7 The evolution of calculus

CHAPTER OBJECTIVES:

6.4 Indefinite integration as anti-differentiation

6.5 Anti-differentiation with a boundary condition to determine the constant of integration; definite integrals; area of the region enclosed by a curve and the *x*-axis or *y*-axis in a given interval; areas of regions enclosed by curves; volumes of revolution about the *x*-axis or *y*-axis

6.6 Kinematic problems involving displacement, velocity and acceleration; total distance travelled

Before you start

You should know how to:

1 Find the derivatives of linear, polynomial, rational, exponential and logarithmic functions. e. g. Differentiate

$y = e^{2x+3}\ln(1-x)^2$

$\frac{dy}{dx} = 2e^{2x+3}\ln(1-x)^2 - e^{2x+3}\frac{2(1-x)}{(1-x)^2}$

Using the product and chain rules

$\frac{dy}{dx} = \frac{2e^{2x+3}((x-1)\cdot\ln((x-1)^2)+1)}{x-1}$

2 Find points of intersection between the graphs of two functions. e. g. find the point where the graphs of the functions $y = e^{-\frac{x}{2}}$ and $y = e^{x+1}$ intersect.

At this point $e^{-\frac{x}{2}} = e^{x+1}$ therefore $-\frac{x}{2} = x + 1$

$x = -\frac{2}{3}, y = e^{\frac{1}{3}}$

so point of intersection is $(-\frac{2}{3}, e^{\frac{1}{3}})$

3 Find the velocity and acceleration given the displacement. e. g. For a displacement function $s(t)$, velocity is the first derivative and acceleration is the second derivative.

Skills check

1 Find the derivatives of these functions.

a $y = x\ln(x)$

b $y = \frac{e^{2x-3}}{\sqrt{2-x}}$

c $y = x^4 - \frac{1}{x^4}$

Use the differentiation rules from Chapter 4.

2 Find the point(s) of intersection of the graphs of these functions.

a $y = 3x - 2$ and $y = x^2 - 2x + 4$

b $y = 1 - x$ and $y = \sqrt{2x+1}$

c $y = \frac{6}{x} + 3x$ and $y = x^3 - 5x$

3 A particle moves along a line so that its displacement at any time t is $s(t) = 3t^4 - t^3 + t$. Find expressions for the velocity and acceleration of the particle at any time t.

▲ Valencia Aquarium

Integral calculus

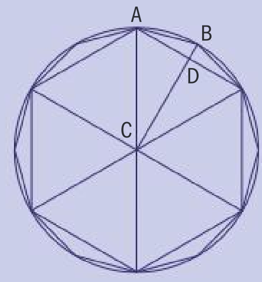
How to calculate surface areas and volumes of regular shapes such as rectangles and cylinders has been common knowledge for thousands of years; but how do architects and engineers calculate areas and volumes of curved spaces, such as the aquarium in Valencia, Spain?

About 2000 years ago, **Archimedes** was one of the first mathematicians to attempt to find the area between a parabola and a chord. His method was to fill the area with shapes whose areas were known, for example triangles. He did this until the space not covered was so small as to be negligible, or in the words of Newton and Leibniz, infinitesimally small. Modern-day mathematicians call this the method of 'exhaustion'.

This chapter looks at integral calculas. In section 7.3 you will see how integration is related to areas under curves.

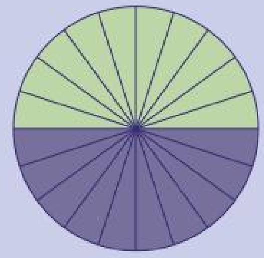
Although we study the derivative first, some of the concepts of integration were known long before differentiation. These ideas were important in the beginnings of fair trade, which depended in part on knowing how to work out areas of regular and irregular shapes.

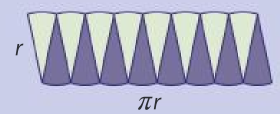
Method of exhaustion



Take a circle and start filling it with isosceles triangles from its center. The sides of the triangles are radii. The altitude, CD = *h*, is shorter than the radius, CB. If we create *n* such triangles, then the sum of the areas of the triangles approximates the area of the circle, $A \approx \sum_{i=1}^{n} \frac{1}{2} b_i h_i$

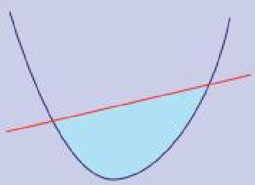
As we increase the number of triangles, the altitudes of the triangles get closer to the length of the radius, and the sum of the bases of the triangles approaches the actual circumference of the circle, so we can write $A \approx \sum_{i=1}^{n} \frac{1}{2} r(b_i) \approx \frac{1}{2} r \cdot (2\pi r) \approx \pi r^2$. We can see that as we increase the number of triangles, the sum of their areas gets closer to the actual area of the circle, until $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} r(b_i) = \pi r^2$





This is an example of the method of exhaustion.

Archimedes figured out the area between a parabola and a chord. How did he do it?



Choose a convenient shape whose area you know how to find, and fill the space between the chord and the parabola with these shapes, to 'exhaustion'!

Perhaps this prompted Leibniz to say, "He who understands Archimedes and Apollonius will admire less the achievements of the foremost men of later times."

Archimedes did not use coordinate axes – this system was invented by **Rene Descartes** in the 17th century.

7.1 Integration as anti-differentiation

The process of finding a function $f(x)$ whose derivative is $f'(x)$ is called anti-differentiation, which relates to **integration**.

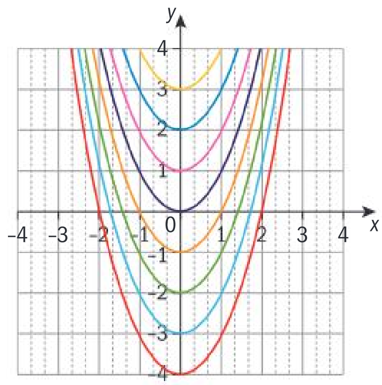
For example, you know that the derivative of x^2 with respect to x is $2x$, hence when you anti-differentiate $2x$ with respect to x you obtain x^2. This, however, is not the only answer, since, for example, $y = x^2 + 3$ also has a derivative of $2x$.

You can easily see that $2x$ is the derivative for any function of the form $y = x^2 + c$, where c is any real number. This set, or family, of all anti-derivatives of a function is called the **indefinite integral** of the function, and c is called the constant of integration.

This can be written using symbols as $\int 2x\,dx = x^2 + c,\ c \in \mathbb{R}$

Mathematical models provide solutions to real-world problems. Analyze mathematical models used to approximate areas and volumes of irregular shapes. Discuss how welll these models approximate the actual areas and volumes of the shapes found through calculus methods.

The integration symbol is an elongated S, and was first used by Leibniz.



▲ Graph of family of curves of $y = x^2 + c$ for different values of c

Notice the lines in the background of the graphs. They form a slope field for the family of curves $y = x^2 + c$, i.e., they show the direction of the tangent lines at the different values of x.

The tangent lines are parallel for corresponding values of x.

Slope fields is a topic in the calculus option.

In general terms

$$\rightarrow \int f(x)\,dx = F(x) + c,\ c \in \mathbb{R}$$

$f(x)$ is called the **integrand**, and x is the variable of integration.

Differentiating x^n:

$x^n \rightarrow$ multiply the coefficient of x by n $\rightarrow$ decrease the power of n by 1 $\rightarrow nx^{n-1}$

Reversing the process:

$nx^{n-1} \rightarrow$ increase the exponent by 1 $\rightarrow$ divide by the new exponent $\rightarrow$ add a constant of integration, c $\rightarrow x^n + c$

In general

$$\rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \neq -1$$

Recall from chapter 4 the constant multiple rule for differentiation,

for c a real number, $f'(cx) = cf'(x)$ provided $f'(x)$ exists.

The reverse is also true, i.e., $\int cf(x)\,dx = c\int f(x)\,dx$.

Example 1

a Find the indefinite integral of $-4x^3$ **b** Find $\int -\frac{5}{x^7}\,dx$

Answers

a $\int -4x^3\,dx = -4\int x^3\,dx$

$= -4\frac{x^4}{4} + c$

$= -x^4 + c$

Differentiate your answer mentally to check your result, differentiating $-x^4$ gives $-4x^3$. Don't forget the constant of integration.

▶ Continued on next page

b $\int -\frac{5}{x^7}\,dx = -5\int x^{-7}\,dx$ $= -5\,\frac{x^{-6}}{-6} + c$ $= \frac{5}{6x^6} + c$	*Again, differentiate your answer to check your result.*

Example 2

Integrate $f(x) = \sqrt[3]{x^2}$

Answer $\int \sqrt[3]{x^2}\,dx = \int x^{\frac{2}{3}}\,dx = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c = \frac{3}{5}x^{\frac{5}{3}} + c$	*Change the radical to a rational exponent and use the power rule.*

Exercise 7A

Find these indefinite integrals, with respect to x.

1 $-2x$ **2** $3x^8$

3 $-5x^4$ **4** $\frac{1}{x^5}$

5 $\sqrt{x^3}$ **6** $\frac{1}{\sqrt{x^3}}$

7 $\frac{2x}{\sqrt{x}}$ **8** $-\frac{\sqrt[4]{x^5}}{7x^3}$

There is another rule that is useful in integrating functions. It is the reverse of the sum and difference differentiation rule.

→ $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example 3

Integrate $1 - \sqrt[4]{x}$ with respect to x.

Answer $1 - \sqrt[4]{x} = 1 - x^{\frac{1}{4}}$ $\int \left(1 - x^{\frac{1}{4}}\right) dx = x - \frac{x^{\frac{5}{4}}}{\frac{5}{4}} + c$ $= x - \frac{4}{5}x^{\frac{5}{4}} + c$	*Change radicals to exponents.* *Integrate term by term.* *Note that* $\int 1\,dx = \int 1 \times x^0\,dx = \frac{x^1}{1} = x + c$

From the family of curves, you can find a specific curve that passes through a given point.

Example 4

If $\frac{dy}{dx} = \left(1 - \frac{1}{x^2}\right)^2$

find y given that the graph of the function passes through the point $(1, 0)$.

