4 Modeling the real world

CHAPTER OBJECTIVES:

6.1 Informal ideas of limit, continuity, and convergence; definition of the derivative from first principles; the derivative interpreted as a gradient function and as a rate of change; finding equations of tangents and normals; identifying increasing and decreasing functions; the second derivative; higher derivatives

6.2 Derivative of x^n; differentiation of sums and multiples of functions; the product and quotient rules; the chain rule for composite functions; related rates of change; implicit differentiation

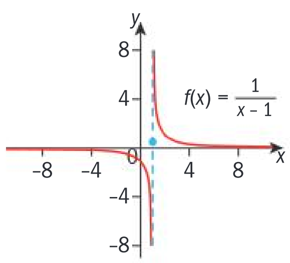
6.3 Local maximum and minimum values; optimization problems; points of inflection with zero and non-zero gradients; graphical behavior of functions including the relationship between the graphs of f, f', and f''

6.6 Kinematic problems involving displacement s, velocity v and acceleration a; total distance traveled

Before you start

You should know how to:

1 Draw graphs of rational functions.
e.g., sketch the graph of $y = \frac{1}{x-1}$, clearly showing any asymptotes as dotted lines.



2 Find the sum of infinite geometric series.
e.g., since, $|r| < 1$, $\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r = \frac{1}{1-\frac{1}{2}} = 2$.

Skills check

1 Sketch the graph of $f(x) = \frac{1}{x-3}$, clearly labeling all intercepts and asymptotes.

2 Find $\sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n$

From abstract models to real-world applications

A mathematical model uses mathematical language and systems of functions to describe, explain, interpret, and predict real-world phenomena. Climate scientists and meteorologists have collected vast amounts of data about weather systems and CO_2 concentrations in the atmosphere over many years. They have created mathematical models that fit the historical data and that they can now use to predict future climate changes.

Mathematical models are used today in all areas of human endeavor, from the natural sciences to the creative arts. In this chapter you will learn how to work with functions that may be derived from real-world situations, such as mechanics and economics.

The global financial crisis of 2008, was mainly due to a mathematical model created by economist David X. Li, to manage financial risk. His model was used in financial institutions throughout the world to assist in the calculation of risk factors in certain investment strategies. Was it a flaw in the model or in its interpretation that caused the crisis?

Mathematical modeling has many beneficial applications. However, what are the possible pitfalls of modeling real-life phenomena? What are the limits of mathematical modeling?

4.1 Limits, continuity and convergence

Zeno of Elea, a philosopher and logician, posed this problem about 2500 years ago. Achilles and a tortoise were engaged in a footrace. Achilles allowed the tortoise a head start of 100 metres. Both started running at a constant speed. Who won the race?

Zeno analysed the problem as follows. After a short time into the race, Achilles arrives at the tortoise's starting point of 100 m. In that time, the tortoise advances further. It then takes more time for Achilles to run this extra distance, in which time the tortoise advances even further. So, whenever Achilles reaches some point that the tortoise has already been at, he still has further to go. Since Achilles has an infinite number of points to cover before he reaches where the tortoise was, Achilles is still trying to win this race today!

It has taken several millennia for mathematicians to arrive at the language and concepts needed to satisfactorily solve this paradox. In this section you will learn some of the mathematics developed by 17th, 18th and 19th-century mathematicians in an attempt to deal with the concepts of time and infinity.

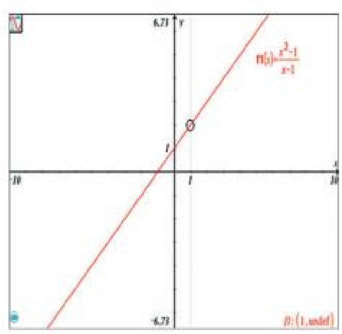
An informal treatment of limits

You can think of a limit as a way of describing the output of a function as the input gets close to a certain value.

The rules for finding limits are quite straightforward, and can be algebraic, graphical, numerical, or a combination of these methods.

As an example, consider the rational function $y = \frac{x^2-1}{x-1}$, $x \neq 1$

This function is not defined at $x = 1$ and its domain is $\{x \mid x \in \mathbb{R}, x \neq 1\}$.



Now, with your GDC, trace along the graph of this function. You will notice that as x gets closer to 1 from the left, the value of the function gets closer to 2. Trace along the graph from the right, and notice that the value of the function likewise gets closer to 2.

This table shows these results.

Approaching $x = 1$ from the left. Approaching $x = 1$ from the right.

x	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$y = \frac{x^2-1}{x-1}$	1.6	1.7	1.8	1.9	undef.	2.1	2.2	2.3	2.4

Approaching 1 in steps or **increments** of 0.1.

You can write this result using this notation: $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

This means that the limit of the functions as x approaches 1 *both* from the left *and* from the right is 2.

Example 1

a Sketch the graph of $y = \frac{2^x - 1}{x}$, $x \neq 0$

b Find $\lim_{x \to 0} \frac{2^x - 1}{x}$, giving your answer to 2 decimal places.

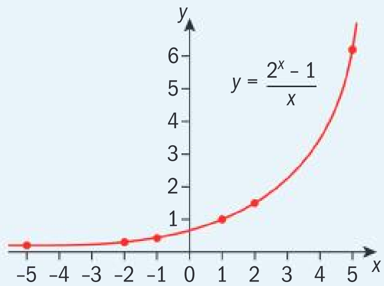
Answers

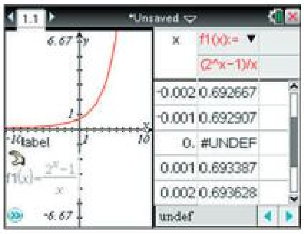
a

Make a table of values

x	−10	−5	−2	−1	0	1	2	5	10
$\frac{2^x - 1}{x}$	0.099	0.194	0.375	0.5	–	1	1.5	6.2	102

Sketch the graph





b $\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.69$

GDC tip! Change the table start and set to obtain a finer approximation of this limit.

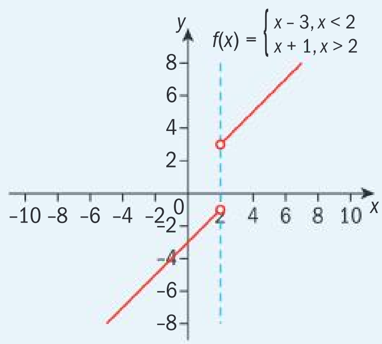
Example 2

a Sketch the function $y = \begin{cases} x - 3, & x < 2 \\ x + 1, & x > 2 \end{cases}$

b Find the value of the function as x gets closer to 2 from the left and from the right.

Answers

a



Since the piecewise function is not defined at x = 2, there are open circles at the points (2, −1) and (2, 3).

b $\lim_{x \to 2^-} = -1$ and $\lim_{x \to 2^+} = 3$

When x approaches 2 from the left, the function gets closer to −1, and when x approaches 2 from the right, the function gets closer to 3

$x \to 2^-$ means x approaches 2 from the left, and $x \to 2^+$ means x approaches 2 from the right.

In Example 1, the limits were the same whether approaching from the left or from the right. In Example 2, the limits are different when approached from the left and from the right. Therefore we say that in Example 2, the function has no limit.

For the limit of a function to exist as x approaches a particular value, the function does not need to be defined at the value but the value of the limit as the function approaches from the left and from the right must be the same.

→ The notation used to say that the limit, L, of a function f exists as x approaches a real value c is $\left(\lim_{x\to c} f(x) = L\right) \Leftrightarrow \left(\lim_{x\to c^+} f(x) = L \text{ and } \lim_{x\to c^-} f(x) = L\right)$ for $L \in \mathbb{R}$.

The double arrow is read 'if and only if'. An 'if and only if' definition or theorem has the form: if p then q and if q then p, where p and q are statements. This means that the two parts of the definition or theorem are equivalent. To prove an 'if and only if' theorem it is necessary to prove both, if p then q and also prove if q then p.

Exercise 4A

Using a GDC, sketch the graph of each function and find the limit, if it exists.

1 $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

2 $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$

3 $\lim_{x\to 2} \begin{cases} 3x-1, & x<2 \\ \dfrac{1}{x^2-1}, & x\geq 2 \end{cases}$

4 $\lim_{x\to 0} \dfrac{|x|}{x}$

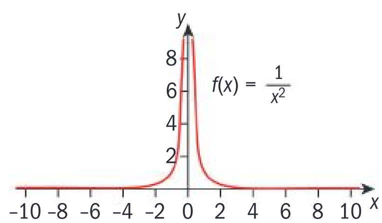
5 $\lim_{x\to 6} (x-6)^{\frac{2}{3}}$

6 $\lim_{x\to 3} \lfloor x \rfloor$

$y = \lfloor x \rfloor$ or $y = \text{int}(x)$ is the floor function. It is defined as the 'largest integer less than or equal to x'. This function will be defined for you in an examination.

Asymptotes and continuity

Does $\lim_{x\to 0} \dfrac{1}{x^2}$ exist? Here is the graph of the function $\dfrac{1}{x^2}$.



You can see that as x approaches 0 from the left and from the right, the values of the function increase without bound, and approach positive infinity. The limit therefore does not exist, since the limit is not a real number.

The line $x = 0$ is the vertical asymptote of this function. We can now define the vertical asymptote of a function.

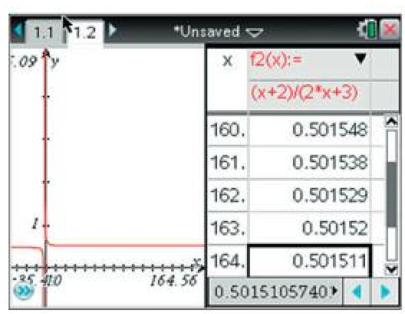
You met vertical asymptotes in Chapter 2.

→ The line $x = c$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either $\lim_{x\to c+} f(x) = \pm\infty$ or $\lim_{x\to c-} f(x) = \pm\infty$.

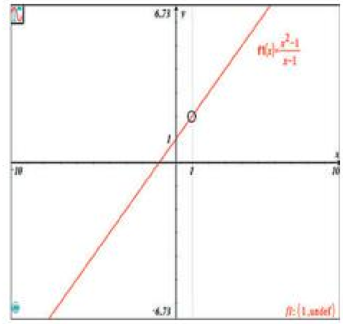
On page 174 you saw that the graph of $y = \dfrac{x^2-1}{x-1}$, $x \neq 1$, is linear.

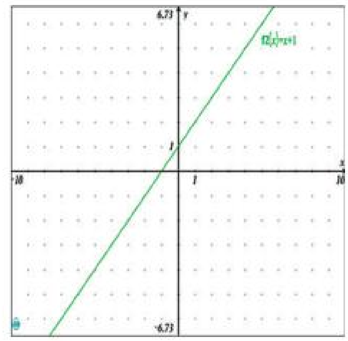
Simplifying, gives $y = \dfrac{x^2-1}{x-1} = x+1$

However, since $x \neq 1$, there is a gap or hole in the function at $x = 1$. For the function $y = x + 1$, however, there is no gap at $x = 1$. Hence, both functions have a limit of 2 as x approaches



1, but the graph of $y = \dfrac{x^2 - 1}{x - 1}$ is discontinuous at $x = 1$. The graph of $y = x + 1$ has no holes anywhere in its domain, so $y = x + 1$ is continuous.





→ A function $y = f(x)$ is **continuous** at $x = c$, if $\lim\limits_{x \to c} f(x) = f(c)$.

The three necessary conditions for f to be continuous at $x = c$ are:

1. f is defined at c, i.e., c is an element of the domain of f.
2. the limit of f at c exists.
3. the limit of f at c is equal to the value of the function at c.

A function that is not continuous at a point $x = c$ is said to be **discontinuous** at $x = c$.

A function is said to be continuous on an open interval I if it is continuous at every point in the interval.

A function is said to be continuous if it is continuous at ***every*** point in its domain.

A function that is not continuous is said to be **discontinuous**.

A polynomial function such as $3x^2 + 2x - 4$ is continuous at every point in its domain.

Example 3

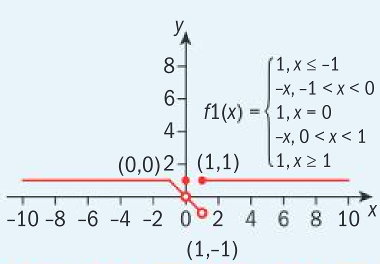
a Sketch the graph of $f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$

b Find the limits, if they exist, as x approaches -1, 0 and 1.

c Determine if f is continuous at $x = -1$, $x = 0$, and $x = 1$.

Answers

a



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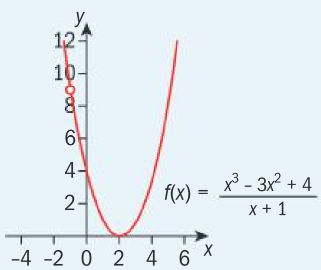
b $\lim_{x \to -1} f(x) = 1$	*As x approaches –1 from the left and from the right, f approaches 1.*
$\lim_{x \to 0} f(x) = 0$	*As x approaches 0 from the left and from the right, f approaches 0.*
$\lim_{x \to 1^-} f(x) = -1, \lim_{x \to 1^+} f(x) = 1$ $\Rightarrow \lim_{x \to 1} f(x) = undef.$	*As x approaches 1 from the left, f approaches –1, and as x approaches 1 from the right, f approaches 1, that is, the limit as x approaches 1 does not exist.*
c $\lim_{x \to -1} f(x) = 1$ and $f(-1) = 1$, hence at $x = 1$ f is continuous. $\lim_{x \to 0} f(x) = 0$, but $f(0) = 1$, hence at $x = 0$, f is discontinuous. $\lim_{x \to 1} f(x) =$ is undefined, and $f(1) = 1$, hence at $x = 1$, f is discontinuous.	

Example 4

$$f(x) = \begin{cases} \dfrac{x^3 - 3x^2 + 4}{x + 1}, & x \neq -1 \\ k, & x = -1 \end{cases}$$

Determine the value of k in order that $f(x)$ be continuous at $x = -1$.

Answer



$\dfrac{x^3 - 3x^2 + 4}{x + 1}$ is equivalent to $(x - 2)^2$ in their respective domains.

$x^3 - 3x^2 + 4 = \dfrac{(x - 2)^2 (x + 1)}{x + 1}$

$= (x - 2)^2$

When $x = -1$, $(x - 2)^2 = 9$

Hence, when $k = 9$,

$f(-1) = \lim_{x \to -1} \dfrac{x^3 - 3x^2 + 4}{x + 1} = 9$

so f is continuous at $x = -1$

For f to be continuous, f (–1) must equal 9.

Exercise 4B

1 $f(x) = \begin{cases} 3-x, & x \geq 1 \\ (x-1)^2, & x < 1 \end{cases}$. Determine if f is continuous at $x = 1$.

2 $f(x) = \begin{cases} x^2+4x+5, & x \leq -2 \\ 2x+5, & x > -2 \end{cases}$. Determine if f is continuous at $x = -2$.

3 $f(x) = \begin{cases} \dfrac{x-1}{|x-1|}, & x \neq 1 \\ 0, & x = 1 \end{cases}$. Determine if f is continuous at $x = 1$.

EXAM-STYLE QUESTIONS

4 Find a value for k such that $f(x) = \begin{cases} x^2-1, & x < 3 \\ 2kx, & x \geq 3 \end{cases}$ is continuous at 3.

5 Find the value of a such that $f(x) = \begin{cases} ax^2-a, & x \geq 3 \\ 4, & x < 3 \end{cases}$ is continuous for all values of x.

6 Determine if these functions are continuous on the set of real numbers. If they are not continuous for all real x, state the values of x for which the function is discontinuous.

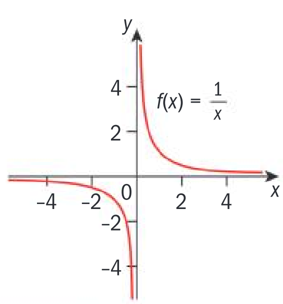
a $f(x) = \dfrac{x^2+1}{x^2-1}$ **b** $f(x) = \dfrac{x+1}{4-x^2}$ **c** $f(x) = \dfrac{x}{x^2+1}$

d $f(x) = \dfrac{x^2+3x+5}{x^2+3x-4}$ **e** $f(x) = \dfrac{x^2+1}{x^3-1}$ **f** $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$

Limits to infinity

Infinity is not a number. It lies beyond all finite bounds. Hence, when discussing the behavior of a function as x approaches positive or negative infinity, written $\pm\infty$, we look for the value that the function approaches as x increases and decreases without bound.

For example, consider the behavior of the function $y = \dfrac{1}{x}$ as x approaches $\pm\infty$.



The equation of the vertical asymptote is $x = 0$. The value of the function approaches 0 as x approaches $\pm\infty$, but is never equal to 0. There is no real number x such that $\dfrac{1}{x} = 0$.

The line $y = 0$ is the horizontal asymptote.

→ The line $y = k$, $k \in \mathbb{R}$, is the horizontal asymptote of $f(x)$ if either $\lim\limits_{x\to\infty} = k$ or $\lim\limits_{x\to-\infty} = k$.

