of 1.5 m. The stool is at the distance x m from the wall.



(a) Show that the angle at which the visitor sees the painting is:

$$\theta = \arctan\frac{2.5}{x} - \arctan\frac{0.5}{x}$$

(b) Find how far from the wall the stool should be placed so that the painting appears as large as possible. Give your answer in the form $\frac{\sqrt{p}}{q}$, where p and q are integers.

[9 marks]

4. (a) Show that $\int \ln x \, dx = x \ln x - x + c$.

(b) An object is initially at the origin, and moves with velocity $v = 3\ln(t+1)$

(i) Find the acceleration of the object after 5 seconds.

(ii) Find an expression for the displacement in terms of t.

(iii) Find the distance travelled by the object in the first 5 seconds.

(c) A second object has velocity given by $v = 8 - t$. It is also initially at the origin.

(i) The second object has greater velocity for $0 \le t \le a$. Find the value of a.

(ii) Find the greatest speed of the second object during the first 20 seconds.

(iii) After how long have the two objects travelled the same distance?

[16 marks]

5. Triangle ABC is made out of a piece of elastic string. Vertices A and B are being pulled apart so that the length of the base AB is increasing at the rate of 3 cms^{-1} and the height, h, is decreasing at the rate of 2 cms^{-1}. Initially, $AB = 20$ cm and $h = 30$ cm.

(a) Show that $AB = 20 + 3t$.

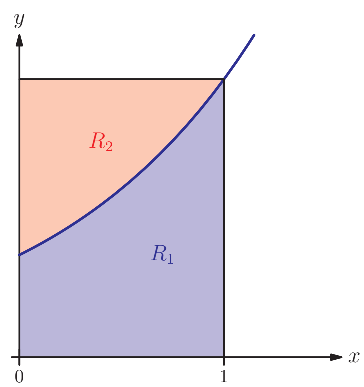
(b) Find an expression for h in terms of t.

(c) Find an expression for the rate of change of the area of the triangle in terms of t. *[12 marks]*

(d) Find the rate at which the area of the triangle is changing when $AB = 26$ and $h = 26$.

6. (a) Use integration by parts to show that $\int (\ln x)^2 \, dx = x\left((\ln x)^2 - \ln x + 1\right)$.

(b) Consider the graph of $y = e^x$ between $x = 0$ and $x = 1$. Regions R_1 and R_2 are defined as shown on the diagram. Region R_1 is rotated around the x-axis and region R_2 is rotated around the y-axis to form volumes V_1 and V_2 respectively. Find the exact value of the ratio $\frac{V_1}{V_2}$. *[14 marks]*



7. Particle A moves in a straight line, starting from O_A, such that its velocity in metres per second for $0 \le t \le 9$ is given by:

$$v_A = -\frac{1}{2}t^2 + 3t + \frac{3}{2}$$

Particle B moves in a straight line, starting from O_B, such that its velocity in metres per second for $0 \le t \le 9$ is given by:

$$v_B = e^{0.2t}$$

(a) Find the maximum value of v_A, justifying that it is a maximum.

(b) Find the acceleration of B when $t = 4$.

The displacements of A and B from O_A and O_B respectively, at time t are s_A metres and s_B metres. When $t = 0$, $s_A = 0$ and $s_B = 5$.

(c) Find an expression for s_A and for s_B, giving your answers in terms of t.

(d) (i) Sketch the curves of s_A and s_B, on the same diagram.

(ii) Find the values of t at which $s_A = s_B$.

[23 marks]

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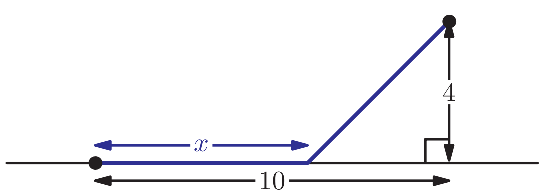
8. John needs to get from his house, which is on the main road, to his friend's house, which is in the field 10 km along the road and 4 km away from the road, as shown in the diagram. John can either cycle along the road, at the speed of 10 kmh^{-1} or walk through the field, at the speed of 5 kmh^{-1}.



John decides to cycle for the first x km and then walk the rest of the way in a straight line.

(a) Show that the time it takes John to get to his friend's house is given by:

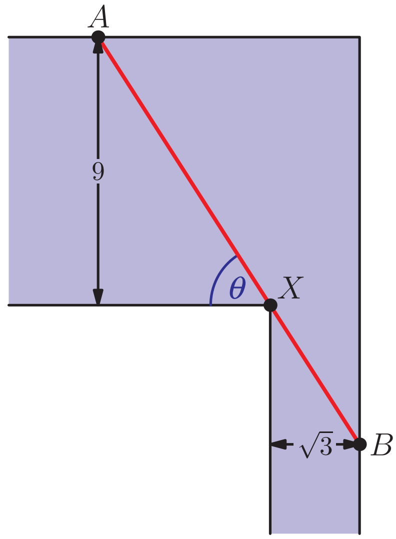
$$T = \frac{x}{10} + \frac{1}{5}\sqrt{16 + (10 - x)^2}$$

(b) John wishes to get to his friend's house in the shortest possible time.

(i) Show that the distance, x, he should cycle satisfies $3(10 - x)^2 = 16$.

(ii) Hence find how far John should cycle. *[10 marks]*

9. A ladder is carried around a corner from a corridor of width 9 m into a corridor of width $\sqrt{3}$ m as shown in the diagram.



(a) AXB is a straight line making angle θ with the first corridor, as shown.

(i) Write AX and XB in terms of θ.

(ii) Find the minimum length of AB.

(b) Find the maximum length of a ladder that can be around the corner.

[8 marks]