Answer

$\left(1-\frac{1}{x^2}\right)^2 = 1-\frac{2}{x^2}+\frac{1}{x^4}$ $= 1-2x^{-2}+x^{-4}$	*Expand.*
$\frac{dy}{dx} = 1 - 2x^{-2} + x^{-4} + c$ $\Rightarrow y = x + \frac{2}{x} - \frac{1}{3x^3} + c$	*Use properties of indices, and integrate term by term. Don't forget the constant of integration.*
At $(1, 0)$, $0 = 1 + 2 - \frac{1}{3} + c$ hence $c = -2\frac{2}{3}$	*Substitute (1, 0) into the equation for y, and find c.*
$\therefore y = x + \frac{2}{x} - \frac{1}{3x^3} - 2\frac{2}{3}$	*Rewrite y with the value of c.*

Exercise 7B

1 Integrate these with respect to x.

a $5x^2 - \frac{1}{5x^2}$ **b** $(x+3)(2x-1)$ **c** $\frac{x^2-1}{x^4}$

d $\left(x+\frac{1}{x}\right)^2$ **e** $\frac{(x+3)(x-4)}{x^5}$ **f** $\sqrt{x} - \frac{5}{\sqrt[3]{x}}$

2 If $\frac{dy}{dx} = (3x^2 - 4)$, find y given that the function passes through the point $(2, -1)$.

3 If $f'(t) = t + 3 - \frac{1}{t^2}$, find f given that the curve goes through the point $(1, -\frac{1}{2})$.

4 If $\frac{dy}{dx} = (2x+3)^3$, find y if $y = 2$ when $x = -1$.

5 Find A in terms of x if $\frac{dA}{dx} = (2x+1)(x^2-1)$, and $A = 0$ when $x = 1$.

6 Find s in terms of t if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$, and $s = 1.5$ when $t = 1$.

7 Find y in terms of x given that $\frac{d^2y}{dx^2} = 6x - 1$, and when $x = 2$, $\frac{dy}{dx} = 4$ and $y = 0$.

EXAM-STYLE QUESTION

8 A particle moves in a straight line such that at time t seconds, its acceleration $a(t) = 6t + 1$. When $t = 0$, the velocity is $2\ \text{m s}^{-1}$, and its displacement from the origin is 1 m. Find expressions for the velocity and the displacement.

In question 4 of exercise 7B, you found the integral of $(2x + 3)^3$ by first expanding and then integrating each term. It would be more efficient to find a method of integration without needing to expand the expression, especially if the power is large.

Investigation – integrating $(ax + b)^n$

Integrate different expressions of the form $(ax + b)^n$, where a, b and n are real numbers, and $a \neq 0$, $n \neq -1$. Predict the integral of all expressions of this form. Prove your conjecture by differentiating your result.

Use the form from the investigation to integrate question 4 of exercise 7B, and then apply your prediction from the investigation.

> Although, strictly speaking $\frac{du}{dx}$ is not a fraction, it conveniently behaves as one, See chapter 4, the chain rule.

In order to integrate $(2x + 3)^3$, let $u = 2x + 3$, and hence $\frac{du}{dx} = 2$, so $dx = \frac{du}{2}$. You can therefore write

$$\int (2x+3)^3\,dx = \int u^3 \frac{du}{2} = \frac{1}{2}\int u^3\,du$$

> The variable in the integrand must be the same as the variable of integration, i.e. here you have u^3 and du.

Integrating u^3 with respect to u,

$$\frac{1}{2}\int u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4} + c = \frac{u^4}{8} + c = \frac{(2x+3)^4}{8} + c$$

Substitute the original expression for u.

The result obtained from the investigation is called the compound formula.

→ $$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ a \neq 0$$

> The compound formula can be used for linear functions only.

Example 5

Integrate $\sqrt{1-2x}$ with respect to x.

Answer

Solution 1:

Let $u = 1 - 2x$, then $\frac{du}{dx} = -2$, and $dx = \frac{du}{-2}$. Hence — *By substitution*

$$\int \sqrt{1-2x}\,dx = -\frac{1}{2}\int u^{\frac{1}{2}}\,du = -\frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{1}{3}u^{\frac{3}{2}} + c$$

$$\int \sqrt{1-2x}\,dx = -\frac{1}{3}(1-2x)^{\frac{3}{2}} + c$$

Solution 2:

$$\int (1-2x)^{\frac{1}{2}}\,dx = \frac{1}{-2\left(\frac{3}{2}\right)}(1-2x)^{\frac{3}{2}} + c$$

$$= -\frac{1}{3}(1-2x)^{\frac{3}{2}} + c$$

Use the compound formula.

> There is a more advanced integration by substitution method in chapter 9.

Using the compound formula is quicker and easier than using the method of substitution.

Example 6

Find $\int \frac{3}{\sqrt[3]{4-5x}}\,dx$.

Answer

$$\int \frac{3}{\sqrt[3]{4-5x}}\,dx = 3\int (4-5x)^{-\frac{1}{3}}\,dx$$

Apply compound formula.

$$= \frac{3}{-5\cdot\frac{2}{3}}(4-5x)^{\frac{2}{3}} + c$$

$$= -\frac{9}{10}(4-5x)^{\frac{2}{3}} + c$$

Exercise 7C

Integrate these with respect to x.

1 $(3x-1)^7$ **2** $-2\sqrt{2x+1}$ **3** $\frac{1}{(4x-1)^5}$

4 $\frac{2}{\sqrt[4]{3-x}}$ **5** $\frac{2}{(2-5x)^{\frac{1}{3}}} + \sqrt[3]{1-x}$ **6** $4\sqrt{2-3x} - 6(3x+2)^{\frac{2}{3}}$

Integration of exponential functions

In chapter 5 you learned how to differentiate exponential functions. In particular, $\frac{d}{dx}(e^x) = e^x$.

$y = e^x$ is the only function whose gradient function is equal to the function itself for all x in the domain.

Therefore

→ $\int e^x\,dx = e^x + c$

Furthermore, it is easy to confirm that

→ $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

Use substitution:

let $u = ax + b$, then $\frac{du}{dx} = a$, or $dx = \frac{du}{a}$

Hence $\int e^{ax+b}\,dx = \int e^u \frac{du}{a} = \frac{1}{a}\int e^u\,du = \frac{1}{a}e^u + c = \frac{1}{a}e^{ax+b} + c$

Example 7

Find $\int 4e^{-2x}\,dx$

Answer

$$\int 4e^{-2x}\,dx = 4\int e^{-2x}\,dx = \frac{4}{-2}e^{-2x} + c = -2e^{-2x} + c$$

Example 8

Integrate $\sqrt{e^{5-3x}}$ with respect to x.

Answer

$$\sqrt{e^{5-3x}} = (e^{5-3x})^{\frac{1}{2}} = e^{\frac{5}{2}-\frac{3}{2}x}$$

Write $\sqrt{e^{5-3x}}$ using exponents.

$$\int e^{\frac{5}{2}-\frac{3}{2}x}\,dx = -\frac{2}{3}e^{\frac{5}{2}-\frac{3}{2}x} + c$$

Recall also that $\frac{d}{dx}(2^x) = 2^x\ln(2)$

Hence, $\int 2^x\ln(2)\,dx = \ln(2)\int 2^x\,dx = 2^x + c$

If you now want to integrate 2^x, you need to divide by $\ln(2)$, since $\ln(2)$ is not part of the integral. That is, $\int 2^x\,dx = \frac{1}{\ln(2)}2^x + c$

If you now differentiate the result, you obtain 2^x.

Using the compound formula, you can also integrate 2^{3x-1} with respect to x. In particular

→ $\int m^{ax+b}\,dx = \frac{1}{a\ln(m)}m^{ax+b} + c$, where m is a positive real number, $a \neq 0$.

Example 9

Find $\int 2^{3x-1}\,dx$.

Answer

$$\int 2^{3x-1}\,dx = \frac{1}{3\ln(2)}2^{3x-1} + c$$

Exercise 7D

In questions 1 to 6, integrate with respect to x.

1 $-5e^{-2x}$ **2** $\frac{1}{e^{3x+2}}$ **3** $\sqrt[3]{e^x} - \frac{2}{e\sqrt{e^{2x}}}$

4 3^x **5** $\frac{1}{3^{2x}}$ **6** 4^{1-x}

7 Use the method of substitution to derive the compound rule for exponential functions, to show that for a real positive number m, for $a \neq 0$ $\int m^{ax+b}\,dx = \frac{1}{a\ln(m)}m^{ax+b} + c$

Integration and logarithmic functions

In chapter 5 you differentiated logarithmic functions.

For $x > 0$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$ so for $x > 0$, $\int \frac{1}{x}\,dx = \ln x + c$

For $x < 0$, $\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$ so for $x < 0$, $\int \frac{1}{x}\,dx = \ln(-x) + c$

The two above statements can be combined into

→ $\int \frac{1}{x}\,dx = \ln|x| + c$

Similarly, using the compound formula,

→ $\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|ax+b| + c,\ a \neq 0$

You can confirm this result by differentiation.

Example 10

Find $\int \frac{3}{1-2x}\,dx$

Answer

$$\int \frac{3}{1-2x}\,dx = 3\int \frac{1}{1-2x}\,dx = -\frac{3}{2}\ln|1-2x| + c$$

Exercise 7E

Integrate with respect to x, $x \neq 0$.

1 $\frac{1}{3x}$ **2** $-\frac{6}{x}$ **3** $\frac{1}{2-3x}$

4 $\frac{5}{3-5x}$ **5** $-2(4 + 3x)^{-1}$

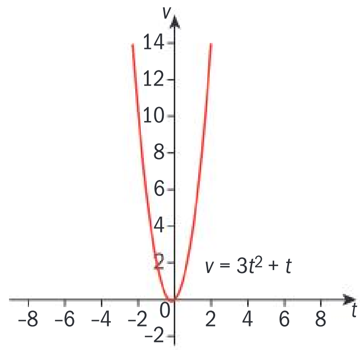
7.2 Definite integration

As you have seen in the previous section, the result of indefinite integration is a family of functions. The process of **definite integration**, however, results in a numerical answer.

All applications of the definite integral used later in this chapter require the numerical evaluation of an integral.

In chapter 4 you worked on kinematic problems. Since velocity is the rate of change of the displacement with respect to time, to obtain the velocity you differentiate the displacement function. Hence, to obtain the displacement from the velocity function, you reverse the process, and anti-differentiate, or integrate the velocity function.

Consider an example. The velocity of a particle at any time t, in seconds, is given by $3t^2 + t\ \text{m s}^{-1}$. Find the total distance traveled from $t = 1$ s to $t = 2$ s.



In order to find the total distance traveled, see if the particle changed direction anywhere in the interval [1, 2]. The graph of the function $f(t) = 3t^2 + t$ shows that the velocity is positive throughout this interval, so the particle did not change direction.

You can evaluate the displacement at $t = 1$ and $t = 2$, and the distance traveled will be the difference of these two values.

See Chapter 4, Example 36.