Example 5

Sketch the graph of $y = \dfrac{x}{\sqrt{x^2+1}}$ for $-20 \le x \le 20$, show clearly any asymptote(s).

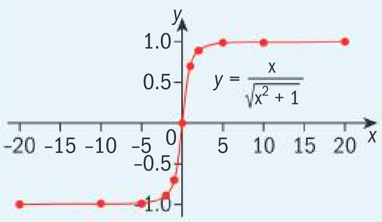
Check your answer on a GDC.

Answer

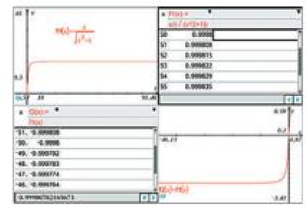
x	–20	–10	–5	–2	–1	0
$\dfrac{x}{\sqrt{x^2+1}}$	–0.999	–0.995	–0.981	–0.894	–0.707	0
x	1	2	5	10	20	
$\dfrac{x}{\sqrt{x^2+1}}$	0.707	0.894	0.981	0.995	0.999	

As x increases in the positive direction y approaches 1.

As x decreases in the negative direction y approaches –1.



Points on the graph are the values from the table.



$\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$

The horizontal asymptotes are $y = 1$ and $y = -1$

Example 6

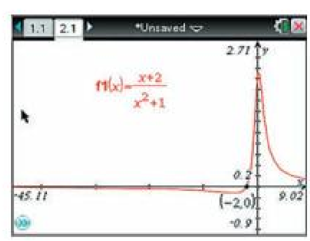
Sketch the graph of $f(x) = \dfrac{x+2}{x^2+1}$ for $-20 \le x \le 20$, show clearly any asymptote(s).

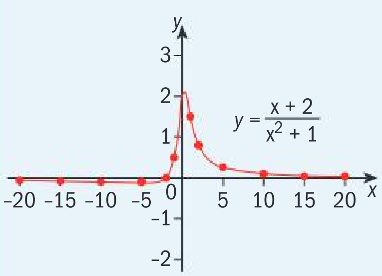
Check your answer on a GDC.

Answer

x	–20	–10	–5	–2	–1	0
$\dfrac{x+2}{x^2+2}$	–0.004	–0.084	–0.199	0	0.5	2
x	1	2	5	10	20	
$\dfrac{x+2}{x^2+2}$	1.5	0.8	0.269	0.119	0.005	

Notice that when $x = -2$, $y = 0$





Points on the graph are the values from the table.

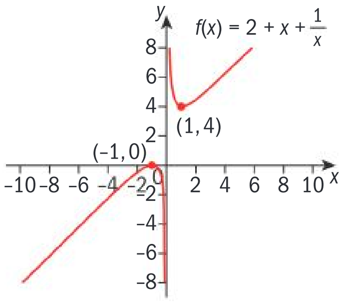
$\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$

The horizontal asymptote is $y = 0$, the x-axis.

The horizontal asymptote tells you the behavior of the function for very large values of x. However unlike the vertical asymptote, the function can assume the value of the horizontal asymptote for small values of x as happened in Example 6 at $x = 2$.

Other asymptotes

While **not** explicitly on the syllabus, it is useful to know that some asymptotes are neither vertical nor horizontal. For example, consider the graph of $f(x) = \frac{x^2+2x+1}{x} = 2+x+\frac{1}{x}$

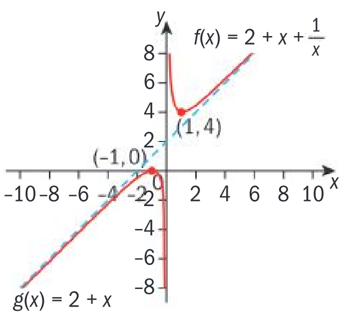


The point (1, 4) is a local minimum of the function. The point (−1, 0) is a local maximum.

You will notice that there is a slant, or oblique, asymptote which passes between the local minimum and maximum points. As x approaches $\pm\infty$ the function resembles ever more closely the straight line $y = x + 2$

On the graph, the line $y = 2 + x$ is an asymptote to the function $f(x) = 2 + x + \frac{1}{x}$

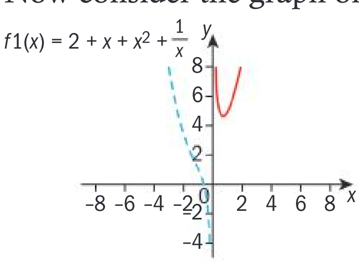
For very large values of x the value of $\frac{1}{x}$ is very small



As is clear from the graph, the difference, $\frac{1}{x}$, between the full function, $\frac{x^2+2x+1}{x}$ and its slant asymptote, $x + 2$, becomes vanishingly small as $x \to \pm\infty$.

The word asymptote comes from the Greek *asymptotos*, meaning 'not falling together'.

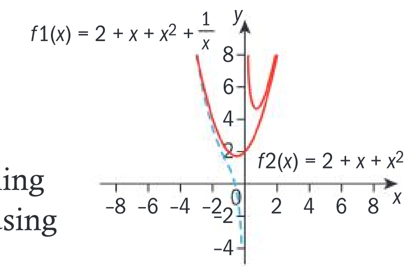
Now consider the graph of $f(x) = 2 + x + x^2 + \frac{1}{x}$



We know that there is a vertical asymptote at $x = 0$. The limit of this function as x approaches ∞ is $\lim_{x\to\infty}\left(2+x+x^2+\frac{1}{x}\right) = 2 + x + x^2$.

The curve $2 + x + x^2$ is an asymptote to the function.

Hence, an asymptote can be defined more generally as a line tangent to a curve at infinity.



Finding limits algebraically

Up to now we have been finding limits graphically and confirming our results numerically. We can find some limits algebraically using these properties of limits.

→ Properties of limits as $x \to \pm\infty$

Let L_1, L_2, and k be real numbers and $\lim_{x\to\pm\infty} f(x) = L_1$ and $\lim_{x\to\pm\infty} g(x) = L_2$. Then,

1. $\lim_{x\to\pm\infty}(f(x) \pm g(x)) = \lim_{x\to\pm\infty} f(x) \pm \lim_{x\to\pm\infty} g(x) = L_1 \pm L_2$
2. $\lim_{x\to\infty}(f(x)\cdot g(x)) = \lim_{x\to\infty} f(x)\cdot \lim_{x\to\infty} g(x) = L_1 \cdot L_2$
3. $\lim_{x\to\pm\infty}(f(x) \div g(x)) = \lim_{x\to\pm\infty} f(x) \div \lim_{x\to\pm\infty} g(x) = L_1 \div L_2$, provided $L_2 \neq 0$.
4. $\lim_{x\to\pm\infty} kf(x) = k\lim_{x\to\pm\infty} f(x) = kL_1$
5. $\lim_{x\to\pm\infty}[f(x)]^{\frac{a}{b}} = L^{\frac{a}{b}}_1$, $\frac{a}{b} \in \mathbb{Q}$ (in simplest form), provided $L^{\frac{a}{b}}_1$ is real.

You will only be required to use informal methods to find limits in the exam

These properties also hold when finding the limit as $x \to c$, $c \in \mathbb{R}$.

Example 7

Find the horizontal asymptote of $y = \dfrac{x+2}{2x+3}$

Answer

$\lim_{x\to\infty}\dfrac{x+2}{2x+3} = \lim_{x\to\infty}\dfrac{1+\frac{2}{x}}{2+\frac{3}{x}}$	*Divide numerator and denominator by largest power of x.*
$\lim_{x\to\infty}\dfrac{1+\frac{2}{x}}{2+\frac{3}{x}} = \lim_{x\to\infty}\left(1+\dfrac{2}{x}\right) \div \lim_{x\to\infty}\left(2+\dfrac{3}{x}\right)$	*Apply limit property 3: the limit of a quotient is the quotient of the limits.*
$\lim_{x\to\infty}\left(1+\dfrac{2}{x}\right) = \lim_{x\to\infty} 1 + \lim_{x\to\infty}\dfrac{2}{x} = 1+0 = 1$ $\lim_{x\to\infty}\left(2+\dfrac{3}{x}\right) = \lim_{x\to\infty} 2 + \lim_{x\to\infty}\dfrac{3}{x} = 2+0 = 2$	*Apply limit property 1: the limit of a sum is the sum of the limits.*
Hence, $\lim_{x\to\infty}\dfrac{x+2}{2x+3} = \dfrac{1}{2}$, and the horizontal asymptote is $y = \dfrac{1}{2}$.	*Remember that the line y = k is an horizontal asymptote if* $\lim_{x\to\infty} f(x) = k$.

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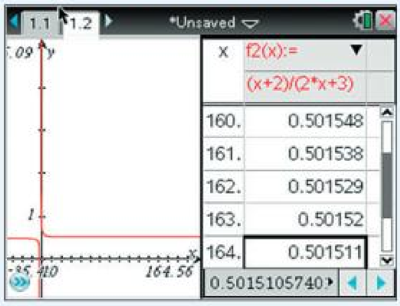
Graphing the function confirms the limit graphically and numerically:



As shown in Example 7, when finding limits of rational algebraic expressions, it is often useful to divide the numerator and denominator by the largest power of x. For example, when finding $\lim_{x\to\infty}\frac{x+3}{x^2+1}$, dividing both numerator and denominator by x^2 gives

$$\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}}$$

Using the properties of limits it is easy to verify that $\lim_{x\to\infty}\frac{x+3}{x^2+1} = 0$

Similarly, $\lim_{x\to\infty}\frac{-2x^3+2}{x^3-x}=\lim_{x\to\infty}\frac{-2+\frac{2}{x^3}}{1-\frac{1}{x^2}}=-2$

Hence, the horizontal asymptote is $y = -2$

You may wish to confirm this result using your GDC.

Investigation – graphs of $x^n + y^n = 1$

Graph the equation $x^2 + y^2 = 1$ using a graphing program. How would you enter the same equation in your GDC in order to see the same shape as the software produces?

Now graph $x^4 + y^4 = 1$. How does it compare with the graph of $x^2 + y^2 = 1$? Experiment with different even values of n for $x^n + y^n = 1$. What do you notice? From your observations, conjecture the shape of the graph of $x^n + y^n = 1$, when n is an even number, and n approaches infinity.

Investigation – graphs of polynomials

Graph functions of the type $\frac{P_n(x)}{Q_m(x)}$, such that n and m are positive integers, that represent the degree of the polynomial function.

Investigate the limit of the polynomial functions as x approaches $\pm\infty$ when

a $n < m$ **b** $n = m$ **c** $n > m$

Make a conjecture regarding the horizontal asymptotes of your functions, and justify your conjecture for the different cases in **a**, **b** and **c**.

Possible examples are, $P_2(x) = x^2 + 3$ $Q_4(x) = 3x^4 + x^2 + 1$

Exercise 4C

1 Find the required limit algebraically, if it exists.

a $\lim_{x\to 4}\left(\frac{x+3}{x-3}\right)$ **b** $\lim_{x\to -2}\left(\frac{x^2+x-2}{x+2}\right)$ **c** $\lim_{x\to -2}\left(\frac{x^6-64}{x^3-8}\right)$

d $\lim_{x\to 0}\frac{x^2-1}{x^2-x}$ **e** $\lim_{x\to 1}\frac{x^2-1}{x^2-x}$ **f** $\lim_{x\to 1}\frac{1}{1+\frac{1}{1-x}}$

g $\lim_{x\to 0}\frac{(2+3x)^2-4(1+x)^2}{6x}$ **h** $\lim_{x\to a}\frac{x^2-a^2}{x-a}$

2 Find the limit of $f(x)$ algebraically as x approaches $+\infty$, if it exists.

a $\frac{2x}{x+2}$ **b** $\frac{3x^2}{x^2-1}$ **c** $\frac{2x^2+x-1}{3x^2+5x-1}$

d $\frac{5x^2}{4x^3+2}$ **e** $\frac{x-1}{x^2-3x+5}$ **f** $\frac{\sqrt{4x+3}+2\sqrt{1+x}}{\sqrt{x}}$

3 Find, algebraically, any horizontal asymptotes of these functions.

a $\frac{3x^2-x+5}{x^2-4}$ **b** $\frac{2x}{4x-1}$ **c** $\frac{2x^2-3x+1}{x^3}$

d $\frac{x^2+1}{1-x^2}$ **e** $\frac{2x^3-3x}{2x^2+1}$

Convergence of sequences

The concept of limits can be used to describe the value that a sequence approaches as its index approaches a certain value.

Sequences were introduced in Chapter 1

You know that $\lim_{x\to\infty}\frac{1}{x}=0$. Now consider $\lim_{n\to\infty} a_n$, $n\in\mathbb{Z}^+$, $a_n=\frac{1}{n}$

Write out the terms of this sequence:

$$1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots$$

If m and n are positive integers, then when $m<n$, $\frac{1}{m}>\frac{1}{n}$. Hence, as the number of terms in the sequence increases, the value of the expression $\frac{1}{n}$ decreases, until it is very close to 0.

Hence, $\lim_{n\to\infty}\frac{1}{n}=0$, and we say that the sequence converges to 0.

You can investigate this graphically and numerically with the GDC.

If the sequence has a finite limit, then the sequence is said to be **convergent**, otherwise it is **divergent**.

The properties of limits of sequences are the same as those for limits of functions.

Example 8

Find $\lim_{n \to \infty} \frac{n^2 + 3n}{2n^2 + 1}$, $n \in \mathbb{Z}^+$

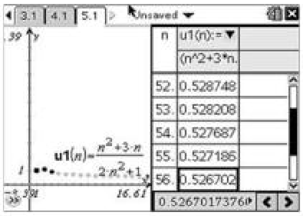
Answer

$$\lim_{n \to \infty} \frac{n^2 + 3n}{2n^2 + 1} = \lim_{n \to \infty} \frac{1 + \frac{3}{n}}{2 + \frac{1}{n^2}} = \frac{1}{2}$$

Divide both numerator and denominator by n^2, and use the properties of limits.

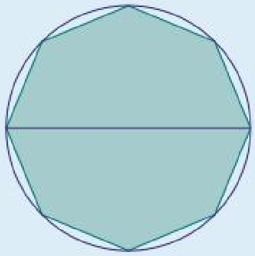
Hence, the sequence converges to $\frac{1}{2}$.

Put your GDC in sequence mode. Confirm this result graphically and numerically on the GDC.



Investigation – inscribed polygons

Consider a polygon inscribed in a circle.
You can form a sequence of rational numbers by taking the ratio of the perimeter of a regular polygon to its diameter.



Begin with an equilateral triangle in a circle. Calculate its perimeter and write the ratio of its perimeter to its diameter. Do the same for regular polygons of up to 10 sides. Formulate a conjecture.
Test your conjecture by calculating the same ratio for polygons with many sides, e.g., 60, 80, 100, etc.

Determine the limit to infinity of your sequence, and justify your answer.

Convergence of series

In Chapter 1 you learned that if a geometric series has a finite sum, it **converges** to its sum.

Recall the formula for finding the sum of a finite geometric series,
$S_n = \frac{u_1(1 - r^n)}{1 - r}$

→ For a geometric series, $\sum_{n=0}^{\infty} u_1 r^n = \lim_{n \to \infty} \frac{u_1(1 - r^n)}{1 - r}$

When $-1 < r < 1$, $\lim_{n \to \infty} r^n = 0$ and the series converges to $S = \frac{u_1}{1 - r}$

Consider the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Writing out this series, you obtain $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$. Since $|r| < 1$, this infinite geometric series has a finite sum, $S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$. The series converges to 1.

If the series does not have a finite sum, the series **diverges**.

Example 9

Determine whether the series $\sum_{n=0}^{\infty}\frac{5^n+4^n}{6^n}$ converges.

Answer

$$\sum_{n=0}^{\infty}\frac{5^n+4^n}{6^n}=\sum_{n=0}^{\infty}\frac{5^n}{6^n}+\sum_{n=0}^{\infty}\frac{4^n}{6^n}=\sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n+\sum_{n=0}^{\infty}\left(\frac{4}{6}\right)^n$$

This is the sum of two geometric series.

$$\sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n=\frac{1}{1-\frac{5}{6}}=6 \text{ and } \sum_{n=0}^{\infty}\left(\frac{4}{6}\right)^n=\frac{1}{1-\frac{4}{6}}=3$$

Find limits seperately.

Hence, $\sum_{n=0}^{\infty}\frac{5^n+4^n}{6^n}=9$, so the series converges to 9.

Using limit property 1, the limit of a sum is the sum of the limits.

Exercise 4D

1 Determine whether these sequences converge.

a $\lim_{n\to\infty}\frac{n+1}{n}$ **b** $\lim_{n\to\infty}\frac{n+1}{2n+1}$ **c** $\lim_{n\to\infty}\frac{n^2-n}{2n^2+\sqrt{n}}$

d $\lim_{n\to\infty}\frac{1-n^3}{n^2+1}$ **e** $\lim_{n\to\infty}\frac{n^2+1}{1-n^3}$

2 Determine whether each series converges. If it converges, determine its sum.

a $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$ **b** $\sum_{n=1}^{\infty}\left(\frac{\pi}{3.14}\right)^n$ **c** $\sum_{n=1}^{\infty}5\left(\frac{1}{3}\right)^n$

d $\sum_{n=1}^{\infty}\frac{3}{10^n}$ **e** $\sum_{n=1}^{\infty}\frac{2^n-3^n}{7^n}$ **f** $\sum_{n=1}^{\infty}4(-0.6)^{n-1}$

EXAM-STYLE QUESTIONS

3 A geometric series has $u_1 = 35$ and $r = 2^x$.

a Find the values of x for which the series is convergent.

b Find the value of x for which the series converges to 40.