Integrate the velocity function to get the displacement function:

$$\frac{ds}{dt} = 3t^2 + t \Rightarrow s = t^3 + \frac{t^2}{2} + c$$

Evaluate the displacement at $t = 1$ and $t = 2$:

When $t = 1$, $s = 1.5 + c$, and when $t = 2$, $s = 10 + c$

$10 + c - (1.5 + c)$
$= 8.5$

Subtracting these two values for s gives 8.5 m as the total distance traveled between $t = 1$ and $t = 2$.
The constant of integration cancels out when subtracting.

You will study more applications of definite integration later in the chapter.

There is a special notation for evaluating a definite integral in this manner.

upper limit

$$\int_1^2 (3t^2 + t)\, dt = \left[t^3 + \frac{t^2}{2}\right]_1^2 = \left(8 + \frac{2^2}{2}\right) - \left(1 + \frac{1}{2}\right) = 8.5$$

lower limit

evaluate at upper limit

evaluate at lower limit

Write the integral in square brackets, with upper and lower limits as shown. Since c always cancels out, you don't need to write it.

If a function f is continuous on an interval $[a, b]$, then its definite integral exists over this interval. Here are some properties of definite integrals.

The proofs of some of these properties are beyond the scope of this course.

Properties of definite integrals

If the integral of f with respect to x in the interval $[a, b]$ exists, then

1 $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2 $\displaystyle\int_a^a f(x)\,dx = 0$

3 $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4 $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5 $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

You can test these properties using the particle example. For example, testing the first property,

$$\int_2^1 (3t^2 + t)\,dt = \left[t^3 + \frac{t^2}{2}\right]_2^1 = \left(1 + \frac{1}{2}\right) - \left(2^3 + 2\right) = -8.5$$

Use the particle example to test properties 2 to 5 of definite integrals.

Example 11

Evaluate $\displaystyle\int_0^1 (x^2 + 4x + 2)\,dx$

Answer

$$\int_0^1 (x^2 + 4x + 2)\,dx = \left[\frac{x^3}{3} + 2x^2 + 2x\right]_0^1$$

Use property 4.

$$= \left(\frac{1}{3} + 2 + 2\right) - 0$$

$$= 4\frac{1}{3}$$

Example 12

Evaluate $\displaystyle\int_{-1}^0 \frac{3}{1-2x}\,dx$

Answer

$$\int_{-1}^0 \frac{3}{1-2x}\,dx = -\frac{3}{2}\left[\ln|1-2x|\right]_{-1}^0$$

Take out $-\frac{3}{2}$ as a factor.

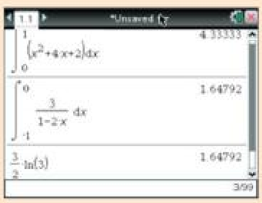
$$= -\frac{3}{2}[\ln 1 - \ln 3]$$

Use property 3.

$$= \frac{3}{2}\ln(3)$$

ln(1) = 0

You can confirm the results of Examples 11 and 12 on a GDC:



Exercise 7F

Evaluate these definite integrals. Check your results using a GDC.

1 $\int_1^3 \left(3x + \frac{1}{x^2}\right)dx,\ x \neq 0$

2 $\int_0^2 3\sqrt{4x+1}\,dx$

3 $\int_{-1}^2 -2e^{1-3x}\,dx$

4 $\int_1^3 3(2^{x+1})\,dx$

5 $\int_{-2}^0 2(1-3x)^5\,dx$

6 $\int_1^4 \frac{1-\sqrt{x}}{\sqrt{x}}\,dx,\ x \neq 0$

The properties of the definite integral are based on the assumption that the integral exists the specific bounds of integration.
Before integrating you need to check if f within is continuous in the given interval.

Example 13

Evaluate $\int_{-1}^0 \frac{1}{1-2x}dx$

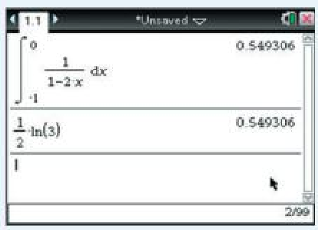
Answer

f is continuous in the interval [–1, 0].

$$\int_{-1}^0 \frac{1}{1-2x}dx = -\frac{1}{2}\left[\ln|1-2x|\right]_{-1}^0$$

$$= -\frac{1}{2}[\ln(1) - \ln(3)] = \frac{\ln 3}{2}$$

Confirming on the GDC:



If f is not continuous in the interval of integration, it is possible to obtain a numerical answer, but this answer is invalid.

Example 14

Evaluate $\int_{-e}^e \frac{1}{1-2x}dx$

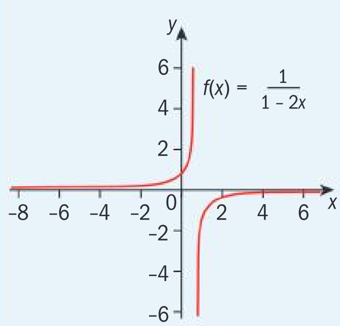
Answer

$f(x)$ has a vertical asymptote at $x = \frac{1}{2}$

f is not continuous in the interval [–e, e], since $\frac{1}{2} \in [-e, e]$.

This integral has no solution.

Graph of $\frac{1}{1-2x}$



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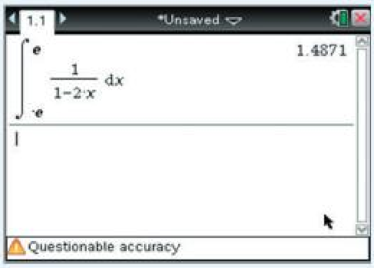
$$\int_{-e}^{e} \frac{1}{1-2x}dx = -\frac{1}{2}[\ln|1-2x|]_{-e}^{e}$$

$$= -\frac{1}{2}(\ln|1-2e| - \ln|1+2e|)$$

$$= -\frac{1}{2}\ln\frac{|1-2e|}{|1+2e|}$$

This result, however, is meaningless since the basic condition necessary is not met, namely, continuity throughout the integrating interval. The GDC integrates it numerically, so the GDC has made a mistake! It does state though that the accuracy is questionable. Some GDCs may give a 'divide by zero' error here.

Although the integral has no solution, you could still proceed and integrate and get a number.



Exercise 7G

Evaluate these integrals, if possible.

1 $\int_{-1}^{0}(2r-1)^4\,dr$

2 $\int_{0}^{4}\frac{1-\sqrt{s}}{\sqrt{s}}ds$

3 $\int_{0}^{2}\frac{x+1}{x^2-1}dx$

4 $\int_{0}^{1}\frac{dx}{(2x+1)^3}$

5 $\int_{-2}^{-1}\frac{1}{x+1}dx$

6 $\int_{0}^{1}\left(\frac{3}{3x+4}-\frac{2}{x+1}\right)dx$

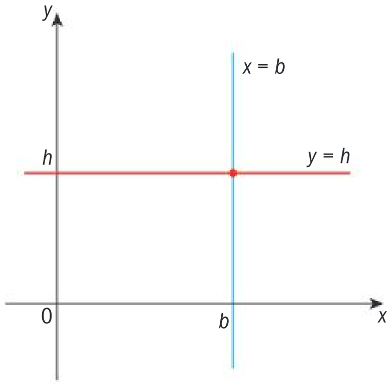
7 $\int_{-1}^{1}\frac{e^x+4}{e^x}dx$

8 $\int_{0}^{2}10^x\,dx$

7.3 Geometric significance of the definite integral

Areas between graphs of functions and the axes

Consider a rectangle in the first quadrant formed by the lines $y = h$, $x = b$, and the points (b, h) and the x- and y-axes.

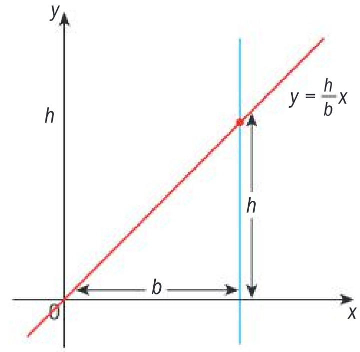


The area of the rectangle is bh. The definite integral of $y = h$ between $x = 0$ and $x = b$ is

$$\int_{0}^{b} h\,dx = h[x]_0^b = hb$$

Integration gives the area under the line $y = h$ between $x = 0$ and $x = b$

Now, consider a right-angled triangle in the first quadrant formed by the lines $y = \frac{h}{b}x$, $x = b$, and the points (b, h) and the x-axis.



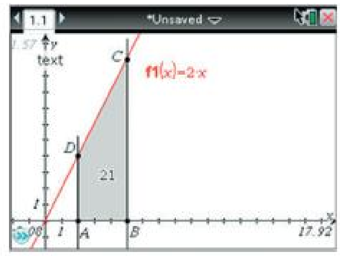
The geometric formula for the area is $\frac{1}{2}\,bh$.

The definite integral of y between $x = 0$ and $x = b$ is

$$\int_0^b \frac{h}{b}x\,dx = \frac{h}{b}\int_0^b x\,dx = \frac{h}{b}\left[\frac{x^2}{2}\right]_0^b = \frac{h}{b}\left[\frac{b^2}{2} - 0\right] = \frac{1}{2}bh$$

Integration gives the area of the triangle.

Consider △OBC formed by the line $y = 2x$, the x-axis, and the line $x = 5$.
Find the area enclosed by the lines $x = 5$ and $x = 2$.



Geometrically it is clear that the area of the shaded part is the difference between the areas of △OBC and △OAD.

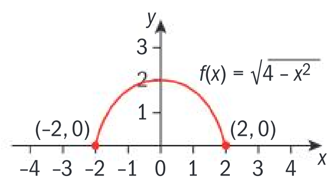
Area of △OBC is $\frac{1}{2}(5\times 10) = 25$

Area of △OAD is $\frac{1}{2}(2 \times 4) = 4$

The difference of the areas is $25 - 4 = 21$ square units.

Use integration:

$$\int_2^5 2x\,dx = \left[x^2\right]_2^5 = 5^2 - 2^2 = 21$$

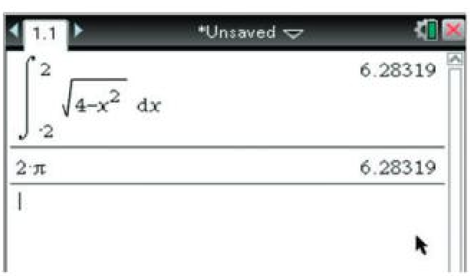


Consider the area under the curve of the graph of $y = \sqrt{4 - x^2}$
You may recognize this as the equation of a semicircle whose center is the origin, and whose radius is 2.

Using the formula for the area of a semicircle,

$$A = \frac{\pi r^2}{2}, \text{ then } A = \frac{4\pi}{2} = 2\pi$$

Now, compare this to the result of $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$, using the GDC.



The examples show the relationship between the definite integral and areas of familiar shapes.