4 Find the set of values of x for which the series $\sum_{n=0}^{\infty}\left(\frac{3x}{x+1}\right)^n$ converges.

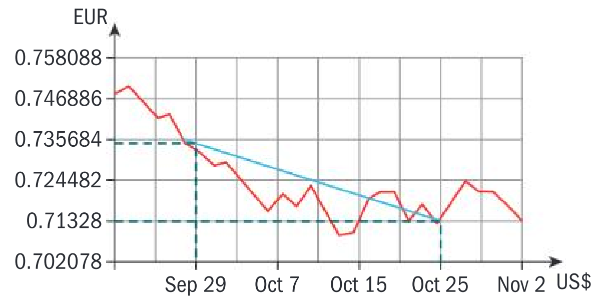
4.2 The derivative of a function

In mathematics, the derivative is the rate at which one quantity changes with respect to another. The process of finding the derivative is called **differentiation**. These ideas are central to the area of mathematics called the calculus.

Calculus was the result of centuries of work and debate. **Isaac Newton** (1642–1726) said "If I have seen further it is by standing on the shoulders of giants". There is evidence that both Newton and **Leibniz** (1646–1716) developed calculus independently within the same ten-year period, approximately 1665 to 1675.

Average rates of change

The graph shows the exchange rate of the euro to the US$ over the indicated time period in 2010. What was the average daily drop in exchange rate from September 29 to October 25?



Source: www.*x*-rates.com

The average daily drop in the exchange rate

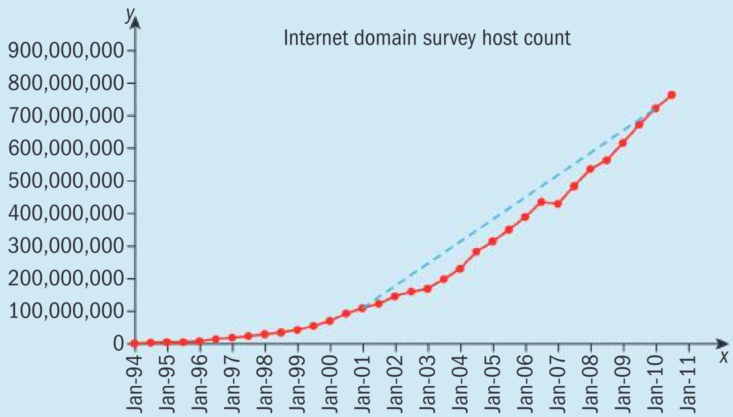
$$= \frac{\text{total change in the rate}}{\text{change in the time period}}$$

The change in the exchange rate over the indicated time period is approximately $0.735684 - 0.71328$, or -0.02356. The number of days between October 25 and September 29 is 26. Hence, the average daily exchange rate drop, $-0.2356 \div 26$, is about 0.001.

Graphically, the average rate of change between two points is the gradient of the line joining the two points.

Example 10

This graph shows the growth of internet domains on the world wide web since 1994. Estimate the average yearly growth between January 2001 and January 2010.



Source: Internet Systems Consortium (www.isc.org)

Answer

The average yearly growth is the gradient of the secant line joining the points (2001,100 000 000) and (2010,800 000 000).

$$\text{Average yearly growth} = \frac{800000000 - 100000000}{9}$$

$$\approx 77.8 \text{ million domains yearly}$$

A secant line joins two points on a curve.

Example 11

A ball rolling toward the edge of a ping-pong table is d cm from the edge at any time t seconds, $t > 1$, and $d = -t^2 + t + 6$. Find the average speed of the ball between the first and third second.

Answer

Average speed = total distance ÷ total time

$$\frac{\left[-(3)^2+3+6\right]-\left[-(1)^2+1+6\right]}{3-1} = -3 \text{ cm } s^{-1}$$

The speed of the ball is 3 cm s^{-1}.

→ In general, the average rate of a function f between two input values x_1 and x_2 is given by

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}, \text{ or } \frac{\Delta y}{\Delta x}$$

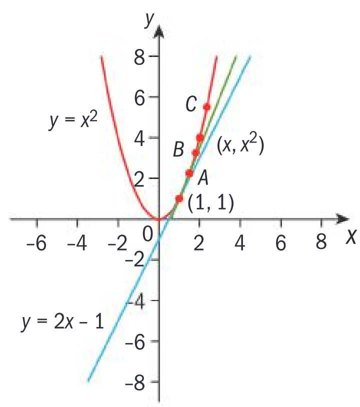
(read as 'the change in y divided by the change in x' where Δ is the Greek letter delta.)

Since speed refers to how fast an object is moving, it is always positive. Velocity refers to the rate at which an object changes its position, hence it can be positive or negative. For example, if I move forward at a rate of 2 km/h and then return at the same rate, my speed is always the same, but the direction in which I'm moving has changed. Moving forward I have a positive velocity, whereas returning I have a negative velocity.

The rate of change, $\frac{\Delta y}{\Delta x}$, at a point, is the gradient of the graph at the point.

If a function is linear, the gradient between any two points is the same, hence the rate of change, $\frac{\Delta y}{\Delta x}$, between any two points is the same, and will be the same for the rate of change at any particular point.

This changes, however, for a curve. Consider the graph of the function $y = x^2$, and the rate of change, $\frac{\Delta y}{\Delta x}$, between any point on the curve and the point (1, 1).



The gradient of any of the secant lines is $\frac{\Delta y}{\Delta x} = \frac{x^2-1}{x-1} = x+1,\ x \neq 1$

What is the gradient at $x = 1$, since according to this definition, x cannot equal 1? You can see geometrically that as the points move closer to (1,1) the secant lines approach a line which is a tangent to the curve at (1,1).

Now take a point on the curve arbitrarily close to the point (1, 1), whose x coordinate is $1 + h$, where h is a very small quantity, $h \neq 0$. The corresponding y-coordinate is $(1 + h)^2$. You can now find the gradient between the two points (1, 1) and $(1+h, (1+h)^2)$:

$$\frac{\Delta y}{\Delta x} = \frac{(1+h)^2-1^2}{(1+h)-1} = \frac{1^2+2h+h^2-1^2}{h} = \frac{2h+h^2}{h} = 2+h$$

The limit of this expression as h approaches 0 is $\lim_{h\to 0}(2+h) = 2$.

Hence, the gradient of the tangent at the point (1, 1) is 2.

→ The gradient of a curve $y = f(x)$ at the point $(a, f(a))$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided this limit exists.

Example 12

Find the gradient of the curve $y = x^2$ at the point $x = -2$.

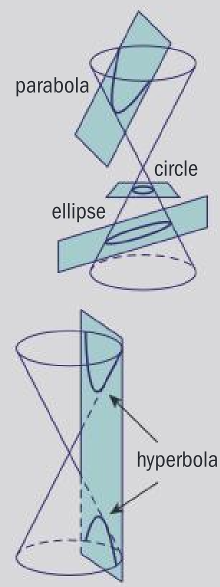
Answer	
$\frac{\Delta y}{\Delta x} = \frac{(-2+h)^2 - (-2)^2}{(-2+h)-(-2)} = \frac{4-4h+h^2-4}{h}$	*Use the definition for gradient of a curve at a point.*
$= \frac{-4h+h^2}{h} = \frac{h(-4+h)}{h} = -4+h$	*Simplify.*
$\lim_{h \to 0} (-4+h) = -4$	*Evaluate the limit.*

Example 13

Find the points on the curve $y = \frac{1}{x}$ such that the gradient at these points is $-\frac{1}{9}$

Answer	
Consider the point $\left(a, \frac{1}{a}\right)$ and a neighboring point $\left(a+h, \frac{1}{a+h}\right)$	*Use the definition for gradient of a curve at a point, then simplify.*
$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{a+h} - \frac{1}{a}}{(a+h)-a} = \frac{\frac{a-(a+h)}{a(a+h)}}{h} = \frac{\frac{-h}{a^2+ah}}{h}$	
$= \frac{-1}{a^2+ah}$	*Evaluate the limit.*
$\lim_{h \to 0} \frac{-1}{a^2+ah} = -\frac{1}{a^2}$	
$-\frac{1}{a^2} = -\frac{1}{9}$, hence $a = \pm 3$ The points are $(3, \frac{1}{3})$ and $(-3, -\frac{1}{3})$.	*Set the expression equal to the gradient, and solve for a.*

In geometrical terms, a curve is a set of points under a specific condition. For example, a circle is a set of points equidistant from a fixed point called its center. Recall from chapter 2 that other geometrical curves, such as the parabola, ellipse and hyperbola are obtained when a plane intersects a cone at different angles. Many curves have been either discovered, or invented for the solution of special problems, for example, in mechanics.



Find examples of real-life problems where conic sections are used, and model them.

Exercise 4E

1 Find the gradient of the curve at the given value of x.

a $y = 2x^2 - 1$ at $x = 1$

b $y = \frac{2}{x}$ at $x = -2$

c $y = x^3$ at $x = 1$

d $y = -x^2$ at $x = 1$

e $y = \frac{x}{x+1}$ at $x = 0$

f $y = \frac{1}{x^2}$ at $x = 2$

2 Find the point on the curve $y = \frac{1}{x^2}$ such that the gradient at the point is 2.

3 Find the point on the curve $y = 2x^2 + \frac{1}{x}$ and then the point on the curve whose gradient is 3.

Investigation – gradients

a Find the gradient to $y = x^n$, n a positive integer, at different points along the curve. You have already found two such values for $y = x^2$, at $x = 1$ and $x = -2$.

b Conjecture a rule to find the gradient of the tangent to $y = x^n$ at any point on its curve.

You have developed the definition of the gradient of a point on a curve, and looked at some examples. In the investigation above you derived a rule for the gradient function of the given curve for all points on the curve. The derivative of a function at each point along its curve can now be found.

→ The **derivative**, or **gradient function**, of a function f with respect to x is the function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

If f' exists, then f has a derivative at x, or is **differentiable** at x. ($f'(x)$ is read f dash, or f prime, of x.) Another notation for the derivative is $\frac{dy}{dx}$, the derivative of the function $y = f(x)$ with respect to x.

A function is differentiable if the derivative exists for all x in the domain of f.

Example 14

Find $f'(x)$ given that $f(x) = 2x^2 + x$, and hence find the gradient of the function at $x = -3$.

Answer

$f'(x) = \lim_{h\to 0}\dfrac{2(x+h)^2+(x+\text{h})-(2x^2+x)}{\text{h}}$	*Use the definition of the derivative,*
$= \lim_{h\to 0}(4x+1+2h)$	*then simplify,*
$= 4x + 1$	*then evaluate the limit.*
$f'(-3) = 4(-3) + 1 = -11$	*Evaluate f' at $x = -3$.*

Example 15

If $f(x) = \sqrt{x}$, find $f'(x)$, and then find the gradient to the curve at $x = 4$.

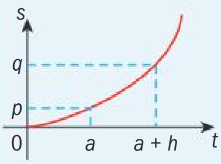
Answer

$f'(x) = \lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$	*To simplify use the difference of two squares.*
$= \lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$	*Multiply by* $\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$ *Place h as shown so as to arrive at the next result.*
$= \lim_{h\to 0}\dfrac{\frac{(x+h)-x}{h}}{\sqrt{x+h}+\sqrt{x}}$	
$= \lim_{h\to 0}\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$	
$= \dfrac{1}{2\sqrt{x}}$	
When $x = 4$, the gradient to the curve is $\dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$	

Example 16

A particle moves in a straight line so that its position from its starting point at any time t, in seconds, is given by $s = 4t^2$, where s is in metres. The particle passes through a point P when $t = a$ and then sometime later it passes through point Q when $t = a + h$. Find the average velocity as the particle travels from point P to point Q, and deduce its velocity at the instant it passes through P.

Answer



Sketch a graph.

$P(a, 4a^2)$ and $Q(a + h, 4(a + h)^2)$

Average velocity = $\dfrac{\textit{total distance traveled}}{\textit{total traveling time}}$

Average velocity $= \dfrac{4(a+h)^2 - 4a^2}{(a+h) - a}$

$= \dfrac{4(a^2 + 2ah + h^2) - 4a^2}{h}$

$= \dfrac{4h^2 + 8ah}{h}$

$= \dfrac{h(4h+8a)}{h}$

$= 4h + 8a\,\text{m s}^{-1}$

To find velocity at P find $\lim\limits_{h \to 0}(4h + 8a)$.

Velocity at $P = 8a\,\text{m s}^{-1}$

The velocity of the particle when $t = a$ is the **instantaneous velocity**.

Some functions do not have a derivative at every point in their domain. You can easily prove that if a function is differentiable at $x = c$, then it will be continuous at $x = c$. In other words, **differentiability at a point implies continuity at the point**.

Let f be differentiable at $x = c$. You want to show that $\lim\limits_{x \to c} f(x) = f(c)$.

Since f is differentiable at $x = c$, and the point $x = c$ is excluded from the limit $x \to c$,

$$\lim_{x \to c}\left[f(x) - f(c)\right] = \lim_{x \to c}\left[\frac{f(x) - f(c)}{x - c} \cdot x - c\right] = \lim_{x \to c}\frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c) = f'(c) \cdot 0 = 0.$$

Since $\lim\limits_{x \to c}\left[f(x) - f(c)\right] = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} f(c) = \lim\limits_{x \to c} f(x) - f(c)$ it follows that $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} f(c)$, hence f is continuous at c.

Now consider the converse, i.e., if a function is continuous at $x = c$, it is differentiable at $x = c$. To find a counter – example you need to find a function that is continuous at $x = c$, but whose left and right limits as x approaches c are either not equal or do not exist.

One such function that you are familiar with is $y=|x|$. You know that for all $x < 0$, the function is equivalent to $y = -x$, hence the gradient of all points to the left of $x = 0$ is -1. For all $x > 0$, the function is equivalent to $y = x$, hence the gradient of all points to the right of $x = 0$ is 1. Using mathematical notation,

$$\lim_{h\to 0^+}\frac{|x+h|-|x|}{h}=-1 \text{ and } \lim_{h\to 0^-}\frac{|x+h|-|x|}{h}=1$$

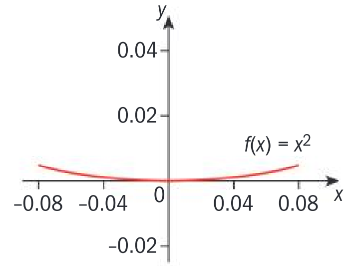
Since the left and right limits differ, the function does not have a derivative at $x = 0$.

→ If a function is differentiable at c, it is continuous at c.

A function that is continuous at c may not be differentiable at c.

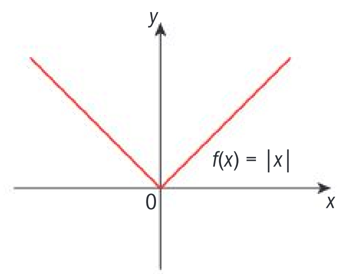
Linearity

A visual approach to deciding if a function is differentiable at the point is local linearity at the point. If you zoom in with your GDC at a point on a function that is differentiable, for example, x^2 at $x = 0$, the function seems to 'flatten' at this point. The more you zoom in, the more linear it appears at $x = 0$.



Test this visual approach on several functions at points that have a derivative.

When you perform the zoom test on $y=|x|$ at $x = 0$, it will remain unchanged regardless of how closely you zoom in.



Exercise 4F

1 Find the gradient function of the given curve, and then the value of the gradient to the curve at the given point.

a $y = x^2 + 2x + 1$ at $x = 0$

b $y = x^3 - 1$ at $x = 1$

c $y = \frac{2}{x}$ at $x = 3$

d $y = \sqrt{x-1}$ at $x = 2$

e $y = \sqrt{x+3}$ at $x = 1$

f $y = \frac{1}{\sqrt{x}}$ at $x = 4$

EXAM-STYLE QUESTION

2 A particle moves in a straight line so that its position from its starting point after t seconds is $12 - 5t^2$. If the particle passes through point A when $t = a$, and point B when $t = a + h$, find

a the average velocity of the object as it moves from A to B

b the velocity as it passes through point A.

Investigation – functions that are not differentiable

Find examples of functions that are continuous at a point in an open domain, but not differentiable at this point, given these conditions.

a A function that is defined piecewise.
b A function with a point such that the secant lines approach $+\infty$ from one side, and $-\infty$ from the other side, at this point.
c A function with a point such that the secant lines approach either $+\infty$ or $-\infty$ from both sides, at this point.

Equations of tangents and normals

If m is the gradient at a point (x_1, y_1) on a curve the equation of the tangent at that point is

$(y - y_1) = m(x - x_1)$

Gradient-point formula: $y - y_1 = m(x - x_1)$

You can also find the normal to the function at a particular point, since the gradient of the normal is the negative reciprocal of the gradient of the tangent.