In Chapter 9 you will learn how to integrate integrals of this kind analytically.

You are now ready to formalize one of the most astonishing and important results of Newton's and Leibniz's work: the connection between differentiation and integration. The theorem justifies the procedures for evaluating definite integrals, and is still regarded as one of the most significant developments of modern-day mathematics.

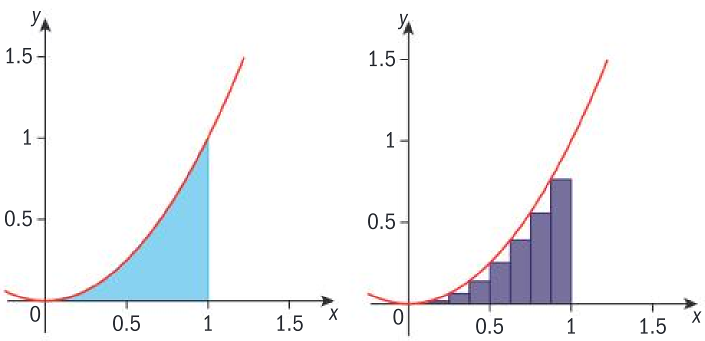
> → The fundamental theorem of calculus
>
> If f is continuous in $[a, b]$ and if F is any anti-derivative of f on $[a, b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

▲ **Augustin-Lewis Cauchy** (1789–1857) formalized the fundamental theorem of calculus.

Both Newton and Leibniz approached the calculus intuitively. The fundamental theorem of calculus, however, was actually formalized and proved by Augustin-Louis Cauchy (1789–1857). His proof elegantly joined the two branches of differential and integral calculus. Cauchy's last words before he died were indeed self-prophetic, "Men pass away, but their deeds abide".

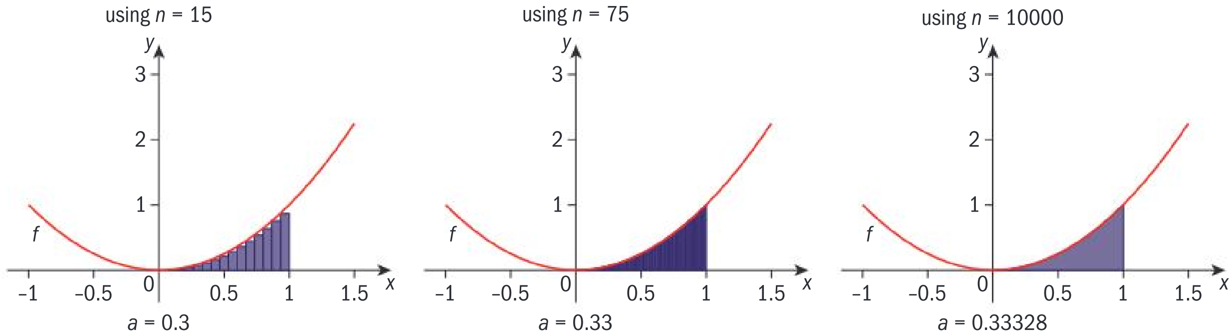
Areas of irregular shapes

Look at the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ in the diagrams. On the left is the actual area and on the right is an approximation of this area, using rectangles of base 0.125 and height x^2. Notice that the error in the approximation is the total area of the white space between the curve and the rectangles. You can use the method of exhaustion to fill the space with more rectangles of smaller width.



Newton approached the problem of finding areas by viewing the area function as the inverse of the tangent, i.e., the area function depended on the ratio of the difference of the y-values to the x-values, $\frac{dy}{dx}$, and employed the use of infinite series. Leibniz, on the other hand, approached the problem by summing the of areas of infinitely thin rectangles, hence the use of an elongated S, the integral symbol.

Using graphing software, it is easy to change n, the number of rectangles under the curve.
Using 15 similar rectangles, the approximation of the area under the curve is 0.3 square units.

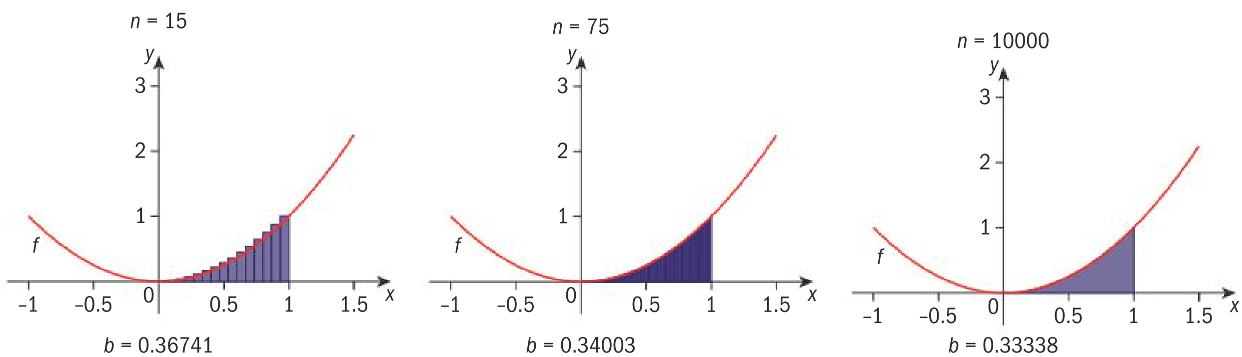


We get a better approximation when $n = 75$.

When $n = 10\,000$, the area is about 0.333 sq. units.

You have considered rectangles below the curve, the so-called lower bound sum. You can also approximate the area by drawing rectangles above the curve, the upper bound sum. This time, the error in the approximation is the sum of the areas of the purple spaces above the curve.

Again, consider the upper bound sum with 15, 75 and then 10 000 rectangles:



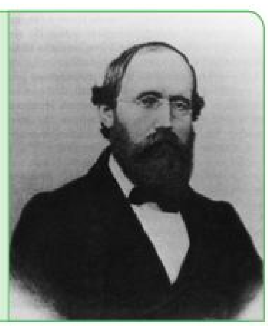
When $n = 15$, the area is approximately 0.367 sq. units.

When $n = 75$, the area is approximately 0.340 sq. units.

When $n = 10\,000$, the area is approximately 0.333 sq. units.

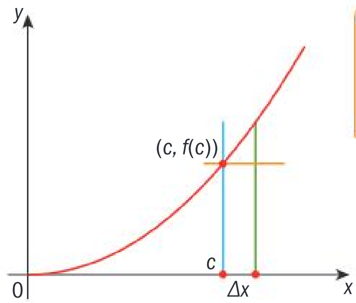
As the number of rectangles increases, the approximate area approaches the actual area.

This method of approximating the area under a curve is called Riemann sums, after the German mathematician **Georg Friedrich Bernhard Riemann** (1826–1866).



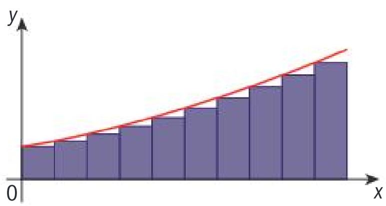
Mathematicians developed different methods to approximate the area under the curve of a graph. Explore some of these methods, and analyze the error of the approximations of the areas that these methods produce.

If f is continuous in the interval $[a, b]$, to find the area under the curve of $f(x)$ from $x = a$ to $x = b$, you can divide $[a, b]$ into n sub-intervals of equal length, $\frac{(b-a)}{n}$, and call this $\triangle x$. In each sub-interval, select the height of the rectangle such that a corner of the rectangle is on the curve, and call this $f(c)$.



$\triangle$ is the Greek upper case 'delta'. $\triangle x$ is "delta x".

Then, the area under the curve of i such sub-intervals is approximated by $\sum_{i=1}^{n} f(c_i)\triangle x_i$



As $\triangle x$ approaches 0, the number of rectangles n approaches infinity and the approximate area approaches the actual area. You can now ready to define the area under a curve as a definite integral.

→ If the integral of f exists in the interval $[a, b]$, and f is non-negative in this interval, then the area A under the curve $y = f(x)$ from a to b is $A = \int_a^b f(x)\,dx$

Example 15

Find the area bounded by the graph of $y = x^3$, $x = 0$, $x = 2$, and the x-axis.

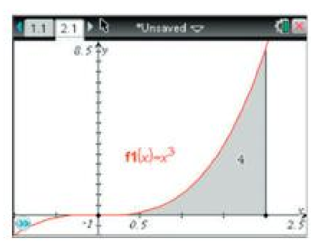
Answer

Since $y = x^3$ is non-negative in the interval [0, 2]

$A = \int_0^2 x^3 dx = \left[\frac{x^4}{4}\right]_0^2 = \frac{2^4}{4} = \frac{16}{4} = 4$ sq. units

Area = 4 sq. units

Confirm on the GDC. The area is entirely below the x-axis.



Example 16

Find the area of a triangle formed by $f(x) = \begin{cases} x+4, & -4 \le x \le -1 \\ 2-x, & -1 \le x \le 2 \end{cases}$

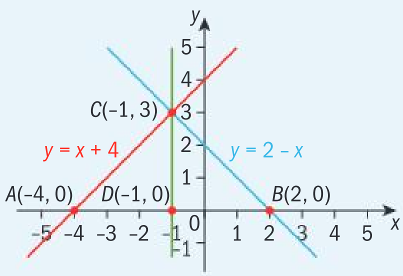
a using the formula for the area of a triangle

b by integration.

Answers

a

Sketch the graph.



$A = \frac{1}{2} \times 6 \times 3 = 9$ sq. units

Area $= \frac{1}{2}bh$

b Since both functions are non-negative in the interval [–4, 2]

Area of ΔACD

Divide the triangle into two smaller triangles.

$$= \int_{-4}^{-1} (x+4)\,dx = \left[\frac{x^2}{2} + 4x\right]_{-4}^{-1} = \left(\frac{(-1)^2}{2} + 4(-1)\right) - \left(\frac{(-4)^2}{2} + 4(-4)\right)$$

$$= -3.5 + 8 = 4.5 \text{ sq. units}$$

Area of ΔBCD

Integrate to find the area of each triangle.

$$= \int_{-1}^{2} (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_{-1}^{2} = \left(2(2) - \frac{(2)^2}{2}\right)_{-1}^{2} - \left(2(-1) - \frac{(-1)^2}{2}\right)_{-1}^{2}$$

$$= 2 + 2.5 = 4.5 \text{ sq. units}$$

Hence, area of ΔABC = 4.5 + 4.5 = 9 sq. units

Add the areas.