Example 17

Given $f(x) = 3x^2 - 2$, find

a the gradient to the curve at $x = 1$
b the equation of the tangent to the curve at $x = 1$
c the equation of the normal to the tangent at $x = 1$

Answers

a $f'(x) = \lim_{h \to 0} \frac{[3(x+h)^2 - 2] - (3x^2 - 2)}{h}$ — *Find the gradient function.*

$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2 - 3x^2 + 2}{h}$

$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} 6x + 3h = 6x$

Hence, at $x = 1$, the gradient is 6. — *Evaluate the gradient function at $x = 1$.*

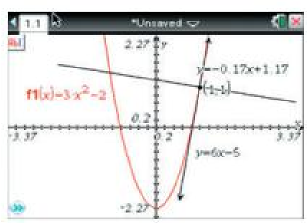
Equation of tangent is
$y - 1 = 6(x - 1)$ or $y = 6x - 5$ — *The gradient of the tangent is 6 and it goes through the point (1, 1).*

c Equation of normal is

$y - 1 = -\frac{1}{6}(x - 1)$

or $y = -\frac{x}{6} + \frac{7}{6}$

The gradient of the normal to the tangent is $-\frac{1}{6}$ and the normal goes through the point (1, 1).



Exercise 4G

1 Given $f(x) = 9 - x^2$, find

a the gradient of the curve at $x = -1$

b the equation of the tangent to the curve at $x = -1$

c the equation of the normal to the curve at $x = -1$

2 Find the points on the curve $y = \frac{1}{x-1}$ whose gradient is -1, and find the equations of the tangents through these points.

3 Find any points on the following curves that have horizontal tangents, i.e., tangents parallel to the x-axis.

a $y = 4 - 3x - 3x^2$ **b** $y = x^3 + 1$ **c** $y = \frac{1}{x}$

d $y = x^2 - 3x$ **e** $y = \sqrt{x}$

4 Find the equations of the tangent and normal to the curve $y = x + \frac{1}{x}$ at $x = 1$

4.3 Differentiation rules

Derivative of a constant function

The graph of $f(x) = c$, $c \in \mathbb{R}$, is a straight line whose equation is $y = c$ and it is parallel to the x-axis. Its gradient is therefore 0 for all x. Hence,

→ If $f(x) = c$, and $c \in \mathbb{R}$, then $f'(x) = 0$

Positive integer powers of x

From the investigation on the derivative of $y = x^2$, you will probably have conjectured that its gradient function is $2x$. What happens with higher powers of x? Here is a rule developed from first principles.

Using the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h}$$

$$= \lim_{h \to 0}((x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}) = nx^{n-1}$$

Use the algebraic identity (see Chapter 1).
$a^n - b^n = (a-b)$
$(a^{n-1} + n^{n-2}b + \ldots$
$+ ab^{n-2} + b^{n-1}),\ n \in \mathbb{Z}^*$
with $a = x + h$ and $b = x$.

There are n terms, each having the limit x^{n-1} as h approaches 0.

→ If n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$

This is the power rule.

You have already seen this result in the investigation in the previous section, that is, if $f(x) = x^2$, then $f'(x) = 2x$

The power rule holds for all n, where n is a real number, and this result will be used without proof.

Constant multiple of a function

→ For $c \in \mathbb{R}$, $(cf)'(x) = cf'(x)$ provided $f'(x)$ exists.

The sum and difference of functions

Let $f(x)$ be the sum of two functions in x whose derivatives exist, i.e., $f(x) = u(x) + v(x)$. Then,

$$f'(x) = \lim_{h\to 0}\frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} + \lim_{h\to 0}\frac{v(x+h)-v(x)}{h}$$

$$= u'(x) + v'(x)$$

→ If $f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$

The proof for the difference of two functions is left as an exercise for you.

The proof for negative integer powers of x is left to the student as an exercise. This result can be extended to all real number powers of x.

Example 18

Differentiate $y = \frac{1}{4}x^5 - x^3 + 5x^2 - \frac{1}{2}x + 3$ with respect to x.

Answer

$\frac{dy}{dx} = \frac{5}{4}x^4 - 3x^2 + 10x - \frac{1}{2}$	*The derivative of a sum is the sum of the derivatives. The first four terms use the power and constant multiple rules, and the last term uses the constant function rule.*

When the function is given as y, write the derivative as $\frac{dy}{dx}$

Example 19

Find $f'(x)$ if $f(x) = \frac{2x^4 - 3x^3 + 1}{x^2}$, $x \neq 0$

Answer

$f(x) = \frac{2x^4}{x^2} - \frac{3x^3}{x^2} + \frac{1}{x^2}$	*Write as a sum.*
$= 2x^2 - 3x + x^{-2}$	*Simplify.*
$f'(x) = 4x - 3 - 2x^{-3} = 4x - 3 - \frac{2}{x^3}$	*Differentiate each term.*

When the function is given as $f(x)$, write the derivative as $f'(x)$.

Example 20

Find the equation of the normal to the curve $f(x) = -2x^3 + x - 1$ at $x = 0$.

Answer

At $x = 0$, $f(0) = -1$, so the point on the curve is $(0, -1)$.	
$f'(x) = -6x^2 + 1$	*Differentiate $f'(x)$ to get the gradient function.*
$f'(0) = 1$	*Evaluate $f'(x)$ at $x = 0$*
Gradient of normal $= -\frac{1}{1} = -1$	*Find gradient of normal.*
$y - 1 = -1(x - 0)$	*Use $y - y_1 = m(x - x_1)$*
$y = -x + 1$	*Simplify.*

Exercise 4H

1 Find $\frac{dy}{dx}$ for each function.

a $y = 4 - x - 3x^2$

b $y = 2x^4 - 3x + 1$

c $y = 4x^3 - \frac{1}{x^3} + 2x^2 + \frac{2}{3x^2}$

d $y = \frac{2 - 3x^2 + 5x^4}{x}$

2 Find the equation of the tangent to the curve $y = 2(3x^2 - 2x)$ at $x = 1$

3 Find the equation of the normal to the curve $y = \frac{x-3}{x}$ at the point $x = -1$

The chain rule

The function $y = (2x - 1)^3$ is a polynomial so it is differentiable for all x. To differentiate this function, expand it, and then use the sum and difference rules.

This function,
$y = (2x - 1)^3$
$= 8x^3 - 4x^2 + 6x - 1$
Hence
$\frac{dy}{dx} = 24x^2 - 4x + 6$

$y = (2x - 1)^3$ is a composite function where
$y = f(g(x))$, $g(x) = (2x - 1)$, and $f(x) = x^3$

Let $u = g(x)$, then $y = u^3$

Consider the relationship $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2$ (since $u = 2x - 1$)

then using the relationship above

$$\frac{dy}{dx} = 3u^2 \cdot 2 = 6u^2$$

Replacing u

$$\frac{dy}{dx} = 6(2x - 1)^2$$

This is as example of the chain rule.

$\frac{dy}{du}$ and $\frac{du}{dx}$ are not fractions, hence this relationship is not arrived at by cancelling du. Since, however, these are rates of change, we can intuitively see that if, for example, y changes twice as fast as u and u changes three times as fast as x, then y would change 6 times as fast as x.

→ If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x)$ is differentiable at x. Furthermore, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Another definition for the chain rule is

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Example 21

Differentiate $y = (1 - 3x)^7$ with respect to x.

Answer	
Let $u = 1 - 3x$, then $\frac{du}{dx} = -3$	*Define u and find* $\frac{du}{dx}$
Hence, $y = u^7$ and $\frac{dy}{du} = 7u^6$	*Write y in term of u.* *Find* $\frac{dy}{du}$
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 7u^6 \cdot (-3)$	*Use the chain rule.*
$\frac{dy}{dx} = 7(1 - 3x)^6(-3) = -21(1 - 3x)^6$	*Substitute for u and simplify.*

Example 22

Differentiate $\sqrt{3x^2 - 4}$

Answer	
$(f \circ g)(x) = \sqrt{3x^2 - 4}$ for $f(x) = \sqrt{x}$ and $g(x) = 3x^2 - 4$	*Find f and g for the composite function.*
$f'(g(x)) = \frac{1}{2}(g(x))^{-\frac{1}{2}} = \frac{1}{2}(3x^2 - 4)^{-\frac{1}{2}}$	*Differentiate f (g)*
$g'(x) = 6x$	*Differentiate g(x)*
$(f \circ g)'(x) = \frac{1}{2}(3x^2 - 4)^{-\frac{1}{2}} \cdot 6x$	*Apply the chain rule.*
$= \frac{3x}{\sqrt{3x^2 - 4}}$	*Simplify.*

You can use the chain rule to show that the derivative of an odd function is an even function. Recall the definition of an odd function, i.e., if f is odd, then $f(-x) = -f(x)$

Hence, $-f'(x) = f'(-x)(-1)$, and it follows that $f'(-x) = f'(x)$

f' is therefore an even function.

Similarly, if f is an even function, then $f(-x) = f(x)$. Hence, $f'(-x)(-1) = f'(x)$, and it follows that $f'(-x) = -f'(x)$, and therefore f' is an odd function.

Odd and even functions are discussed in Section 2.2

Exercise 4I

1 Find $\frac{dy}{dx}$ for each function.

a $y = (2x + 3)^5$

b $y = \sqrt{2 - 3x}$

c $y = \frac{2 - 3x^2 + 5x^4}{x}$

d $y = \frac{-3}{\sqrt{5x^2 + 1}}$

e $y = \left(\frac{x}{1 - \sqrt{x}}\right)^3$

2 Find the equation of the tangent to the curve $y = \sqrt{3x^2 - 2x}$ at $x = 1$

3 Find the equation of the normal to the curve $\frac{x - 3}{x}$ at the point $x = 1$

4 Find the point of the curve $\frac{1}{3x^2 - 6x + 1}$ where the tangent to the curve is parallel to the x-axis.

5 Find the derivative, with respect to x, of the function $y = \sqrt{1 - \sqrt{x}}$

Product rule

You use the chain rule to differentiate composite functions. To find the derivative of a product of functions, you use the product rule.

The derivative of $y = x^2$ is $2x$
Rewrite x^2 as $x \cdot x$. The derivative of x is 1

Thus the product of the derivatives of the component functions is $1 \times 1 = 1 \neq 2x$. Thus, in general, the derivative of a product of functions is not equal to the product of the derivatives of the functions.

You can derive the product rule from first principles.

Let $f(x) = u(x)\,v(x)$, where $u(x)$ and $v(x)$ are differentiable functions.

Then

$$f'(x) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)v(x+h) - [u(x+h)v(x) - u(x+h)v(x)] - u(x)v(x)}{h}$$

Insert the expression in square brackets – this is equal to 0

$$= \lim_{h\to \infty}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]$$

Factorize, and then rearrange.

$$= \lim_{h\to 0} u(x+h).\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + \lim_{h\to 0} v(x).\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$= u(x)v'(x) + v(x)u'(x)$$

Hence, if $f(x) = u(x)v(x)$, where $u(x)$ and $v(x)$ are differentiable functions then

$f'(x) = u(x)v'(x) + v(x)u'(x)$

→ If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
where u and v are functions of x and differentiable.

Another way of writing this is:

If $f(x) = u(x)v(x)$, where $u(x)$ and $v(x)$ are differentiable functions then $f'(x) = u(x)v'(x) + v(x)u'(x)$.

Example 23

Find $f'(x)$ if $f(x) = (2x + 3)(4 - 3x)$

Answer

Let $u(x) = 2x + 3$, then $u'(x) = 2$	*Define u and v.*
Let $v(x) = 4 - 3x$, then $v'(x) = -3$	*Find* $\frac{du}{dx}$ *and* $\frac{dv}{dx}$
$f'(x) = -3(2x + 3) + 2(4 - 3x)$ $= -1 - 12x$	*By the product rule:* $f'(x) = u(x)v'(x) + v(x)u'(x)$

Example 24

Find the equation of the tangent to the curve $y = \frac{x^2+1}{x+1}$, $x \neq -1$ at (0, 1).

Answer

$y = (x^2 + 1)(x + 1)^{-1}$	*Change quotient to a product.*
Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$	
Let $v = (x + 1)^{-1}$, then $\frac{dv}{dx}(x) = -(x + 1)^{-2}$	
$\frac{dy}{dx} = 2x(x + 1)^{-1} - (x^2 + 1)(x + 1)^{-2}$	*Use product rule:* $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$= (x + 1)^{-2}[2x(x + 1) - (x^2 + 1)]$	*Factorize.*
$\frac{2x^2+2x-x^2-1}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$	
$f'(0) = -1$	*Evaluate* $f'(x)$ *at* $x = 0$
$y = -x + 1$	*Use gradient point formula:* $y - y_1 = m(x - x_1)$

Exercise 4J

Differentiate these functions, with respect to x.

1 $y = (x - 1)(x + 3)^3$

2 $y = (2x - 3)^2(4x + 1)^3$

3 $y = \frac{x+1}{x-1}$

4 $y = x\sqrt{1-2x}$

5 $y = \frac{1}{x^4 - 3x + 1}$

6 $y = (x-1)^4(3x-2)^{\frac{2}{3}}$

7 Find the equations of the tangent and normal to the curve $f(x) = (x^2 + 1)(x^2 + 3)$ at the point (–1, 4).

Changing a quotient to a product in order to differentiate, is not always straightforward, so you need the quotient rule.

The quotient rule

If $y = \frac{u}{v}$, where u and v are both differentiable functions in x, then

$$f'(x) = \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{h}$$

Add and subtract $v(x)u(x)$

$$= \lim_{h\to 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}$$

Factorize

$$= \frac{\lim_{h\to 0} v(x) \cdot \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} - \lim_{h\to 0} u(x) \cdot \lim_{h\to 0} \frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0}\left[v(x+h)v(x)\right]}$$

Take the limits in both numerator and denominator.

$$= \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$

Hence, if $u(x)$ and $v(x)$ are differentiable functions, and

$f'(x) = \frac{u(x)}{v(x)}, v(x) \neq 0$ then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$

→ If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

where u and v are differentiable functions of x.

An alternative way of writing this is:

if $u(x)$ and $v(x)$ are differentiable functions, and

$f(x) = \frac{u(x)}{v(x)}$, $v(x) \neq 0$ then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$

Example 25

Using the quotient rule, differentiate $y = \frac{x^2+1}{x+1}$ $x \neq -1$

Answer

Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$

Let $v = (x + 1)$, then $\frac{dv}{dx} = 1$

Hence,

$$\frac{dy}{dx} = \frac{2x(x+1)-(x^2+1)}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$$

This is the same function as in Example 24 – and answer, using the quotient rule is the same.

Example 26

Differentiate $y = \frac{x^2+1}{x^2-1}$, $(x \neq \pm 1)$, with respect to x, and hence find the derivative at $x = 2$

Answer

Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$

Let $v = x^2 - 1$, then $\frac{dv}{dx} = 2x$

Find $\frac{dv}{dx}$ and $\frac{du}{dx}$

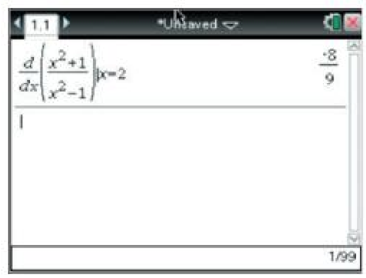
$$\frac{dy}{dx} = \frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

Use the quotient rule.

At $x = 2$, $\frac{dy}{dx} = -\frac{8}{9}$

Check your answer using the GDC.



Exercise 4K

1 Differentiate these functions, with respect to x:

a $y = \frac{x^2-7}{x^3}$

b $y = \frac{x}{\sqrt{x^2+1}}$

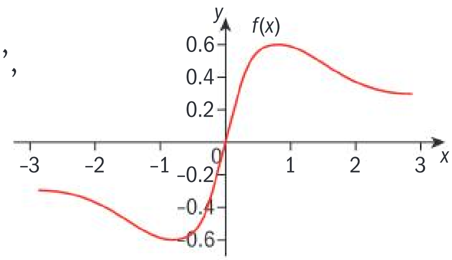
c $y = \frac{1}{x^4-3x+1}$

d $y = \frac{1+\sqrt{x}}{1-\sqrt{x}}$

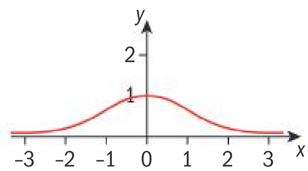
e $y = \sqrt{\sqrt{x}-x}$

f $y = \frac{1}{x^3\sqrt{1-2x+3x^2}}$

2 Find the gradient of the curve known as 'Newton's Serpentine', $y = \frac{4x}{x^2+1}$, at $x = -1$



3 Find the equation of the normal to the curve known as 'The Witch of Agnesi', $y = \frac{8}{4+x^2}$, at $x = 1$



4 Find $f'(x)$ if $f(x) = \sqrt[3]{\left(1-\frac{1}{2+x}\right)^2}$

Throughout history famous curves have often been given special names, such as the two in questions 2 and 3. What is the significance of the name given to a curve? What properties of curves, if any, do the names highlight? Find applications of these famous curves, or real-life situations that the curves model.

Higher derivatives

If f is a differentiable function, then $f'(x)$ is the derivative of $f(x)$. Similarly, if $f'(x)$ is a differentiable function, $f''(x)$ is the derivative of $f'(x)$. Since multiple dash or prime notation begins to lose its efficiency after about the third derivative, for higher derivatives we write $f^{(n)}(x)$.

Using $\frac{dy}{dx}$ notation, we write

$$f'(x) = \frac{dy}{dx} \quad f''(x) = \frac{d^2y}{dx^2} \quad f'''(x) = \frac{d^3y}{dx^3} \quad f^{(n)}(x) = \frac{d^ny}{dx^n} \quad n = 4, 5, \ldots$$

f'' is 'f double dash', or 'f double prime', of x. $f^{(n)}(x)$ is the nth derivative of f with respect to x.

Example 27

Find the first five derivatives of $f(x) = x^4 - 3x^2 + 2x - 1$

Answer

$f'(x) = 4x^3 - 6x + 2$
$f''(x) = 12x^2 - 6$
$f'''(x) = 24x$
$f^{(4)}(x) = 24$
$f^{(5)}(x) = 0$

When using a superscript for a derivative the numbers are placed in brackets, as shown.

For $n \geq 5$, $f^{(n)}(x) = 0$

Example 28

A particle moves in a straight line so that its position from a fixed point after t seconds is given by $s(t) = 3t + 5t^2 - t^3$, s in cm.

a Find the velocity of the particle at $t = 2$.

b If the acceleration is the derivative of the velocity, find the acceleration of the particle at $t = 2$.

Continued on next page

Answers	
a $s'(t) = 3 + 10t - 3t^2$	*Differentiate s(t).*
$s'(2) = 11 \text{ cm s}^{-1}$	*Evaluate v(t) = s′(t) at t = 2*
b $s''(t) = 10 - 6t$	*Differentiate s′(t).*
$s''(2) = -2 \text{ cm s}^{-2}$	*Evaluate a(t) = s″(t) at t = 2*

Exercise 4L

1 If $f(x) = 4x + 1 + \frac{1}{x}$, find $f''(x)$.

2 If $f(x) = x^4 - 2x - 1$, find $f'(0)$ and $f''(-1)$.

3 If $f(x) = x^4 - 4x^3 + 16x - 16$, find x such that $f(x) = f'(x) = f''(x) = 0$

4 $f(x) = x^4 + rx^2 + sx + t$ passes through the point $(-1, 16)$. At this point, $f''(x) = -f'(x) = 16$. Find the values of r, s, and t.

5 A particle moves in a straight line such that its position at any time t is $s(t) = (t - 4)^3(3 - 2t)^2$ metres. Find

a the velocity after 4 seconds

b the acceleration after 4 seconds

c the jerk of the particle after 1 second.

The derivative of the acceleration is called the 'jerk'.

6 Given $f(x) = \frac{1}{x}$, find f', f'', f''', $f^{(4)}$, $f^{(5)}$ and hence find an expression for $f^{(n)}(x)$. Prove your result using the method of mathematical induction.

Investigation – Leibniz's formula

$f(x) = uv$ is the product of two functions in x. You can find $f'(x)$ using the product rule,

$f'(x) = u'v + uv'$

You can find $f''(x)$, using the product rule.

$$f''(x) = (u'v)' + (uv')'$$
$$= (u''v + u'v') + (u'v' + uv'')$$
$$= u''v + 2u'v' + uv''$$

Now find the 3rd derivative and 4th derivatives. Note the similarity to the binomial formula, and conjecture a formula for $f^{(n)}(x)$. Use this formula to find the 5th derivative of $f(x)$.

The general case for $f^{(n)}(x)$ is called **Leibniz's formula**.

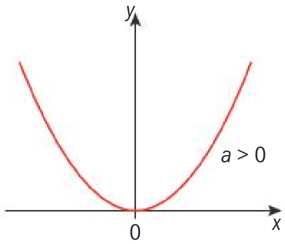
Graphical meaning of the derivative

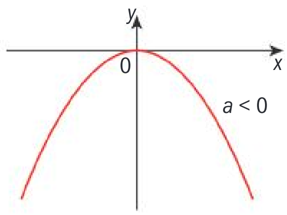
Local maximum and minimum points

Look at a quadratic function whose leading coefficient is

i positive ($a > 0$) or **ii** negative ($a < 0$).

What is the gradient of the parabola at its vertex?

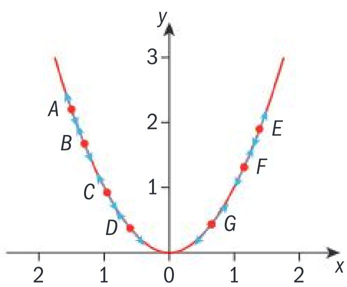




In both cases the tangent to the vertex is parallel to the x-axis. This means that the gradient of the targent to the vertex of a quadratic function is 0.

A point where the derivative of a function is zero is sometimes called a **stationary point**.

i For $a > 0$ the vertex is a **minimum point** (the curve is concave upwards). What are the signs of the gradients of the points on the left and right of this vertex?

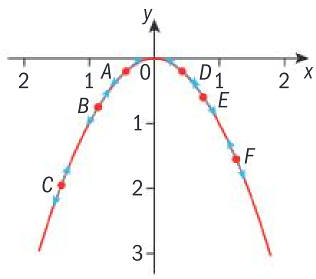


To the left of the minimum point the gradients of the tangents to points A, B, C and D are all negative. Also, the function is **decreasing** in the interval where the **gradients are negative**.

To the right, the gradients of the tangents to points E, F and G are all positive. The function is **increasing** in the interval where the **gradients are positive**.

→ Hence, the gradients of the points change from negative to positive in going through the minimum point.

ii For $a < 0$ the vertex is a **maximum point** (the curve is concave downwards).



To the left of the **maximum point** the gradients of the tangent to points A, B, and C are all positive. In the interval where the **gradients are positive**, the function is **increasing**.

To the right, the gradients of the tangent to points D, E, and F are all negative. In the interval where the **gradients are negative**, the function is **decreasing**.

→ The gradients of the points change from positive to negative in going through the maximum point.

Example 29

Find the maximum and minimum points on the curve $y = 2x^4 - 4x^2 + 1$

Answer

$\frac{dy}{dx} = 8x^3 - 8x$

$8x^3 - 8x = 0$, hence $8x(x^2 - 1) = 0$

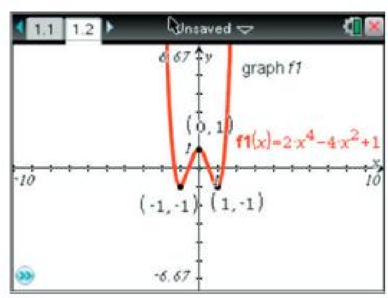
and $x = 0, \pm 1$

Set the first derivative equal to 0 and solve.

Test values to the left and right of these, using a sign diagram:

values of x	$x < -1$	$-1 < x < 0$	$1 < x < 1$	$x > 1$
sign of $\frac{dy}{dx}$	–	+	–	+

Since the gradients go from negative, through zero at $x = \pm 1$, to positive at $x = \pm 1$ the function has local minimum points. At $x = 0$ the gradient goes from positive, through zero, to negative and is therefore a maximum.



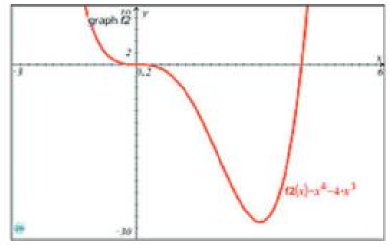
Exercise 4M

1 Find any maximum and minimum points of these functions and classify them as such.

a $y = x^2 - 3x + 1$ **b** $y = -2x^3 + 6x^2 - 3$

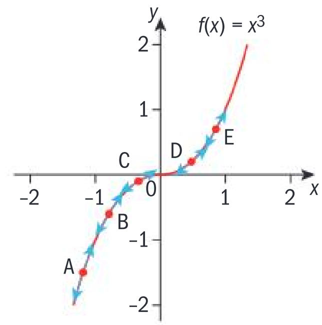
c $y = 3x^4 - 2x^3 - 3x^2 + 4$ **d** $y = x^4 - 4x^3$

Points of inflexion



In question **1d** of Exercise 4N, there was a point on the curve whose gradient was 0, but it was neither a maximum nor a minimum point. Unlike at a maximum, or minimum, the sign of the gradient did not change when going through the point. If you look at its graph, you will see that the graph changes from concave up to concave down at this point, $x = 0$.

Now look at the graph of the function $y = x^3$



No sign change in gradients going from the left to the right of the point of inflexion.

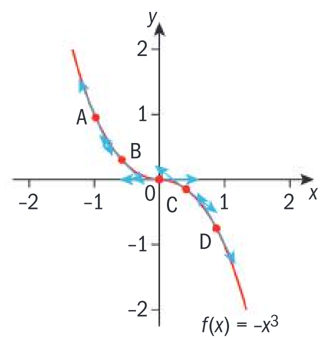
At the point (0, 0), the gradient is 0, and this is neither a maximum nor a minimum of the function. At this point the curve changes from being concave downwards to being concave upwards. The point where the concavity of a curve changes is called a **point of inflextion**. At a point of inflexion the tangent line at the point crosses the curve. A horizontal point of inflexion has a gradient of 0.

→ A point whose gradient is equal to 0 is either a maximum, minimum, or horizontal point of inflexion.

First derivative test

On both sides of the point of inflexion in the graph of $y = x^3$ the gradients are **positive**, i.e., there is **no change in sign**. Since the gradients are positive for all x, the function is **increasing** throughout its domain ($x \neq 0$).

Now look at the graph of $y = -x^3$



No sign change in gradients going from the left to the right of the point of inflexion.

On both sides of the point of inflextion the gradients are negative, i.e., there is **no change in sign**. Since the gradients are **negative** for all x, the function is **decreasing** throughout its domain ($x \neq 0$).

→ Consider the function $f(x)$ and suppose that $f'(c) = 0$.
To determine if the point $x = c$ is a maximum, minimum or horizontal point of inflexion, make a sign table and test values of $f(x)$ to the left and right of c.

- If the signs of gradients change from negative to positive, then f has a minimum at $x = c$.
- If the signs of the gradients change from positive to negative, then f has a maximum at $x = c$.
- If there is no sign change, then f has a horizontal point of inflexion at $x = c$.

→ Let $f(x)$ be continuous on $[a, b]$ and differentiable on $[a, b]$.

- If $f' > 0$ for all $x \in]a, b[$, then f increases on $[a, b]$
- If $f' < 0$ for all $x \in]a, b[$, then f decreases on $[a, b]$

Example 30

a Find and classify any maxima, minima or points of inflexion of the function $f(x) = x^3 - 3x + 1$
b State the intervals where f is increasing and where f is decreasing.
c Sketch the graph of the function.

Answers

a $f'(x) = 3x^2 - 3$
Set $f'(x) = 0$
$3x^2 - 3 = 0 \Rightarrow x = \pm 1$

Differentiate, set $f'(x) = 0$ and solve for x.

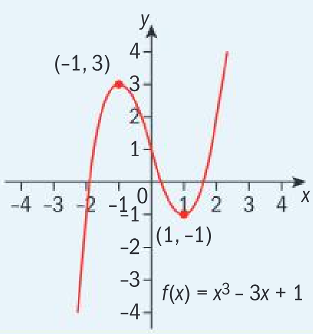
Make a sign diagram.

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f'(x)$	+	0	–	0	+
f	increasing	stationary	decreasing	stationary	increasing

$f(-1) = 3$ and $f(1) = -1$
Hence, the cubic has a maximum at $(-1, 3)$ and a minimum $(1, -1)$

b f is increasing at $]-\infty, -1[\cup]1, \infty[$
f is decreasing at $]-1, 1[$

c



Sketch the curve.
Label the maximum and minimum points.
Set f(x) = 0 to find where the curve crosses the x-axis.
Find f(0), where the curve crosses the y-axis.

The cubic has a positive leading coefficient so as $x \to -\infty, f(x) \to -\infty$ and as $x \to +\infty$, $f(x) \to +\infty$. Given the general shape of a cubic this means that the only possibilities are a maximum followed by a minimum or a single point of inflexion.

Example 31

Find and classify any maxima, minima or points of inflexion of $y = x^4 + 2x^3$, and the intervals where the function is increasing or decreasing.

Answer

a $\frac{dy}{dx} = 4x^3 + 6x^2$
$4x^3 + 6x^2 = 0 \Rightarrow 2x^2(2x + 3) = 0 \Rightarrow x = 0, x = -\frac{3}{2}$

Differentiate.

Set $\frac{dy}{dx} = 0$

x	$x < -\frac{3}{2}$	$x = -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x = 0$	$x > 0$
$\frac{dy}{dx}$	–	0	+	0	+
f	decreasing	stationary	increasing	stationary	increasing

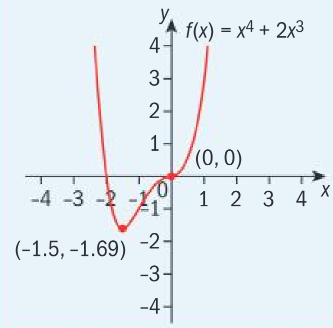
Continued on next page

Hence, $f(x)$ has a minimum at $\left(-\frac{3}{2}, -\frac{27}{16}\right)$ and a horizontal point of inflexion at (0, 0).

The function is increasing at $\left]-\frac{3}{2}, 0\right[\cup \left]0, \infty\right[$

and is decreasing at $\left]-\infty\ -\frac{3}{2}\right[$

Sketch the graph of the quartic function to confirm these results.

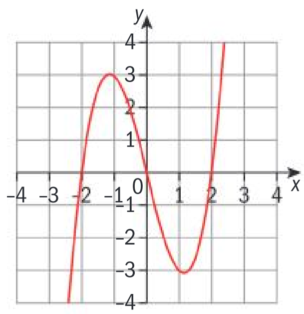


Exercise 4N

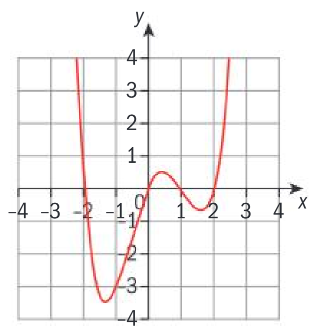
1 Use the graphs to estimate where f' is

i 0 **ii** positive **iii** negative.

a



b



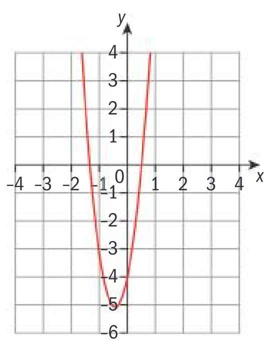
2 Use these graphs of gradient functions for a function f to determine

i where f has any maxima, minima or points of inflexion

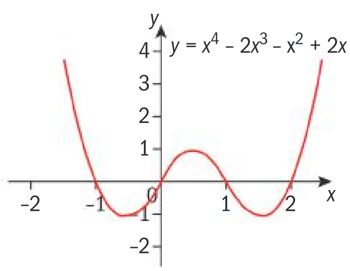
ii intervals where f is increasing

iii intervals where f is decreasing.

a



b



3 For each function, find:

i any stationary points, and justify your results

ii any intervals where f is increasing

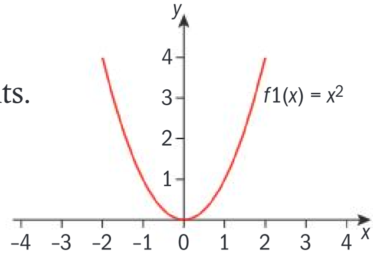
iii any intervals where f is decreasing.

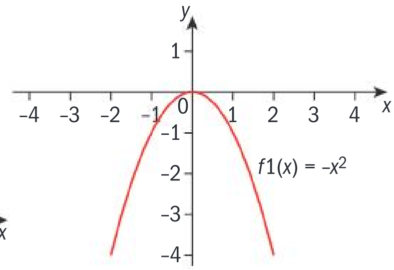
a $y = -3x^2 + 6x - 1$ **b** $y = x\sqrt{2-x^2}$ **c** $y = \dfrac{x}{x^2+1}$

d $y = x^{\frac{1}{3}}(x-2)$ **e** $y = x^2\sqrt{2-x^2}$

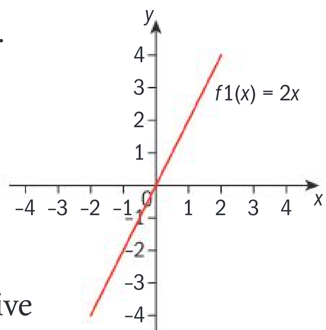
Second derivative test for maximum and minimum points

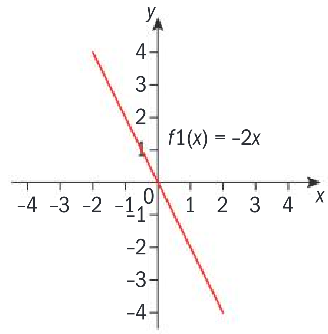
This is another test to determine the nature of maximum and minimum points. Here are the graphs of $y = x^2$ and its derivative.