Alternative solution

ΔACD ≡ ΔBCD (RHS), so area of ΔABC = 2 × Area of ΔACD

Area of ΔACD = 4.5 sq. units

Area of ΔABC = 2 × 4.5 = 9 sq. units

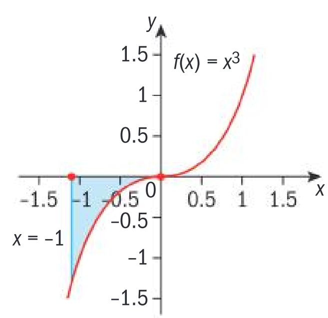
Notice that the triangle is symmetrical about the line CD.

Now look at areas below the x-axis, for example, the area above the graph of $y = x^3$, between $x = -1$ and $x = 0$.

Calculating the integral

$$A = \int_{-1}^{0} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{0} = -\frac{1}{4}$$

Since area is positive, take the absolute value: $A = \frac{1}{4}$ sq. unit

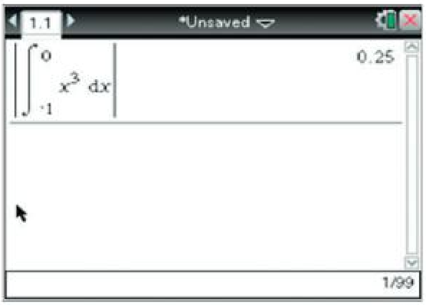


→ When f is negative for all $x \in [a, b]$, then the area bounded by the curve and the lines $x = a$ and $x = b$ is $\left|\int_a^b f(x)\mathrm{d}x\right|$

For the area below the x-axis for $y = x^3$

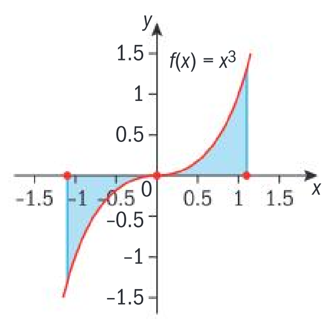
$$A = \left|\int_{-1}^{0} x^3\mathrm{d}x\right| = \left|\left[\frac{x^4}{4}\right]_{-1}^{0}\right| = |-0.25| = 0.25$$

Confirming this result on the GDC:



This confirms numerically using the absolute value of the function.

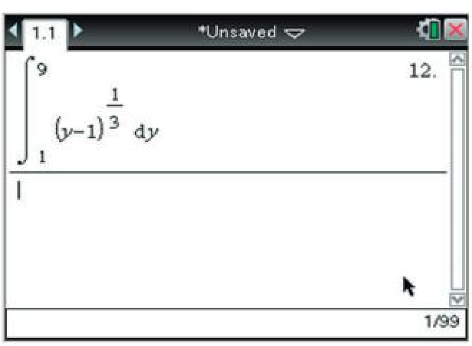
Now look at the area bounded by the graph of $y = x^3$, $x = -1$, $x = 1$, and the x-axis.



Since the area is partly above and partly below the x-axis, you have to integrate the functions in the two intervals separately.

$$A = \left|\int_{-1}^{0} x^3\mathrm{d}x\right| + \int_0^1 x^3\mathrm{d}x = |-0.25| + 0.25 = 0.5$$

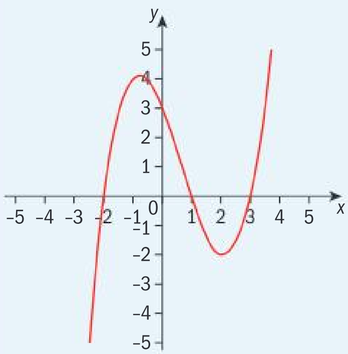
You can also evaluate this area graphically on the GDC by graphing $y = |x^3|$. To evaluate the integral numerically on the GDC, enter the integral of the absolute value of the function. This eliminates the need for separating the integrals.



Example 17

Find the area of the region bounded by the graph of the function $y = \frac{1}{2}(x-1)(x+2)(x-3)$ and the x-axis and confirm your answer graphically on the GDC.

Answer



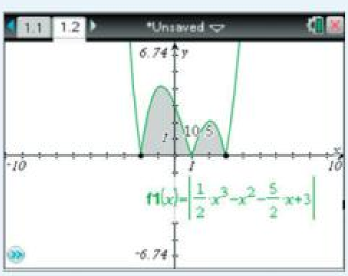
Graph the function on your GDC. Since part of the graph lies below the x-axis, integrate the function separately in the intervals where it is above and below the x-axis.

$$A = \left| \int_{-2}^{1} \left(\frac{1}{2}(x-1)(x+2)(x-3) \right) dx + \int_{1}^{3} \left(\frac{1}{2}(x-1)(x+2)(x-3) \right) dx \right|$$

$$A = \int_{-2}^{1} \left(\frac{1}{2}x^3 - x^2 - \frac{5}{2}x + 3 \right) dx \quad + \left| \int_{1}^{3} \left(\frac{1}{2}x^3 - x^2 - \frac{5}{2}x + 3 \right) dx \right|$$

$$= \left[\frac{x^4}{8} - \frac{x^3}{3} - \frac{5x^2}{4} + 3x \right]_{-2}^{1} \quad + \left| \left[\frac{x^4}{8} - \frac{x^3}{3} - \frac{5x^2}{4} + 3x \right]_{1}^{3} \right|$$

$$= \frac{63}{8} + \left| -\frac{8}{3} \right| = \frac{253}{24} = 10.5 \text{ sq. units to 3 sf.}$$



On the GDC, enter the absolute value of the function and the interval itself as the lower and upper bound.

$A = 10.5$ sq. units

→ The total area of $f(x)$ in an interval $[a, b]$, where its graph is partly above and partly below the x-axis is $A = \int_a^b |f(x)|\, dx$.

Investigation – odd and even functions

In Example 15 $f(x) = x^3$ is an odd function.
Choose different odd functions continuous in an interval $[a, b]$.
For your examples, find $\int_{-a}^{a} f(x)\,dx$
Make a conjecture and justify it.
Does your conjecture hold when applying this definite integral to areas? Explain.
Do the same for even functions continuous in an interval $[a, b]$.

Exercise 7H

Graph the functions on your GDC. Find the areas by integration. Then check your answer on your GDC.

Find the area of the region bounded by the graph of the function, the x-axis, and the given lines.

1 $y = x^4 - x$, $x = -1$ and $x = 1$

2 $y = x^2 - 2x - 3$, $x = -1$ and $x = -3$

3 $y = x^2 - 2x - 3$, $x = -3$ and $x = 1$

In questions 4–11, find the area of the region bounded by the graph of the function, the x-axis, and the given lines.

4 $y = e^x - 3$, $x = 0$, $x = 3$

5 $y = x^4 + 3x^3 - 3x^2 - 7x + 6$, $x = -3$; $x = 1$

6 $y = \sqrt{4 - x}$, $x = 0$, $x = 4$

7 $y = \frac{1}{x^2} + 1$, $x = \frac{1}{2}$, $x = 5$

8 $y = 2^x$, $x = 1$, $x = 2$

9 $y = 2e^{-x+1} - 1$, $x = 0$, $x = 3$

10 $y = \frac{1}{x+2}$, $x = -1$, $x = 2$

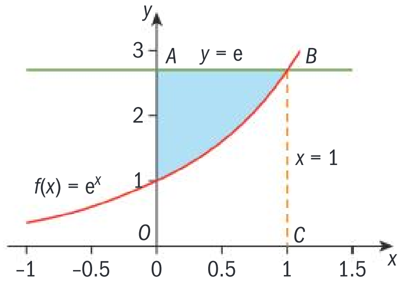
11 $y = \frac{2}{3-4x}$, $x = 1$, $x = 3$

12 Find the area of the region bounded by the graph of $y = -x^3 + 6x^2 + x - 30$, its x-intercepts, and the x-axis.

13 Find the area of the region enclosed by $y = \begin{cases} x^2, & 0 \le x < 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}$ and the x-axis.

14 Find the area of the region enclosed by $y = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ x^2, & 1 \le x \le 2 \end{cases}$ and the x-axis.

The graph shows the region bounded by the graph of the function $y = e^x$; the y-axis, and the line $y = e$.



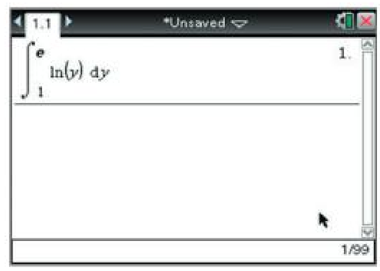
You can find this area by first finding the area of the region below the curve bounded by the graph of the function, the x-axis, and the lines $x = 0$ and $x = 1$.
Then subtract this area from that of the rectangle $OABC$, which is e sq. units.
Hence, the area of the desired region is

$$e - \int_0^1 e^x \, dx = e - \left[e^x\right]_0^1 = e - (e-1) = 1 \text{ sq. unit}$$

You can also obtain the result by rearranging to make x the subject and then integrating with respect to y, from $y = 1$ to $y = e$.

If $y = e^x$ then $x = \ln(y)$, and, $A = \int_1^e \ln(y)dy = 1$ sq. unit

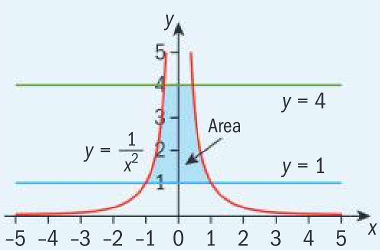
Since you don't yet know how to integrate ln(y) analytically (this will be covered in chapter 9), use the GDC to confirm the result.



Example 18

Find the area of the region bounded by the graph of the function $y = \frac{1}{x^2}$ and the lines $y = 1$ and $y = 4$.

Answer



Graph the function.

$x = \sqrt{\frac{1}{y}}$

Make x the subject.

$A = 2\int_1^4 \sqrt{\frac{1}{y}} dy$

Integrate with respect to y.

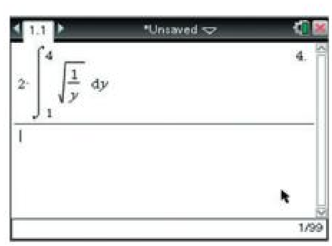
$\int \sqrt{\frac{1}{y}} dy$ gives the area to the right of the y-axis.

By symmetry, A is double the area on the right of the y-axis.

$A = 2\left[2y^{\frac{1}{2}}\right]_1^4 = 4\left(4^{\frac{1}{2}} - 1\right)$

$= 4$ sq. units

Confirm on the GDC.

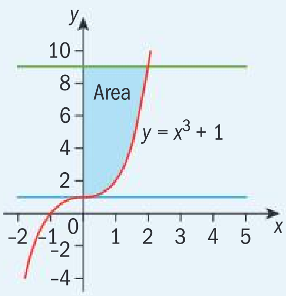


Why does the alternative method used here not work for Example 18?

Example 19

Find the area of the region bounded by the graph of the function $y = x^3 + 1$, the y-axis, and the lines $y = 1$ and $y = 9$.

Answer



Graph the function on your GDC and identify the area.

$x = \sqrt[3]{y-1}$

Make x the subject.

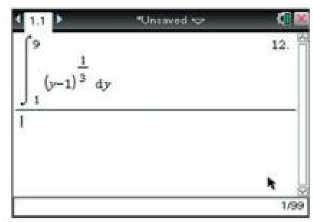
$A = \int_1^9 (y-1)^{\frac{1}{3}}\,dy$

Integrate with respect to y.

$= \left[\dfrac{3(y-1)^{\frac{4}{3}}}{4}\right]_1^9$

$= \dfrac{3(8)^{\frac{4}{3}}}{4} - 0 = 12$ sq. units

Confirm on the GDC.



Alternative solution

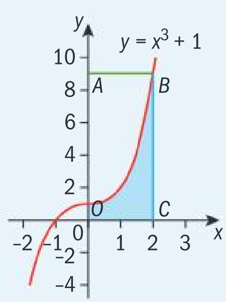
Area of rectangle OABC = 9 × 2 = 18

Area above curve

$= 18 - \int_0^2 (x^3 + 1)\,dx$

$= 18 - \left[\dfrac{x^4}{4} + x\right]_0^2 = 18 - 6$

$= 12$ sq. units



Exercise 7I

Find the areas of the regions bounded by the function, the y-axis, and the given lines.

1 $y = x^2 + 1, \quad y = 1, y = 10$

2 $y = \sqrt{x}, \quad y = 0, y = 4$

3 $y = \sqrt{4 - x}, \quad y = 0, y = 2$

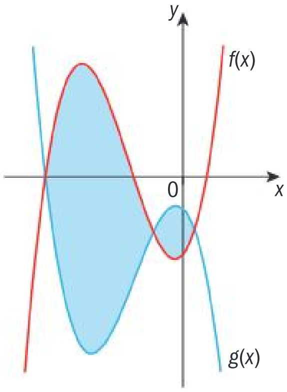
4 $y = 4 - x^2, y = 3, y = 4$

5 $y = \dfrac{1}{\sqrt{-x+4}}, y = \dfrac{1}{2}, y = 2$

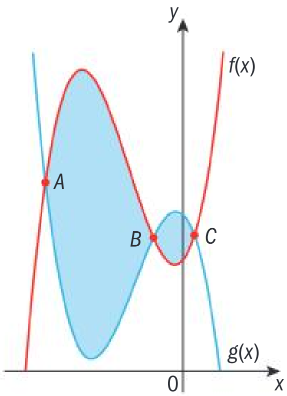
Areas of regions between curves

The graph shows two curves $f(x)$ and $g(x)$.

The regions bounded by the two curves are shaded.



Translate both graphs vertically so that both areas are above the x-axis.



A translation of both graphs by the same amount in the same direction preserves the original area.

The area between points A and B is the difference of the areas under the curves $f(x)$ and $g(x)$ from A to B.

→ If functions f and g are continuous in the interval $[a, b]$, and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area between the graphs of f and g is

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$$

Similarly the area between points B and C is the difference of the areas under the curves $g(x)$ and $f(x)$ between points B and C,

$$A = \int_b^c g(x)\,dx - \int_b^c f(x)\,dx = \int_b^c (g(x) - f(x))\,dx$$

To find the total area between A and C, add the areas of the two regions.

Example 20

Find the area enclosed by the graphs of the curves

$f(x) = \frac{1}{2}x^3 + 2x^2 + 2x - \frac{1}{2}$ and $g(x) = -\frac{1}{2} + 3x + 2x^2 - \frac{1}{2}x^3$

Answer

$\frac{1}{2}x^3 + 2x^2 + 2x - \frac{1}{2} = -\frac{1}{2} + 3x + 2x^2 - \frac{1}{2}x^3$

$x^3 - x = 0 \Rightarrow x(x+1)(x-1) = 0$

$x = 0, \pm 1$

$A = \int_{-1}^{0} [f(x) - g(x)]\,dx + \int_{0}^{1} [g(x) - f(x)\,dx]$

$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$

$A = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$

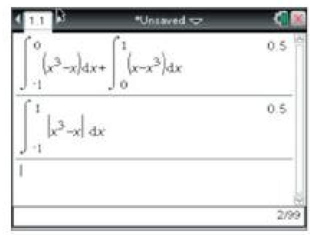
$A = \frac{1}{4} + \frac{1}{4} = \frac{1}{4}$ sq. unit

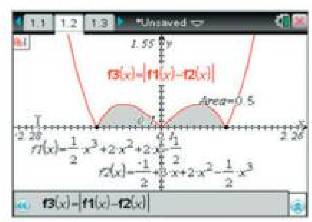
Let f(x) = g(x) to find the points of intersection of the two curves.

Since the leading coefficient of f(x) is positive and the leading coefficient of g(x) is negative, we know that in the interval [–1, 0], f(x) > g(x) and in the interval [0, 1], g(x) > f(x).

If we are not sure which function is greater in the given interval, it is sufficient to place the integrals in an absolute value sign.

Check your answers on the GDC.





The total area of the regions enclosed by the graphs of two functions f and g that intersect at $x = a$, $x = b$ and $x = c$, $a < b < c$ is

$$A = \int_{a}^{c} \left| f(x) - g(x)\,dx \right|$$

In area problems, a region may be not be entirely enclosed between two functions. The next example highlights this case.

Example 21

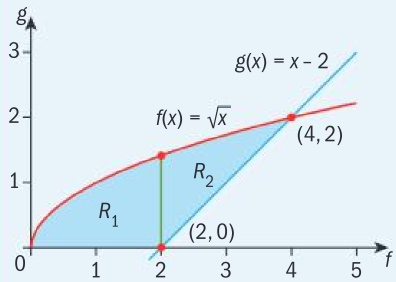
Find the area of the region in the first quadrant that is enclosed by $y = \sqrt{x}$, the x-axis, and the line $y = x - 2$.

Answer

$A = R_1 + R_2$

$$R_1 = \int_0^2 \sqrt{x}\ \mathrm{d}x = \left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^2 = \frac{4\sqrt{2}}{3}$$

Sketch the graph.



$$R_2 = \int_2^4 (\sqrt{x} - (x-2))\,\mathrm{d}x$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2(4)^{\frac{3}{2}}}{3} - \frac{4^2}{2} + 2(4) - \left(\frac{2(2)^{\frac{3}{2}}}{3} - \frac{2^2}{2} + 2(2)\right)$$

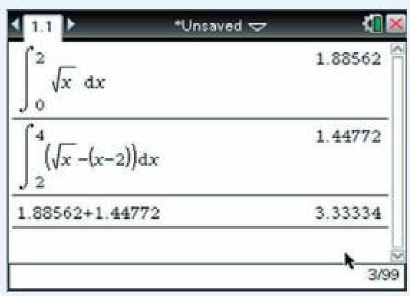
$$= \frac{16}{3} - \left(\frac{4\sqrt{2}}{3} + 2\right) = \frac{10 - 4\sqrt{2}}{3}$$

$$\therefore\ A = \frac{4\sqrt{2}}{3} + \frac{10 - 4\sqrt{2}}{3}$$

$$= \frac{10}{3}$$

= 3.33 sq. units to 3 sf

Check on a GDC.



Exercise 7J

In questions 1–11, find the area of the region enclosed by the graphs of the curves.

Do not use a GDC for questions 1–6.

1 $y = 2 - x^2$ and $y + x = 0$

2 $y = x^3$ and $y = x^2$

3 $y = 4 - x^2$ and $y = 2 - x$

4 $y = |x|$ and $y = x^{\frac{2}{3}}$

5 $y = 16 - x^2$ and $y = x^2 - 4x$

6 $y = x^4 - 2x^2$ and $y = 2x^2$

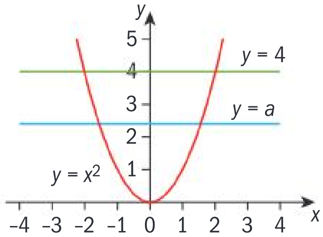
7 $y = 2x^3 + 5x^2 + x - 2$ and $y = 8 - 4x - 20x^2 - 8x^3$

8 $y = x^4 - 4$ and $y = \frac{1}{1+x}$, for $x > 0$

9 $y = e^{1-x} - 1$; $y = \sqrt{x}$; $x = 4$

EXAM-STYLE QUESTION

10 In this graph, the regions bounded by the curve $y = x^2$ and the lines $y = 4$ and $y = a$ is equal to the region bounded by the curve $y = x^2$ and $y = a$. Find the value of a.



In questions 11–13, find the area of the region whose boundary is defined by the functions or lines.

11 $y = 2 - x$ and $y = x^2$

12 $y = e^x$, $y = e^{-x}$, $x = \pm 1$

13 $y = \frac{1}{x}$, $y = x^{\frac{2}{3}}$, x-axis and $x = 3$

Areas and kinematics

At the beginning of section 7.2, you found the total distance traveled by a particle in a given time interval by integrating the velocity function, evaluating the displacement at the end points of the interval, and then subtracting these results. The velocity in this case was positive throughout this interval.

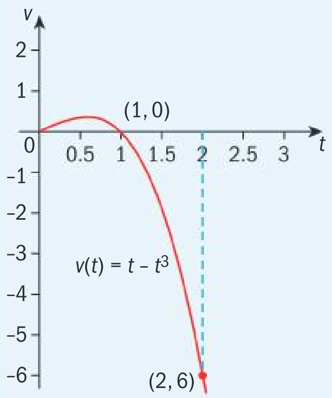
Consider a similar problem where the velocity changes direction within the given interval. The velocity function will be partly above and partly below the t-axis.

Example 22

A particle moves in a straight line such that its velocity at any time t can be modeled by $v(t) = t - t^3$ ms^{-1}.
Find the total distance traveled by the particle in the time interval [1, 2]

Answer

Sketch the function to see if it is entirely above or below the t-axis, or if part of the graph is below and part above the t-axis.