The gradients go from negative to positive at the minimum point, the second derivative is positive, and $f''(x) = 2$. When the graph of f is concave up its second derivative is positive.





Here are the graphs of $y = -x^2$ and its derivative.

The gradients go from positive to negative at the maximum point, the second derivative is negative, and $f''(x) = -2$. When the graph of f is concave down, the second derivative is negative.

- If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$
- If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$

If both $f'(c) = 0$ and $f''(c) = 0$ the test is inconclusive. You shall study this later on in this section.

Example 32

Find and classify all maxima, minima and horizontal points of inflexion of the function $y = 3 + x + \frac{1}{x}$

Confirm your findings with a sketch graph of the function.

Answer

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Set $\frac{dy}{dx} = 0$ and

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

solve for x

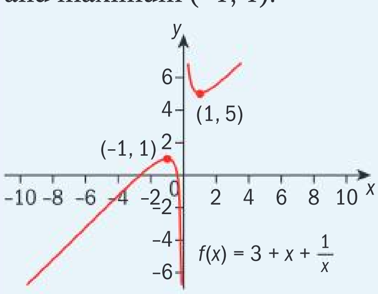
$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

Using the second derivative test.

$f''(1) = 2 > 0 \Rightarrow f$ has a minimum at $x = 1$

$f''(-1) = -1 < 0 \Rightarrow f$ has a maximum at $x = -1$

Hence, the stationary points are: minimum (1, 5) and maximum (–1, 1).



The graph of the function confirms the result.

Exercise 40

1 Find and classify any maxima, minima and horizontal points of inflexion of these functions.

a $y = 2x^3 + 3x^2 - 12x - 3$ **b** $y = -x^4 + 2x - 1$

c $y = x^5 - 5x$ **d** $y = \frac{12}{x^2 + 2x - 3}$

e $y = \frac{3x + 3}{x(3 - x)}$

Do this exercise analytically and confirm your results on a GDC.

EXAM-STYLE QUESTION

2 For each of these functions

i Find the coordinates of any maxima, minima or horizontal points of inflexion, and state their type. Justify your answers.

ii Indicate the intervals where f is increasing, and the intervals where f is decreasing.

iii Sketch the curve, showing clearly the points you have found in **i**, as well as the intercepts and any asymptotes.

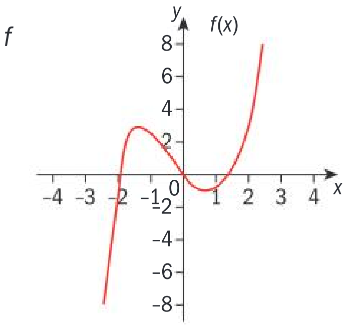
a $y = x^5 - 5x^4$ **b** $y = \frac{1 - x}{x^2 + 8}$

4.4 Exploring relationships between f, f′ and f″

Points of inflexion and concavity

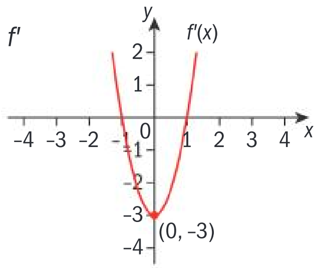
Here are the graphs of the cubic $f(x) = x^3 - 3x + 1$, its first derivative $f'(x) = 3x^2 - 3$ and second derivative $f''(x) = 6x$.

Notice that there is a point on f which corresponds to a stationary point on f' at $x = 0$. At this point, f' has a gradient of 0. This is evident from the graph of f''.



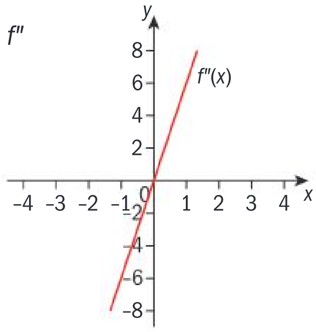
$f(x)$ has a point of inflexion at $x = 0$

This is a non-horizontal point of inflexion, since the first derivative at this point is not equal to 0. At this point f changes concavity from concave down to concave up.



Where f has a point of inflexion at $x = 0$, f' has a minimum.

See the next page for the graph of f''.



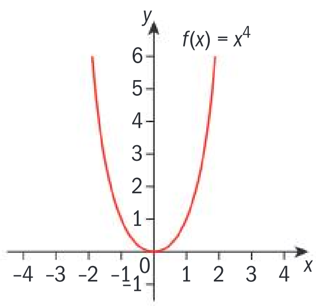
Where f has a point of inflexion at $x = 0$, f'' is negative on the left and positive on the right.

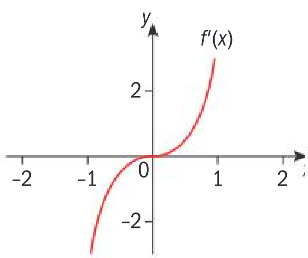
- f is concave down in an open interval if for all x in the interval, $f''(x) < 0$
- f is concave up in an open interval if for all x in the interval, $f''(x) > 0$

- If a curve has a point of inflexion its second derivative will be 0 at this point.

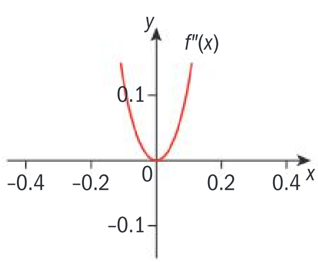
However if a curve has a point whose second derivative is 0 that point is not always a point of inflexion.

For example, consider $f(x) = x^4$





Here, $f'(x) = 4x^3$, and $f'(x) = 0$ for $x = 0$.
At $(0, 0)$ $y = x^4$ has a minimum.
$f''(x) = 12x^2$, and $f''(x) = 0$ for $x = 0$, but $(0, 0)$ is clearly not a point of inflexion.



Example 33

Consider the function $f(x) = 2x^4 - 4x^2 + 1$

a Find any maxima, minima or horizontal points of inflection.

b Find the intervals where the function is **i** decreasing **ii** increasing.

c Find the intervals where the function is **i** concave up **ii** concave down.

d Sketch the function, indicating any maxima, minima and points of inflexion.

Answers

a $f'(x) = 8x^3 - 8x$

$8x^3 - 8x = 0$, hence $8x(x^2 - 1) = 0$, and $x = 0$, or $x = \pm 1$.

$f''(x) = 24x^2 - 8$

$f''(0) = -8$, $f''(0) < 0$, hence at $x = 0$, f has a maximum.

$f''(-1) = 16$, $f''(-1) > 0$, hence at $x = -1$ f has a minimum.

$f''(1) = 16$, $f''(1) > 0$, hence at $x = 1$, f has a mimimum.

The stationary points are therefore (0, 1), (–1, –1), and (1, –1).

Set $f'(x) = 0$ to find any stationary points.

Using the second derivative test.

b Sign diagram

x	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
sign of f′	–	+	–	+
behavior of f	decreasing	increasing	decreasing	increasing

i f is decreasing in the intervals $]-\infty, 1[\cup]0, 1[$

ii f is increasing in the intervals $]-1, 0[\cup]1, \infty[$

c $24x^2 - 8 = 0$, hence $x = \pm\frac{1}{\sqrt{3}}$

To find any points of inflexion, set $f''(x) = 0$

x	$x < -\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	$x > \frac{1}{\sqrt{3}}$
sign of f″	+	–	+
concavity of f	concave up	concave down	concave up

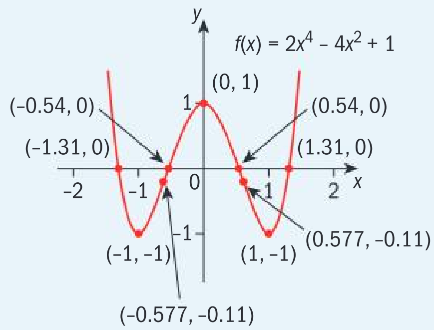
$f''(x) < 0$ means concave down.

$f''(x) > 0$ means concave up.

i f is concave up on the interval $\left]-\infty, -\frac{1}{\sqrt{3}}\right[\cup \left]\frac{1}{\sqrt{3}}, \infty\right[$

ii f is concave down on the interval $\left]-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right[$

d



Although the question does not ask for zeros in the sketch, you should indicate on the graph where they are.

$f(x) = 2x^4 - 4x^2 + 1$
$f(0) = 1$

Exercise 4P

1 For each function

i find any points of inflexion

ii determine the intervals where the function is concave up or concave down.

Justify your answers.

a $y = x^3 - x$ **b** $y = x^4 - 3x + 2$ **c** $y = \sqrt{4x - x^2}$

d $y = (x-1)^{\frac{2}{3}}$ **e** $y = \dfrac{3x^2}{x-1}$

EXAM-STYLE QUESTION

2 Here is the graph of f' for a function f.

From the graph, indicate

a the x-coordinates of any points where the gradient of $f(x)$ is zero and determine the nature of the points

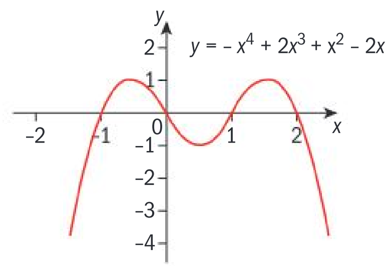
b the intervals where f is

i increasing **ii** decreasing

c the intervals where f is

i concave up **ii** concave down.

d Sketch f on a copy of the graph.



Investigation – cubic polynomials

For the cubic polynomial $y = ax^3 + bx^2 + cx + d$, define the conditions for a, b, and c such that the cubic has stationary points, and show that the cubic always has a point of inflection.

4.5 Applications of differential calculus: kinematics

Kinematics is the study of how objects move. A particle moving in a straight line is the simplest type of motion. To describe simple linear motion you need a starting point, a direction, and a distance.

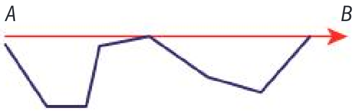
- The path is the set of points between the start and end location.
- The displacement describes the difference in the particle's position between its start and end points.
- The distance is the length of the path between two points on the path the particle has traveled.

Although Leibniz and Newton discovered calculus at about the same time, Newton was led to calculus when studying motion. In fact, he used the term 'fluxions' for derivatives. Leibniz's notation for the derivative $\frac{dy}{dx}$ is the one we use today. Newton's notation $\dot{y}$, with a dot over the dependent variable, was less popular.

For example, if you jog once around a standard track the distance you will cover is 400 m, but your displacement is 0, since your start and end positions are the same.

Displacement and distance are related. In physics terms, displacement is a vector quantity that measures the change in position between a start and an end point, and distance is a scalar. In the diagram below, the path of a particle follows the squiggly line from A to B. The length in metres of this path is the distance traveled.

The displacement is given by the vector $\overrightarrow{AB}$. Hence, the displacement is always less than or equal to the distance traveled.



Instantaneous velocity and acceleration

- Velocity at a particular instant is the derivative of the position function, $s(t)$ with respect to time.

  $v(t) = \frac{ds}{dt}$ or $s'(t)$
- Acceleration is the derivative of the velocity with respect to time, $a(t)$. It is therefore the second derivative of the position function. If a particle's velocity at time t is $v(t)$, then

  $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ or $s''(t)$
- When v and a have the same sign, the particle is speeding up (accelerating).
- When v and a have opposite signs, the particle is slowing down (decelerating).

The derivative of the acceleration is called the jerk. It is exactly what you experience when there is a sudden change in acceleration, i.e., you feel your body being moved abruptly. For example, if you are in an airplane when there is a sudden change in acceleration, your drink might spill, or items might fall off trays. It is interesting that the travel sickness that many people experience is often due to jerk, even jerk that is not noticeable. Note that the jerk as the derivative of acceleration is not explicity on the HL syllabus but you may be asked to calculate a higher derivative.

Example 34

A particle moves in a horizontal line so that its position from a fixed point after t seconds is s metres, where $s = 5t^2 - t^4$
Find the velocity and acceleration of the particle after 1 second.
Is the particle speeding up or slowing down at $t = 1$?

Answer

$v = \frac{ds}{dt} = 10t - 4t^3$	*Differentiate s to find velocity.*
At $t = 1$, $v = 6\ ms^{-1}$	*Evaluate at t = 1*
$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 10 - 12t^2$	*Differentiate v to find acceleration.*
At $t = 1$, $a = -2\ ms^{-1}$	*Evaluate at t = 1*
At $t = 1$ the particle is slowing down.	*The signs of v and a are different which means the particle is slowing down.*

Example 35

A particle moves along a line so that its position at any time t, in seconds, is

$s(t) = t^3 - 7t^2 + 11t - 2.5$

a Find the velocity and acceleration at any time t.

b Find the times when the particle is at rest, and the times when it is speeding up and slowing down. Justify your answers.

c Find the values of t when the particle changes direction.

d Find the total distance traveled in the first 3 seconds.

Answers

a $v(t) = s'(t) = 3t^2 - 14t + 11$

$a(t) = v'(t) = 6t - 14$

b $3t^2 - 14t + 11 = 0$, $t = 1$ or $t = \frac{11}{3}$

When the particle is at rest v(t) = 0

$6t - 14 = 0$, $t = \frac{7}{3}$

When the particle has constant velocity, a(t) = 0

t	$0 < t < 1$	$1 < t < \frac{11}{3}$	$t > \frac{11}{3}$
sign of v	+	–	+

When $a = 0$, $t = 2\frac{1}{3}$

t	$t < \frac{7}{3}$	$t > \frac{7}{3}$
sign of a	–	+

v and a have different signs for $0 < t < 1$ and $\frac{7}{3} < t < \frac{11}{3}$

In these time intervals the particle is slowing down. This makes sense since these are the times just before the particle comes to rest.

v and a have the same signs for $1 < t < \frac{7}{3}$ and $t > \frac{11}{3}$

At these time intervals the particle is speeding up.

From the sign diagram for $v(= s')$, s is increasing on the intervals $0 < t < 1$ and $t > \frac{11}{3}$, hence the particle is moving to the right. During the interval $1 < t < \frac{11}{3}$, s is decreasing, hence the particle is moving to the left.

c $t = 1$ and $t = \frac{11}{3}$

d The total distance traveled in the first 3 seconds is

$[s(1) - s(0)] + [s(3) - s(1)]$.

$s(1) - s(0) = 2.5 - (-2.5) = 5$

In the first second the particle moves 5 m to the right.

$s(3) - s(1) = -5.5 - 2.5 = -8$

Between the first and third seconds, the particle moves 8 m to the left.

Hence the particle has covered a total distance of 13 m.

Since the particle changes direction at t = 1, find the distance traveled from t = 0 to t = 1, and from t = 1 to t = 3, separately.

Exercise 4Q

1 At any time t, in seconds, a diver's position after diving off a board can be modeled by the function $s(t) = -5t^2 + 5t + 10$, where s is the height, in metres, above the water. Find

- **a** the height of the diving board
- **b** how long it takes the diver to hit the water
- **c** the velocity and acceleration of the diver at impact. Interpret your answers.

2 A detonation in the earth propels a rock straight up. Its height at any time t can be modeled by the function $s(t) = 50t - 15t^2$ where s is in metres. Find

- **a** the maximum height of the rock
- **b** the velocity and speed when the rock is 20 m above the ground, and interpret your answers
- **c** the acceleration of the rock at any time t
- **d** the time taken for the rock to hit the ground again.

3 A particle moves in a straight line such that its displacement t seconds later is s metres, where $s(t) = 7t + 5t^2 - 2t^3$

- **a** Find the initial velocity and acceleration, and interpret your answer.
- **b** Find the velocity and acceleration after 2 seconds, and interpret your answer.

4 A particle moves in a straight line such that its displacement from a fixed point after t seconds is s metres, where $s(t) = 10t^2 - t^3$

- **a** Find the average velocity in the first 3 seconds.
- **b** Find the velocity at $t = 3$ and acceleration at $t = 3$.
- **c** Determine if the particle is speeding up or slowing down at $t = 3$.
- **d** Find the value of t when the direction of the particle changes.

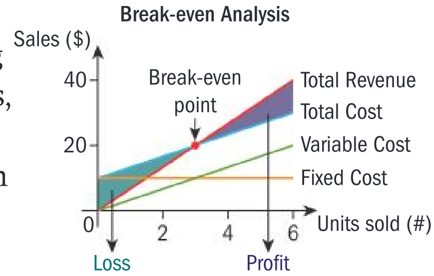
EXAM-STYLE QUESTION

5 The position of a particle at any time t seconds after it starts moving is given by the function $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t$ metres. Find

- **a** the velocity and acceleration at any time t
- **b** the times when the particle is
  i at rest **ii** is speeding up **iii** is slowing down
- **c** the acceleration when the particle's velocity is 0, and interpret your answer
- **d** the times when the particle changes direction
- **e** the total distance traveled in the first 5 seconds.

4.6 Applications of differential calculus: economics

Calculus is applied in basic economic theory in marginal analysis. Economists analyze how small changes, for example, increasing the production of a product by a single unit, affect profits and costs. Marginal analysis quantifies the benefits of performing such an action against the costs. When the benefits, or profits, exceed the cost of the action, you can proceed on this course until this balance changes. The break-even point occurs when the production costs and the total revenues, the amount of income generated before any deductions are made, are the same.