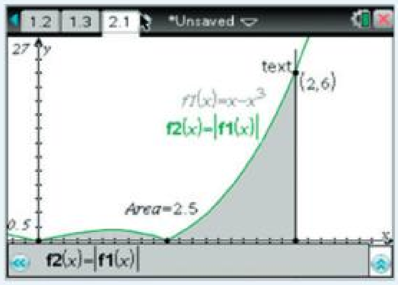
$$d(t) = \int_0^1 (t - t^3)\,dt + \left| \int_1^2 (t - t^3)\,dt \right|$$

Integrate the parts separately above and below the x-axis

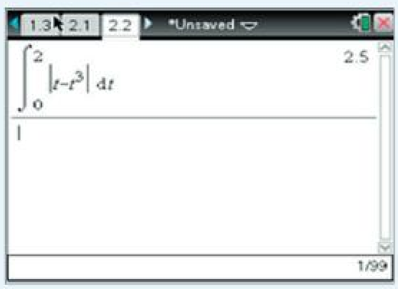
$$= \left[\frac{t^2}{2} - \frac{t^4}{4}\right]_0^1 + \left|\left[\frac{t^2}{2} - \frac{t^4}{4}\right]_1^2\right|$$

$$= \frac{1}{4} + \left|-2 - \left(\frac{1}{2} - \frac{1}{4}\right)\right| = 2.5\,\text{m}$$

Graphically:



Numerically:



The total distance is the integral of the absolute value of the function on the interval [0, 2].

From example 22, you can see that:

→ If v is a velocity function in terms of t, then the total distance traveled between times t_1 and t_2 = $\int_{t_1}^{t_2} |v|\,dt$

Exercise 7K

1 A particle starts from rest and moves in a straight line.
Its velocity at any time t seconds is given by $v(t) = t(t - 4)$ m s^{-1}
Find the distance traveled between the two times when the particle is at rest.

2 A particle moves in a straight line so that after t seconds its velocity is $v(t) = 5 + 4t - t^2$ m.
Find the total distance traveled by the particle

a in the first second

b between the first second and the sixth second.

3 A particle starts from rest and its acceleration, in m s^{-2}, can be modeled by $a(t) = 1 - e^{-2t}$, $0 \le t \le 3$.
Find the distance traveled in the first 3 seconds.

EXAM-STYLE QUESTION

4 The velocity of a particle moving in a straight line is given by $v(t) = 10 + 5e^{-0.5t}$ m s^{-1}

a Show that the acceleration of the particle at any time t is always negative.

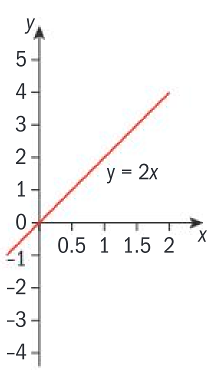
b Find the total distance covered in the first 2 seconds.

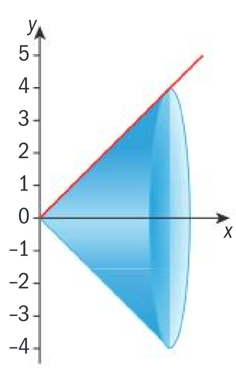
Volumes of solids of revolution

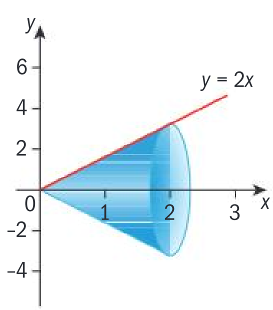
A lathe is a machine that rotates material on its axis to make objects with circular cross-sections and curved sides, such as vases. A variety of materials, such as metal or plastic, may be used.

In mathematics, objects like those made with a lathe are called **solids of revolution**. A solid figure with curved sides is obtained by rotating the curve through 360° about a line; for example, the x-axis.

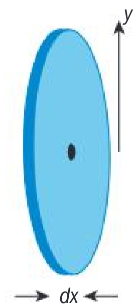
Here is the graph of $y = 2x$ between $x = 0$ and $x = 2$. Rotating the line $y = 2x$ about the x-axis gives a cone.







To find the volume of the cone in an interval $[a, b]$, take cross-sectional slices, as with the area and inscribed rectangles. These slices are 3-D cylinders each with radius y, and height tending to dx so each has volume $\pi y^2 dx$. Then, to find the volume of the cone, add the volume of all the cylinders, i.e., $\sum \pi y^2 dx$. When dx is infinitesimally small,

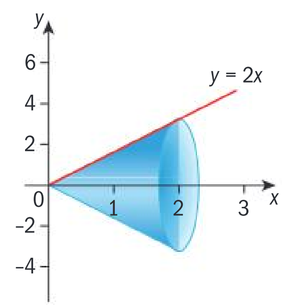


then $v = \int_a^b \pi y^2 dx = \pi \int_a^b y^2 dx$

→ The volume of a solid formed when a function $y = f(x)$, continuous in the interval $[a, b]$, is rotated 2π radians about the x-axis is $V = \pi \int_a^b y^2 dx$

2π radians = 360°

The volume of the cone obtained by rotating the line $y = 2x$ in the interval $[0, 2]$ through 2π radians about the x-axis is



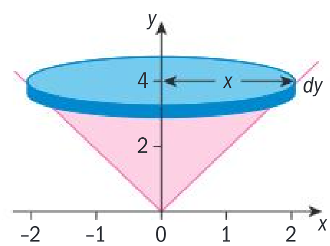
$$V = \pi \int_0^2 (2x)^2 dx = 4\pi \int_0^2 x^2 dx = 4\pi \left[\frac{x^3}{3}\right]_0^2 = 4\pi \left[\frac{8}{3}\right] = \frac{32\pi}{3} \text{ cubic units}$$

Compare this to the result obtained using the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$

$V = \frac{\pi}{3}(4^2)(2) = \frac{32\pi}{3}$ cu. units

Similarly, you can find the volume of the cone formed when the line $y = 2x$ is rotated 2π radians about the y-axis in the same interval.

The cylinders have radius x and height dy.



→ The volume of a solid of revolution formed when $y = f(x)$ in the interval $y = c$ to $y = d$ is rotated 2π radians about the y-axis is $V = \pi \int_c^d x^2 dy$

$f(a) = c \quad f(b) = d$

To find the volume of the cone formed by rotating the line $y = 2x$ about the y-axis, rearrange to give $x = \frac{y}{2}$. The interval $[0, 2]$ on the x-axis corresponds to $[0, 4]$ on the y axis. Then

$$V = \pi \int_0^4 \left(\frac{y}{2}\right)^2 dy = \frac{\pi}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{\pi}{4}\left(\frac{4^3}{3}\right) = \frac{16\pi}{3} \text{ cu. units.}$$

Example 23

Find the volume of the solid formed when the graph of the curve $y = \sqrt{x}$ in the interval [1, 4] is rotated 2π radians about **a** the x-axis **b** the y-axis.

Answers

a $V = \pi\int_1^4 \left(\sqrt{x}\right)^2 dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \pi\left(\frac{16}{2} - \frac{1}{2}\right) = \frac{15\pi}{2}$ cu. units

Use $V = \pi\int_a^b y^2 dx$

b $y = \sqrt{x} \Rightarrow x = y^2$; when $x = 1$, $y = 1$; when $x = 4$, $y = 2$

$V = \pi\int_1^2 y^4 dy = \pi\left[\frac{y^5}{5}\right]_1^2 = \pi\left(\frac{32}{3} - \frac{1}{5}\right) = \frac{31\pi}{5}$ cu.units

Rearrange to make x the subject and find the values of y when x = 1 and x = 4

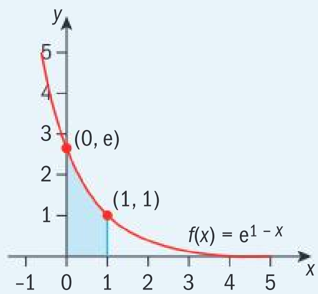
Use $V = \pi\int_c^d y^2 dy$

Example 24

Find the volume of the solid formed when the graph of the curve $y = e^{1-x}$ is rotated 2π radians about the x-axis between $x = 0$ and $x = 1$.

Answer

Sketch the graph.



$V = \pi\int_0^1 (e^{1-x})^2\, dx = \pi\int_0^1 e^{2(1-x)}\, dx$

$= -\frac{\pi}{2}[e^{2(1-x)}]_0^1$

$= -\frac{\pi}{2}(1 - e^2)$

$= \frac{\pi}{2}(e^2 - 1)$ cu. units

$= 10.0$ (3 sf) cu. units

Sangaku are Japanese geometrical puzzles in Euclidean geometry on wooden tablets. They were placed as offerings at Shinto shrines or Buddhist temples during the Edo period (1603–1867) as offerings to the gods .The tablets were created using only Japanese mathematics. For example, the connection between an integral and its derivative (the fundamental theorem of calculus) was unknown, so Sangaku problems on areas and volumes were solved by expansions in infinite series and term-by-term calculation. You may wish to select a Sangaku puzzle, and through research investigate their method of calculating areas and volumes.

Exercise 7L

In questions 1 and 2, find the volume of the solid formed by rotating the region enclosed by the graph of the function and the x-axis, through 2π radians about the x-axis, in the given interval.

1 $y = (x - 1)^2 - 1$, [0, 1]

2 $y = 1 + \sqrt{x}$, [0, 2]

3 When the graph of the function $y = \frac{x^2}{2}$ is revolved 2π radians about the y-axis, it models the shape of a bowl.
Find the volume of the bowl between $y = 0$ and $y = 2$.

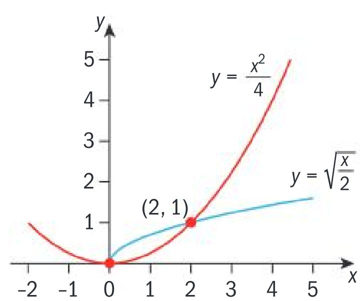
4 A paperweight is modeled by the graph of the function $y = \sqrt{2x - x^2}$ when it is revolved 2π radians about the x-axis between $x = 1$ and $x = 2$.
Find the volume of the paperweight.

5 Find the volume of the solid of revolution formed when the graph of the function $y = x^{\frac{3}{2}}$ is revolved about the y-axis between $y = 1$ and $y = 3$.

6 A wine bottle stopper is modeled by the function $y = \frac{x}{12}\sqrt{36 - x^2}$
Find the volume of the stopper when it is rotated 2π radians about the x-axis between $x = 0$ and $x = 6$.

There are several methods to find the volume of a solid of revolution. Investigate the different methods, such as disc, shell and washer methods, and explore the conditions under which the various methods are employed.

Now look at the volume of a solid formed by the region between two curves. The graph shows the region formed between the curves $y = \sqrt{\frac{x}{2}}$ and $y = \frac{x^2}{4}$



Geometrically, the volume of the region between the two curves rotated 2π radians about the x-axis is the difference in the volumes of the solids formed by the curve and the x-axis.

→ Hence, if $f(x) \geq g(x)$ for all x in the interval $[a, b]$, then the volume of revolution formed when rotating the region between the two curves 2π radians about the x-axis in the interval $[a, b]$ is

$$V = \pi\int_a^b (f(x))^2\,dx - \pi\int_a^b (g(x))^2\,dx, \text{ or}$$

$$V = \pi\int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

It is easy enough to find the points of intersection analytically by setting the two equations equal to each other, and solving for x.

For the two curves $y = \sqrt{\frac{x}{2}}$ and $y = \frac{x^2}{4}$,

$$V = \pi\int_0^2 \left(\sqrt{\frac{x}{2}}\right)^2 dx - \pi\int_0^2 \left(\frac{x^2}{4}\right)^2 dx$$

$$= \pi\int_0^2 \frac{x}{2}\,dx - \pi\int_0^2 \frac{x^4}{16}\,dx = \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^2 - \frac{\pi}{16}\left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{\pi}{2}(2) - \frac{\pi}{16}\cdot\left(\frac{32}{5}\right) = \pi - \frac{2\pi}{5} = \frac{3\pi}{5} \text{ cu. units.}$$

Now rotate the region about the y-axis in the same interval.
Rearrange both equations to make x the subject:

$x = 2y^2$ and $x = 2\sqrt{y}$

For both functions, when $x = 0$, $y = 0$ and when $x = 2$, $y = 1$, so the curves intersect at (0, 0) and (2, 1).

Hence, $V = \pi\int_0^1 (2\sqrt{y})^2 \mathrm{d}y - \pi\int_0^1 (2y^2)^2 \mathrm{d}y$

$$V = 4\pi\int_0^1 y\,\mathrm{d}y - 4\pi\int_0^1 y^4\,\mathrm{d}y = 4\pi\left[\frac{y^2}{2}\right]_0^1 - 4\pi\left[\frac{y^5}{5}\right]_0^1$$

$$= \pi\frac{4}{2} - \pi\frac{4}{5} = \frac{6\pi}{5} \text{ cu. units}$$

→ If x_1 and x_2 are relations in y such that $x_1 \geq x_2$ for all y in the interval $[c, d]$, then the volume formed when rotating the region between the two curves 2π radians about the y-axis in the interval $[c, d]$ is

$$V = \pi\int_d^c x_1^2\mathrm{d}y - \pi\int_d^c x_2^2\mathrm{d}y$$

$$\text{or } V = \pi\int_c^d (x_1^2 - x_2^2)\mathrm{d}y$$

The astronomer **Johann Kepler** (1571–1630) expanded upon Archimedes' work on finding volumes of irregular shapes. Legend has it that at his wedding, Kepler was distracted by the problem of how much wine was in the barrels his guests were being served from. The problem so fascinated him that he dedicated an entire book to its solution. The book, published in 1615, was entitled *Nova stereometria doliorum vinariorum'* or *New volume measurements of wine barrels*.

Example 25

The graphs of $x = \frac{y^4}{4} - \frac{y^2}{2}$ and $x = \frac{y^2}{2}$ completely enclose a region. Find the volume of the solid formed when this region is rotated 2π radians about the y-axis in the interval $[c, d]$, $c, d \geq 0$.

Answer

$$\frac{y^4}{4} - \frac{y^2}{2} = \frac{y^2}{2} \Rightarrow \frac{y^4}{4} - y^2 = 0 \Rightarrow y^2\left(\frac{y^2}{4} - 1\right) = 0 \Rightarrow y = 0,\ y = \pm 2$$

Without a graph it is safer to use the absolute value in the interval.

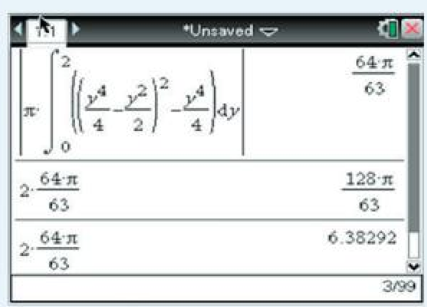
$$V = \pi\left|\int_0^2 \left(\frac{y^4}{4} - \frac{y^2}{2}\right)^2 - \left(\frac{y^2}{2}\right)^2 \mathrm{d}y\right| = \pi\left|\int_0^2 \left(\frac{y^8}{16} - \frac{y^6}{4}\right)\mathrm{d}y\right|$$

$$V = \pi\left[\left|\frac{y^9}{144} - \frac{y^7}{28}\right|\right]_0^2 = \pi\left|\frac{2^9}{144} - \frac{2^7}{28}\right| = \frac{64\pi}{63}$$

The volume formed from $y = -2$ to $y = 0$ is twice the volume from $y = 0$ to $y = 2$, hence the total volume is

$$2 \cdot \frac{64\pi}{63} = \frac{128\pi}{63} = 6.38 \text{ cu. units}$$

Confirm on a GDC:



Exercise 7M

1 Find the volume of the solid formed when the region between the graphs of the functions $y = x$ and $y = \frac{x}{2}$ is rotated through 2π radians about the x-axis between $x = 2$ and $x = 5$.

2 Find the volume of the solid formed when the region between the graphs of the functions $y = x - 4$ and $y = x^2 - 4x$ is revolved 2π radians about the x-axis.

3 Find the volume of the solid formed when the region between the graphs of $y = x$ and $y^2 = 2x$ is revolved 2π radians about the y-axis.

4 Find the volume of the solid formed when the region between the graphs of the functions $y = 2x - 1$, $y = x^{\frac{1}{2}}$, and $x = 0$ is revolved 2π radians about the y-axis.

Review exercise

EXAM-STYLE QUESTION

1 The gradient function of a curve is $\frac{dy}{dx} = ax + \frac{b}{x^2}$. The curve passes through the point (–1, 2), and has a point whose gradient is 0 at (–2, 0). Find the equation of the curve.

2 Calculate the area enclosed by the graphs of $y = x^2$ and $y^2 = x$

3 The region enclosed by $y = 1 + 3x - x^2$ and $y = \frac{2}{x}$ for $x > 0$ is rotated 2π radians about the x-axis. Find the volume of the solid formed.

4 Evaluate

a $\int_1^2 \left(x + \frac{1}{x^2} - \frac{1}{x^4}\right) dx$

b $\int_1^4 \frac{5x^2 - 4}{\sqrt{x}} dx$

c $\int_1^2 \frac{1}{x - 3} dx$

d $\int_1^e \frac{1}{1 - 4x} dx$

Review exercise

1 A particle moves in a straight line so that its velocity after t seconds is $v(t) = t^3 - 4t$ m s^{-1}
Find the total distance traveled in the first 3 seconds.

EXAM-STYLE QUESTION

2 The velocity of a particle moving in a straight line is $v(t) = t^3 - 3t^2 + 2$ m s^{-1}
Find the total distance traveled between the maximum and minimum velocities.

3 Find the total area of the region enclosed by the graph of $y = x^2 - 4 + \frac{3}{x^2}$ and the x-axis.

4 Integrate these where possible with respect to x.

a $\frac{3x^4+6}{x^2}$ **b** $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$

c $\frac{1}{2-3x}$ **d** $\frac{2}{\sqrt{1-4x}}$

e $2e^{-3x}+\sqrt[3]{e^x}$

5 Find the quotient when $2x^2 + 3x$ is divided by $2x - 1$.
Hence, evaluate $\int_1^2 \left(\frac{2x^2+3x}{2x-1}\right)dx$

6 Find the area enclosed by the graph of $y = \frac{1}{(x+1)}$, the y-axis, and the line $y = 5$.

7 Find the area enclosed by the graph of $y=\sqrt{x+1}$, and the x- and y-axes.

EXAM-STYLE QUESTION

8 The area enclosed by the curve $y = 3x(a - x)$ and the x-axis is 4 units^2. Find the value of a.

9 The region between the graphs of $y = 3^x$, $y = 3^{-x}$, and the lines $x = -1$ and $x = 1$ is rotated 2π radians about the x-axis. Find the volume of the solid formed.

CHAPTER 7 SUMMARY

Integration

- $\int f(x)\mathrm{d}x = F(x) + c,\ c \in \mathbb{R}$
- $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1},\ n \neq -1$
- $\int [f(x) \pm g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$
- $\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ a \neq 0$
- $\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c$
- $\int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + c,\ a \neq 0$
- $\int m^{ax+b}\,\mathrm{d}x = \frac{1}{a\ln(m)}m^{ax+b} + c$, where m is a positive real number, $a \neq 0$.
- $\int \frac{1}{x}\mathrm{d}x = \ln|x| + c$
- $\int \frac{1}{(ax+b)}\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c,\ a \neq 0$

Definite integration

- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\int_b^a f(x)\,\mathrm{d}x = 0$
- $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$
- $\int_a^b (f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_b^c f(x)\mathrm{d}x$

The fundamental theorem of calculus

- If f is continuous in $[a, b]$ and if F is any anti-derivative of f on $[a, b]$ then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

Continued on next page

Areas between graphs of functions and the axes

- If the integral of f exists in the interval $[a, b]$, and f is non-negative in this interval, then the area A under the curve $y = f(x)$ from a to b is $A = \int_a^b f(x)\,dx$.
- When f is negative for all $x \in [a, b]$, then the area bounded by the curve and the lines $x = a$ and $x = b$ is $\left|\int_a^b f(x)\,dx\right|$
- If functions f and g are continuous in the interval $[a, b]$, and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area between the graphs of f and g is $A = \int_a^b (f(x) - g(x))\,dx$

Kinematics

- If v is a velocity function in terms of t, then the total distance traveled between times t_1 and t_2 is $\int_{t_1}^{t_2} |v|\,dt$

Volumes of revolution

- The volume of a solid formed when a function $y = f(x)$, continuous in the interval $[a, b]$, is rotated 2π radians about the x-axis is $V = \pi\int_a^b y^2\,dx$.
- The volume of a solid of revolution formed when $x = f(y)$ in the interval $y = c$ to $y = d$ is rotated 2π radians about the y-axis is $V = \pi\int_c^d x^2\,dy$
- If $f(x) \geq g(x)$ for all x in the interval $[a,b]$, then the volume formed when rotating the region between the two curves 2π radians about the x-axis in the interval $[a, b]$ is
  $V = \pi\int_a^b (f(x))^2\,dx - \pi\int_a^b (g(x))^2\,dx$, or $V = \pi\int_a^b ([f(x)]^2 - [g(x)]^2)\,dx$.
- If x_1 and x_2 are relations in y such that $x_1 \geq x_2$ for all y in the interval $[c, d]$, then the volume formed when rotating the region between the two curves 2π radians about the y-axis in the interval $[c, d]$ is $V = \pi\int_d^c {x_1}^2\,dy - \pi\int_d^c {x_2}^2\,dy$
  or $V = \pi\int_c^d ({x_1}^2 - {x_2}^2)\,dy$