Source: www.12manage.com

There are three basic terms in marginal analysis.

- Marginal profit is the rate of change of profit with respect to the number of units produced or sold.
- Marginal revenue is the rate of change of revenue with respect to the number of units sold.
- Marginal cost is the rate of change of cost with respect to the number of units sold.

Here is a summary of the basic terms and corresponding formulae.

x = number of units produced (or sold)

$r(x)$ = total revenue from selling x amount of units

$c(x)$ = total cost of producing x amount of units

$p(x)$ = profit in selling x amount of units

$r'(x)$ = marginal revenue, the extra revenue for selling one extra unit

$c'(x)$ = marginal cost, the extra cost for selling one extra unit

$p'(x)$ = marginal profit, the extra profit for selling one additional unit

The variables are connected by the formula $p(x) = r(x) - c(x)$

Example 36

The profit, in euros, obtained from selling x pairs of shoes can be modeled by $p(x) = 0.000\,25x^3 + 10x$

a Find the marginal profit for a production of 50 pairs of shoes.

b Find the actual gain in profit obtained by raising the production level from 50 to 51 pairs of shoes.

c Comment on your answers to **a** and **b**.

Answers

a $p'(x) = 0.000\,75\,x^2 + 10$

$p'(50) = 0.000\,75\,(50)^2 + 10 = 11.88$ euros

b $p(50) = 0.000\,25\,(50)^3 + 10(50) = 531.25$ euros

$p(51) = 0.000\,25\,(51)^3 + 10(51) = 543.16$ euros

The additional profit is therefore 543.16 – 531.25 = 11.91 euros

c The marginal profit is the extra profit for selling one additional pair of shoes. This approximates very well the actual profit, according to the profit formula, for selling an extra pair of shoes.

In the real world, however, profit does not always work according to the profit model. It is more likely that a company will be able to maintain sales only by lowering prices at some stage. So other models, such as a demand function, need to be introduced in order to make a more accurate marginal analysis.

What are the variables affecting maximum profit, and when does maximum profit occur?

Profit is the difference between revenues and costs, i.e., $p(x) = r(x) - c(x)$. Differentiating, $p'(x) = r'(x) - c'(x)$. Profits will be maximized when $p'(x) = 0$, i.e., $0 = r'(x) - c'(x) \Rightarrow r'(x) = c'(x)$

Maximum profit occurs when the marginal revenue and marginal cost are equal.

Of course, minimum profit can also occur when $p'(x) = 0$. Profit will at any rate occur when marginal revenues and marginal costs are equal.

Example 37

The cost of manufacturing fishing poles, in thousands of units, is modeled by $c(x) = x^3 - 10x^2 + 20x$. The revenue is modeled by $r(x) = 7x + 3$
Find a production level that maximizes profits.

Answer	
$c'(x) = 3x^2 - 20x + 20$ $r'(x) = 7$ $c'(x) = r'(x)$ $\Rightarrow 3x^2 - 20x + 20 = 7$ $\Rightarrow 3x^2 - 20x + 13 = 0$ $\Rightarrow x_1 \approx 0.730,\ x_2 \approx 5.94$	*Differentiate both c(x) and r(x), and set them equal.*
Therefore, maximum profit occurs at a production level of 5 940 fishing poles. Minimum profit occurs at a production level of approximately 730 fishing poles.	*Since p(x) = r(x) – c(x) is a cubic function, if it has a maximum it will also have a minimum.*

The ideal production level is the one that minimizes costs.
If the cost of producing x units is $c(x)$, then the average cost per unit is $\frac{c(x)}{x}$. If the cost of producing x units can be minimized, then it will occur when $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c(x)}{x}\right) = 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c(x)}{x}\right) = \frac{xc'(x) - c(x)}{x^2} = 0$$

$\Rightarrow xc'(x) = c(x)$

$\Rightarrow c'(x) = \dfrac{c(x)}{x}$

Use quotient rule.

If the production cost of x units can be minimized, it will occur when the marginal cost of producing one extra unit is the same as the average cost of producing x units.

Example 38

The cost of manufacturing fishing poles, in thousands of units, is modeled by $c(x) = x^3 - 10x^2 + 20x$. Find a production level, if it exists, that minimizes average cost.

Answer

$c'(x) = 3x^2 - 20x + 20$

$\dfrac{c(x)}{x} = x^2 - 10x + 20$

Hence, $3x^2 - 20x + 20 = x^2 - 10x + 20$

$\Rightarrow 2x^2 - 10x = 0 \Rightarrow 2x(x - 5) = 0$

$\Rightarrow x = 0, x = 5$

For minimum cost, $c'(x) = \dfrac{c(x)}{x}$

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{c(x)}{x}\right) = 2x - 10$

$\dfrac{\mathrm{d}}{\mathrm{d}x}(2x - 10) = 2$

Use the second derivative test.

The only production level that would possibly minimize average cost is $x = 5$.

Since x is in thousands, 5000 fishing poles is the production level necessary to minimize costs.

Since 2 > 0, the second derivative is positive for all x, so x = 5 is a minimum.

In Examples 37 and 38 the maximum profit is for a production level of almost 6000 fishing poles, but to minimize average cost, the production level is 5000 poles.

Hence, these results would have to be analyzed and a decision taken as to which production level the company should aim for.

Ultimately, every company's goal is to maximize profits, which means maximizing revenues. Obviously, a company with minimal production costs, but no revenues, will not make a profit.

Exercise 4R

1 A company manufactures oil tanks for reservoirs. The total weekly cost in euros of producing the tanks can be modeled by $c(x) = 20\,000 + 180x - 0.1x^2$

- **a** Find the marginal cost function.
- **b** Find the marginal cost of producing 100 tanks per week.
- **c** Find the cost of producing 101 tanks, and compare your answer with **b**.

2 A company does market research before producing a new type of memory stick. Initial overheads and fixed costs of production, in euros, for x memory sticks can be modeled by $c(x) = 500 + 3x$. The projected selling price is modeled by $p(x) = 7 - 0.002x$

a Find

i the domain of the price function

ii the marginal cost function, and interpret its meaning

iii the revenue function, and its domain.

b Graph the revenue and cost functions, and find the break-even points, and interpret what they mean.

State the memory stick production levels in the form $a < x < b$, (a and b are integers), that must be met in order for the company to make a profit.

3 A company specializes in making units from rare metals for nuclear plants. The total cost, in dollars, is modeled by $c(x) = 500\,x^2 + 1000$, where x represents hundreds of units. Find the number of units the company should make in order to minimize costs.

4 The cost, in euros, for producing x number of jackets is $c(x) = 400 + 20x - 0.2x^2 + 0.0004\,x^3$

a Find the number of jackets that should be produced to minimize costs.

b Find the number of jackets that should be produced to maximize profits if the revenue function can be modeled by $r(x) = 35x - 3$

c Interpret your answers to **a** and **b**.

4.7 Optimization and modeling

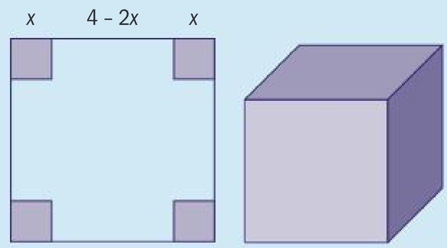
The primary purpose of applied mathematics is to describe, investigate, explain and solve real-world problems. This process is called mathematical modeling. The steps are:

- Identify all variables, parameters, limits and constraints.
- Translate the real-world problem into a mathematical system.
- Solve the mathematical system.
- Interpret the reasonableness of the solution in light of the real-world problem.

Optimization problems deal with finding the most effective solutions to real-world problems, for example, how to minimize the surface area of a container for a required volume.

Example 39

An open box is made from cutting congruent squares from the corners of a 4 m by 4 m cardboard sheet. How large should the squares be so that the box has a maximum volume? What is the maximum capacity of the box?



Answer

Let x be a side of the square in cm

$L = W = 4 - 2x$

$H = x$

Identify any variables.

Express the dimensions of the box in terms of the side of the square, x.

$0 < x < 2$

Identify constraints.

$V = x(4 - 2x)^2$

Write the function for volume.

$V' = (4 - 2x)^2 + 2x(4 - 2x)(-2)$

Differentiate the function.

$= (4 - 2x)^2 - 4x(4 - 2x)$

$V' = 12x^2 - 32x + 16$

Set $V' = 0$, $12x^2 - 32x + 16 = 0$

$4(3x^2 - 8x + 4) = 0$

$4(3x - 2)(x - 2) = 0$

$x = \frac{2}{3}$ or $x = 2$

Since $x < 2$, reject $x = 2$

Solve for x.

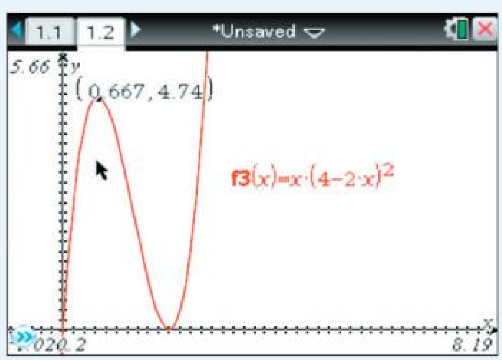
$V'' = 24x - 32$

When $x = \frac{2}{3}$, $V'' = -16 < 0$ hence at $x = \frac{2}{3}$ m, V has a maximum.

Use the second derivative test to check $x = \frac{2}{3}$ gives a maximum.

Hence, $V = \frac{2}{3}(4 - \frac{4}{3})^2 = 4.74\ \text{m}^3$

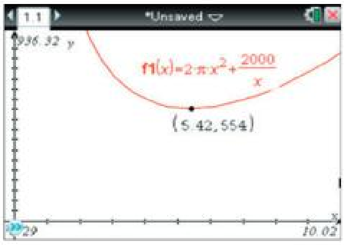
Check your answer on a GDC.



Example 40

You have been asked to design a cylindrical can to hold 1 litre of car oil, with the minimum surface area in order to minimize costs. Find the dimensions of the can.

Answer	
$V = 1$ litre $= 1000\,\text{cm}^3$	*Identify constraints.*
r = radius of base	*Identify variables.*
h = height of can	
$V = \pi r^2 h$	
Surface area $A = 2(\pi r^2) + 2\pi rh$	*Identify function to be optimized.*
$1000 = \pi r^2 h$, hence $h = \frac{1000}{\pi r^2}$	*Use the constraint to reduce the function to be maximized to one unknown.*
$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$	
$= 2\pi r^2 + \frac{2000}{r}$	
$A' = 4\pi r - \frac{2000}{r^2}$	*Differentiate.*
$4\pi r - \frac{2000}{r^2} = 0$	
$4\pi r^3 - 2000 = 0$	
$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$, $h = 10.8$ cm	*Substitute r and h into A.*
$A = 554\ \text{cm}^2$	*Check your answer on the GDC.*



Exercise 4S

1 Find the dimensions of the rectangle with largest area, if its base is on the x-axis, and its upper corners are on the parabola $10 - x^2$.

2 A rectangular plot of land is bounded on one side by a river, and by a fence on the other three sides. Find the largest area that can be enclosed using 800 m of fencing.

3 A stained glass window is to be designed and entered in an annual competition in the UK. The window must be in the shape of a semicircle over a rectangle, such that the diameter is on a side of the rectangle. If the perimeter of the window is to be 4 m, find the dimensions that will result in the rectangular part having the largest possible area.

4 You wish to make an open rectangular box from a 24 cm by 45 cm piece of cardboard, by cutting out congruent squares from its corners and folding up the sides. Find the dimensions of the box of largest volume you can make this way, and find the volume.

5 Find the volume of the largest right circular cone that can be inscribed in a sphere whose radius is 10 cm.

6 A rectangular sheet of tin whose dimensions are l cm by w cm and whose perimeter is 36 cm will be rolled to create a cylinder.

a Find the values of the length and width that will give the greatest volume.

b The same sheet of tin will be revolved about one of its sides to create a cylindrical figure. Find the values of l and w that will give the greatest volume.

4.8 Differentiation of implicit functions

The functions you have studied so far have been explicitly defined, i.e., the dependent variable is defined in terms of the independent variable. The equation of the circle however, whose center is at the origin is $x^2 + y^2 = r^2$, where r is the radius of the circle. In this case, both x and y are implicitly defined.

$y = x^2 + 2$ is an explicitly defined function. $x^3y + xy^3 = 3$ is an implicitly defined function.

You can often change an implicitly defined function into an explicit function by expressing one variable in terms of the other. However, this is not always easy, or possible, and so to analyze such functions you need to differentiate them implicitly.

For example, differentiate $y^2 = 4x$ implicitly with respect to x.

Differentiate the left-hand side with respect to x using the chain rule:

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$$

Differentiate the right-hand side: $\frac{d(4x)}{dx} = 4$

Hence: $2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$

Therefore: $\frac{dy}{dx} = \frac{2}{y}$

In order to check the result, you can (if possible) write the function explicitly, and differentiate.

In this case $y^2 = 4x \Rightarrow y = \pm\sqrt{4x} = \pm 2\sqrt{x}$

Taking $y = 2\sqrt{x}$

$$\frac{dy}{dx} = \frac{d(2x^{\frac{1}{2}})}{dx} = 2\frac{d(x^{\frac{1}{2}})}{dx} = 2.\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} = \frac{2}{2\sqrt{x}} = \frac{2}{y}$$

Likewise, taking $y = -2\sqrt{x}$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{x}} = \frac{2}{-2\sqrt{x}} = \frac{2}{y}$$

Thus you obtain the same result by either differentiating implicitly or explicitly.

Example 41

Find the gradient of the circle $x^2 + y^2 = 1$ at the point (0, 1).

Answer

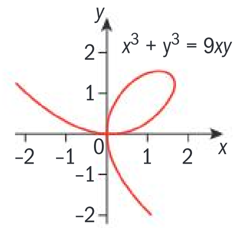
$2x + 2y\frac{dy}{dx} = 0$ — *Differentiate implicitly with respect to x.*

$\frac{dy}{dx} = -\frac{x}{y}$ — *Rearrange.*

Substitute (0, 1) for x and y

$\frac{dy}{dx} = 0$ — *Again, you can check by first writing the equation explicitly and then differentiating.*

As in these examples, it is sometimes possible to check the result of implicit differentiation by defining the function explicitly, and then differentiating. For example, consider the Folium of Descartes whose equation is $x^3 + y^3 = 9xy$, and is graphed here. This is obviously not a function, but consists of three distinct piecewise functions joining to form the graph.



▲ The Folium of Descartes

Example 42

Differentiate the folium of Descartes, $x^3 + y^3 - 9xy = 0$

Answer

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(y + x\frac{dy}{dx}\right) = 0$$

Use the product rule to differentiate xy.

$$3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$

Expand.

$$\frac{dy}{dx}(3y^2 - 9x) = 9y - 3x^2$$

Factorize $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

Solve for $\frac{dy}{dx}$

Example 43

The point $P(2, m)$, where $m < 0$, lies on the curve $2x^2y + 3y^2 = 16$

a Calculate the value of m.

b Find the gradient of the normal to the tangent at P.

Answer

a $2(2)^2(m) + 3m^2 = 16$

$8m + 3m^2 = 16$

Substitute (2, m) into the implicit function for (x, y).

$3m^2 + 8m - 16 = 0$

$(3m - 4)(m + 4) = 0$

Solve the quadratic for m.

$3m - 4 = 0$ or $m + 4 = 0$

$m = \frac{4}{3}$ or $m = -4$

Since $m < 0$, $m = -4$

Reject positive value of m.

b $2\left(2xy + x^2\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$

Differentiate implicitly.

$4xy + 2x^2\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-4xy}{2x^2 + 6y}$

At $(2, -4)$, $\frac{dy}{dx} = \frac{1}{2}$

Substitute your solution for x and y.

Hence, the gradient of the normal at P is -2.

Exercise 4T

1 Find $\frac{dy}{dx}$ by differentiating implicitly with respect to x.

a $3y^2 + x^2 = 4$ **b** $y^4 = x^3 + 1$

c $x^2 + y^2 - 3x + 4y = 2$ **d** $2x^2 - 3x^2 y^2 + y^2 = 9$

e $(x + y)^2 = 5 - 2x$ **f** $x^2 = \frac{x - y}{x + y}$

2 Find the equation of the tangent to the curve $x^2 - y^2 = 9$ at the point $(5, 4)$.

3 Find the equation of the normal to the curve $y^2 = 3x + 1$ at the point $(1, -2)$.

4 Find the equations of the tangent and the normal to the curve $x^2 - \sqrt{3}xy + 2y^2 = 5$ at the point $(\sqrt{3}, 2)$.

5 Find the coordinates of the where the gradient is zero points on the curve $x^2 + y^2 - 6x - 8y = 0$

6 Given the curve $3x^2 + 2xy + y^2 = 3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1, -2)$.

7 Given the curve $x^2 + xy + y^2 = 3$, find the x-intercepts and show that the tangents to the curve at the x-intercepts are parallel.

8 A rectangular tank with square base x m and height y m is designed so that the top of the tank is at ground level. The purpose of the tank is to store excess water that runs off from the ground, which has a low porous index. The proposed volume of the tank is 125 m^3. The costs for such a design is modeled by the function $C(x) = 3(x^2 + 2xy) + 8xy$ in local currency. Find the dimensions of the tank that will minimize the costs.

EXAM-STYLE QUESTION

9 Given the curve $x + y = x^2 - 2xy + y^2$

a find $\frac{dy}{dx}$

b show that $1 - \frac{dy}{dx} = \frac{2}{2x - 2y + 1}$

c show that $\frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3$

4.9 Related rates

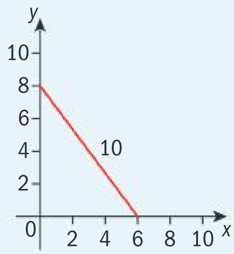
Related rates problems look at the effect that a change in a particular rate has on another rate. For example, when a balloon is filled with air at a certain rate, at what rate is its surface area increasing? Suppose the balloon then begins to lose air at a certain rate, at what rate is its surface area decreasing?

The next example shows the necessary steps to solving related rates problems.

Example 44

A 10 m long industrial ladder is leaning against a wall on a building site. It starts to slip down the wall at a rate of 0.5 ms^{-1}.
How fast is the foot of the ladder moving along the ground when it is 6 m from the wall?

Answer



Sketch a diagram of the problem, naming the variables.

When $x = 6$ and $\frac{dy}{dt} = -0.5$, find $\frac{dx}{dt}$

Write down the given information, and what you are asked to find.

$x^2 + y^2 = 100$

Write down an equation relating the variables.

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

Differentiate with respect to time.

When $x = 6$, $y = 8$

$2(6)\frac{dx}{dt} + 2(8)(-0.5) = 0$

$\frac{dx}{dt} = \frac{2}{3}$

Find any missing values necessary to solve for the required rate.
Substitute and solve.

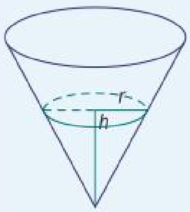
The ladder is moving along the ground at a rate of approximately 0.667 $m\,s^{-1}$

Interpret the answer in the context of the given problem using appropriate units.

Example 45

Water is poured into a conical tank at the rate of 3 m^3 min^{-1}. The tank stands with the point downward. How fast is the water level in the tank rising when the depth is 2 m and the radius of the water surface is 1.5 m?

Answer



Sketch a diagram.
Define the variables.

At time t, r is the radius of the water surface.
h is the depth of water in the tank.

Write down the given information and what you have to find.

Find $\frac{dh}{dt}$ when $\frac{dV}{dt} = 3$, $h = 2$, $r = 1.5$

$V = \frac{1}{3}\pi r^2 h$

Write down a formula connecting the variables.

$\frac{dV}{dt} = \frac{1}{3}\pi(2hr\frac{dr}{dt} + r^2\frac{dh}{dt})$

Differentiate with respect to t.

$\frac{r}{h} = \frac{1.5}{2} \Rightarrow 2r = 1.5h \Rightarrow r = 0.75h$

Similar triangles.

$\frac{dr}{dt} = 0.75\frac{dh}{dt}$ so $r\frac{dr}{dt} = 1.5 \cdot 0.75\frac{dh}{dt}$

Differentiate to get $\frac{dr}{dt}$ *in terms of* $\frac{dh}{dt}$

Alternatively,
$\frac{dr}{dh} = \frac{dr}{dh} \times \frac{dh}{dt} = 0.75\frac{dh}{dt}$

$3 = \frac{1}{3}\pi[(2)(2)(1.125)(\frac{dh}{dt}) + (2.25)\frac{dh}{dt}]$

Substitute values in $\frac{dV}{dt}$ *and solve.*

$\frac{dh}{dt} = 0.424$

When the depth is 2m, the water level is rising at a rate of. 0.424m min^{-1} (3 sf).

Interpret the answer in the context of the problem.

Exercise 4u

1 The area of circle and its radius are related by the formula $A = \pi r^2$. Write an equation relating the rate of change of the area to the rate of change of its radius.

2 The formula for the surface area of closed cylinder is $A = 2\pi r^2 + 2\pi rh$. Write an expression relating the rate of change of the area to the rates of change of both the radius and the height of the cylinder.

3 If l, w, and h are respectively the length, width and height of a rectangular box, express the rate of change of the diagonal of the box in terms of the rates of change of dimensions of the box.

4 The length of a rectangle is increasing at a rate of $2\,\text{cm}\,\text{s}^{-1}$ while its width is decreasing at a rate of $2\,\text{cm}\,\text{s}^{-1}$. When the length and width of the rectangle are 12 cm and 5 cm respectively, find the rate of change of

a the area **b** the perimeter **c** the diagonal.

5 A cube is increasing in volume at a rate of $1.5\,\text{m}^3\,\text{s}^{-1}$. Find the rate at which the surface area of the cube is changing when the cube has a volume of $81\,\text{m}^3$.

6 A ladder 5 m long is leaning against the side of a building. Its base begins to slide away from the wall, and when it is 3 m from the wall, its slides at a rate of $0.5\,\text{m}\,\text{s}^{-1}$. At this point find

a how quickly the top of the ladder is sliding down the wall

b the rate of change of the area between the ladder, the wall and the ground.

7 A spark from a fire burns a hole in a paper napkin. The hole initially has a radius of 1 cm and its area is increasing at a rate of $2\,\text{cm}^2\,\text{s}^{-1}$. Find the rate of change of the radius when the radius is 5 cm.

8 An airplane is flying at an altitude of 8 miles and passes over a radar station. When the airplane is 12 miles from the base of the station, the radar detects that its horizontal distance is changing at a rate of 320 mph. Find how fast the airplane is flying at this point in time.

9 Kim is flying her kite at a height of approximately 10 m. The wind is blowing horizontally at the kite at a rate of $1\,\text{m}\,\text{s}^{-1}$. How quickly must Kim let out the string when the kite is 20 m away from her?

EXAM-STYLE QUESTION

10 Two concentric circles are expanding in size. At time t the radius of the outer circle is 9 m and it is expanding at the rate of $1.2\,\text{m}\,\text{s}^{-1}$. The radius of the inner circle is 1 m and it is expanding at the rate of $1.5\,\text{m}\,\text{s}^{-1}$. Find the rate of change of the area of the ring between the circles, at time t.

11 Consider a ramp modeled by the function $y = \frac{1}{x}$, $x > 0$. A ball slides down the ramp so that the x-coordinate of its position at any time t seconds is increasing by a rate of $f(x)$ units per second. If its y-coordinate is decreasing at a constant rate of 1 unit per second, find $f(x)$.

12 A conical tank with vertex pointed downward has a radius of 10 m at its top and is 24 m high. Water flows out of the tank at a rate of $20\text{m}^3/\text{min}$. How fast is the depth of the water in the tank decreasing when it reaches a depth of 16m?

Review exercise

EXAM-STYLE QUESTION

1 Find the limits, if they exist, of the following.

a $\lim_{x\to 1}\frac{x^2-3}{x+1}$ **b** $\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{x}$ **c** $\lim_{x\to 2}\frac{3^x-1}{x}$

d $\lim_{x\to 0}\frac{3x^2+x^2}{x^2}$ **e** $\lim_{x\to\infty}\frac{5x^2}{2x^3+1}$ **f** $\lim_{x\to-\infty}\frac{7}{x^3+1}$

2 Determine if $y=\begin{cases}x^2+2x, & x\le 2\\ x^3-6x, & x>2\end{cases}$ is continuous at $x = 2$.

3 Determine if the sequence $a_n=\frac{2n^2-3}{n^3-2}$ converges as n tends to $+\infty$.

4 Determine if the series $\sum_{n=0}^{\infty}3\left(\frac{(-1)^n}{5^n}\right)$ converges, and if it does, find its sum.

EXAM-STYLE QUESTIONS

5 Find the values of a for which the series $a^2+\frac{a^2}{1+a^2}+\frac{a^2}{(1+a^2)^2}+\ldots$ is convergent, and find its sum.

6 Given $y=\frac{x^3-2x^2+5}{x^2-x^3}$, find

a its horizontal asymptote

b the points where the curve intersects its horizontal asymptote, for small values of x.

7 Find the equation of the tangent and normal to the curve $y=\frac{2x+1}{x^2+1}$ at $x = 0$

EXAM-STYLE QUESTION

8 Let f be an even function with domain $(-a, a)$, $a > 0$. f is differentiable throughout its domain. Show that the tangent to the graph of f at $x=0$ is parallel to the x-axis.

9 Find any points on the curve $y = x\sqrt{x+1}$ those tangents are parallel to the line $x + y = -3$

10 The normal to the curve $y = \frac{1}{2}(2x^4 - 5x^3 - 5x^2 + 3x)$ at the point where $x = 1$ meets the curve again at point P. Find the coordinates of P.

EXAM-STYLE QUESTION

11 If f is a function such that $f(x) = [g(x)]^3$, $g(0) = -\frac{1}{2}$, $g'(0) = \frac{8}{3}$, find the equation of the tangent to $f(x)$ at $x = 0$.

12 Differentiate y with respect to x.

a $y = (1 - 3x)^7(3x + 5)^3$ **b** $y = \sqrt{(4x^2 - 3x + 1)^5}$

c $y = \frac{x^2 - 3}{\sqrt{x+1}}, x \neq -1$ **d** $y = \sqrt{x + \sqrt{x^2 + 1}}$

e $(x + 2 + (x - 3)^8)^3$

13 Consider the polynomial function $f(x) = ax^3 + 6x^2 - bx$. Determine the values of a and b if f has a minimum at $x = -1$, and a point of inflexion at $x = 1$.

14 Consider the function $y = x - \sqrt[3]{x}$

a Find the intercepts of the function.

b Find any stationary points and distinguish between them.

c Find any points of inflexion.

d Determine the intervals where

i the function increases **ii** the function decreases.

EXAM-STYLE QUESTION

15 Consider the function $y = \frac{2x}{x^2 - 1}$

a Find the vertical and horizontal asymptotes.

b Show that the function is an odd function.

c Show that $\frac{dy}{dx} < 0$ for all x in the domain.

d Sketch the function.

16 Consider the function $f(x) = \frac{(x-3)^2}{x^2 - 3}$

a Find any zeros, intercepts, and asymptotes of f.

b Find any stationary points, and justify your answers.

c Find any points of inflexion.

d Find the intervals where f is

i increasing, **ii** decreasing.

e Sketch the function showing all features found.

17 Given $x = y^5 - y$, find $\frac{dy}{dx}$, if it exists, at the points where $x = 0$

Review exercise

EXAM-STYLE QUESTIONS

1 Find the shortest distance between the point $(1.5, 0)$ and the curve $y = \sqrt{x}$

2 A piece of wire 80 cm in length is cut into three parts: two equal circles and a square. Find the radius of the circles if the sum of the three areas is to be minimized.

3 The radius of a right circular cylinder is increasing at a rate of 3 cm min^{-1} and the height is decreasing at a rate of 4 cm min^{-1}. Find the rate at which the volume is changing when the radius is 9 cm and the height is 12 cm, and determine if the volume is increasing or decreasing.

4 A poster has a total area of 180 cm^2 with a 1 cm margin at the bottom and sides, and a 2 cm margin at the top. Find the dimensions that will give the largest printing area.

5 A particle travels along the x-axis. Its velocity at any point x is $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+2x}$. Find the particle's acceleration at $x = 2$ in terms of x.

CHAPTER 4 SUMMARY

Continuous function

- A function $y = f(x)$ is **continuous** at $x = c$, if $\lim_{x \to c} f(x) = f(c)$.
  The three necessary conditions for f to be continuous at $x = c$ are:

  1 f is defined at c, i.e., c is an element of the domain of f.

  2 the limit of f at c exists.

  3 the limit of f at c is equal to the value of the function at c.

  A function that is not continuous at a point $x = c$ is said to be **discontinuous** at $x = c$.

Properties of limits

- **Properties of limits as $x \to \pm\infty$**
  Let L_1, L_2, and k be real numbers and $\lim_{x \to \pm\infty} f(x) = L_1$ and $\lim_{x \to \pm\infty} g(x) = L_2$. Then,

  1 $\lim_{x \to \pm\infty}(f(x) \pm g(x)) = \lim_{x \to \pm\infty} f(x) \pm \lim_{x \to \pm\infty} g(x) = L_1 \pm L_2$

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2 $\lim_{x\to\infty}(f(x)\cdot g(x)) = \lim_{x\to\infty} f(x)\cdot \lim_{x\to\infty} g(x) = L_1 \cdot L_2$

3 $\lim_{x\to\pm\infty}(f(x)\div g(x)) = \lim_{x\to\pm\infty} f(x)\div \lim_{x\to\pm\infty} g(x) = L_1 \div L_2$,
provided $L_2 \neq 0$.

4 $\lim_{x\to\pm\infty} kf(x) = k\lim_{x\to\pm\infty} f(x) = kL_1$

5 $\lim_{x\to\pm\infty}[f(x)]^{\frac{a}{b}} = L^{\frac{a}{b}}{}_1$, $\frac{a}{b} \in \mathbb{Q}$ (in simplest form),
provided $L^{\frac{a}{b}}{}_1$ is real.

Convergence of series

- For a geometric series, $\sum_{n=0}^{\infty} u_1 r^n = \lim_{n\to\infty}\frac{u_1(1-r^n)}{1-r}$

  When $-1 < r < 1$, $\lim_{n\to\infty} r^n = 0$ and the series converges to $S = \frac{u_1}{1-r}$

The derivative of a function

- The **derivative**, or **gradient function**, of a function f with respect to x is the function $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided this limit exists.
- If f' exists, then f has a derivative at x, or is **differentiable** at x. ($f'(x)$ is read f dash, or f prime, of x.) Another notation for the derivative is $\frac{dy}{dx}$, the derivative of the function $y = f(x)$ with respect to x.
- A function is differentiable if the derivative exists for all x in the domain of f.

Basic Differentiation rules

- If $f(x) = c$, and $c \in \mathbb{R}$, then $f'(x) = 0$
- If n is a positive integer, and $f(x) = x^n$, then $f'(x) = nx^{n-1}$
- For $c \in \mathbb{R}$, $(cf)'(x) = cf'(x)$ provided $f'(x)$ exists.
- If $f(x) = u(x) \pm v(x)$, then $f'(x) = u'(x) \pm v'(x)$

The chain rule

- If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x)$ is differentiable at x. Furthermore, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Another definition for the chain rule is $(f \circ g)'(x) = f'(g(x))\cdot g'(x)$

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The product rule

- If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
  where u and v are functions of x and differentiable.

  Another way of writing this is:

  If $f(x) = u(x)v(x)$, where $u(x)$ and $v(x)$ are differentiable functions then $f'(x) = u(x)v'(x) + v(x)u'(x)$.

The quotient rule

- If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
  where u and v are differentiable functions of x.

  An alternative way of writing this is:
  if $u(x)$ and $v(x)$ are differentiable functions, and
  $f(x) = \frac{u(x)}{v(x)}$, $v(x) \neq 0$ then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$

Higher derivatives

- $f'(x) = \frac{dy}{dx}, f''(x) = \frac{d^2y}{dx^2}, f'''(x) = \frac{d^3y}{dx^3}, f^n(x) = \frac{d^ny}{dx^n} \quad n = 4, 5, \ldots$

Maximum, minimum and horizontal points of inflexion

- A point whose gradient is equal to 0 is either a maximum, minimum, or horizontal point of inflexion.

First derivative test

- Consider the function $f(x)$ and suppose that $f'(c) = 0$.
  To determine if the point $x = c$ is a maximum, minimum or horizontal point of inflextion, make a sign table and test values of $f(x)$ to the left and right of c.
  - If the signs of gradients change from negative to positive, then f has a minimum at $x = c$.
  - If the signs of the gradients change from positive to negative, then f has a maximum at $x = c$.
  - If there is no sign change, then f has a horizontal point of inflection at $x = c$.

Second derivative test

- If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$
- If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$

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Kinematics

- Velocity at a particular instant is the derivative of the position function, $s(t)$ with respect to time.
  $v(t) = \frac{ds}{dt}$ or $s'(t)$
- Acceleration is the derivative of the velocity with respect to time, $a(t)$. It is therefore the second derivative of the position function. If a particle's velocity at time t is $v(t)$, then
  $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ or $s''(t)$
- When v and a have the same sign, the particle is speeding up (accelerating).
- When v and a have opposite signs, the particle is slowing down (decelerating).

Economics

- Marginal profit is the rate of change of profit with respect to the number of units produced or sold.
- Marginal revenue is the rate of change of revenue with respect to the number of units sold.
- Marginal cost is the rate of change of cost with respect to the number of units sold.

If x is the number of units produced a sold:

- $r(x)$ = total revenue from selling x amount of units
- $c(x)$ = total cost of producing x amount of units
- $p(x)$ = profit in selling x amount of units
- $r'(x)$ = marginal revenue, the extra revenue for selling one extra unit
- $c'(x)$ = marginal cost, the extra cost for selling one extra unit
- $p'(x)$ = marginal profit, the extra profit for selling one additional unit

Differentiation of implicit functions

To differentiate $y^2 = 4x$ multiply with respect of x:

LHS: $\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$

RHS: $\frac{d}{dx}(4x) = 4$

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$