9 The power of calculus

CHAPTER OBJECTIVES:

6.1 Definition of a derivative from first principles
6.2 Derivative of sin*x*, cos*x* and tan*x*, sec*x*, csc*x*, cot*x*, arcsin*x*, arccos*x* and arctan*x*.
6.4 Indefinite integral of sin*x* and cos*x*; other indefinite integrals using the results from 6.2; the composites of any of these with a linear function
6.5 Areas of regions enclosed by curves; volumes of revolution about the *x*-axis or *y*-axis
6.6 Kinematic problems involving displacement *s*, velocity *v* and acceleration *a*
6.7 Integration by substitution; integration by parts

Before you start

You should know how to:

1 Transform trigonometric expressions.

e.g. prove $\sin 2\theta \equiv \dfrac{2\tan\theta}{1+\tan^2\theta}$

$$\text{RHS} = \frac{\dfrac{2\sin\theta}{\cos\theta}}{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} = \frac{2\sin\theta\cos\theta}{(\cos^2\theta+\sin^2\theta)}$$

$$= 2\sin\theta\cos\theta = \sin 2\theta$$

2 Apply the product and quotient rules on x^n, e^x and $\ln x$, how to do implicit differentiation and the chain rule.

e.g. Differentiate $f(x) = e^{x^2}\ln(2x-1)$

$$f'(x) = e^{x^2}\cdot 2x \cdot \ln(2x-1) + e^{x^2}\frac{1}{2x-1}\cdot 2$$

$$= 2xe^{x^2}\ln(2x-1) + \frac{2e^{x^2}}{2x-1}$$

Skills check

1 Prove these identities.

a $\cos 2\theta \equiv \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

b $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$

2 Find the derivative of:

a $f(x) = 3e^{2x} - 2x^2$

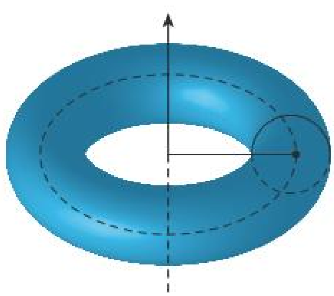
b $g(x) = (x+1)\cdot\ln(x^2+2x+1)$

c $h(x) = \dfrac{e^{x^2}}{x+1}$

Further calculus and applications

The geometrical name for a doughnut shape is a ring torus. It is a solid of revolution, created by rotating a circle about a vertical axis at the centre of the 'hole' in the torus.

The torus was studied by a Greek geometer **Pappus** of Alexandria (290–350 CE).
There are three types of torus called ring, horn and spindle torus. Investigate the properties of these tori (the plural of torus).



You learned about solids of revolution in Chapter 7.

In this chapter you will learn more differentiation and integration techniques, and use these to model and analyze real-world problems. Integration can be used to find the volume of a torus – or the volume of dough needed to make a doughnut.

This chapter will also show you how to solve optimization problems – such as how to calculate the amount of dough needed to create the optimum number of doughnuts for a day's sales, with minimum wastage.

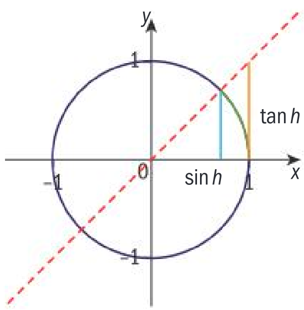
9.1 Derivatives of trigonometric functions

To find derivatives of trigonometric functions we are going to use the definition of the derivative, i.e., differentiate from first principles. First find the values of some trigonometric limits that will appear in the process.

Trigonometric limits

One of the most useful limits that involves trigonometric functions is $\lim\limits_{h\to 0}\frac{\sin h}{h}$.

To determine its value, consider the unit circle.



Notice that the arc whose length is denoted by h can be 'squeezed in' between two vertical line segments that represent $\sin h$ and $\tan h$ values. From the diagram:

$$\sin h \le h \le \tan h$$

$$1 \le \frac{h}{\sin h} \le \frac{1}{\cos h}$$

Divide by $\sin h$, $\sin h \ne 0$.

$$\cos h \le \frac{\sin h}{h} \le 1$$

Use reciprocal values.

$$\lim_{h\to 0^+}\cos h \le \lim_{h\to 0^+}\frac{\sin h}{h} \le \lim_{h\to 0^+}1$$

Take a limit $h \to 0^+$

$$1 \le \lim_{h\to 0^+}\frac{\sin h}{h} \le 1$$

Use properties of limits and $\cos 0 = 1$.

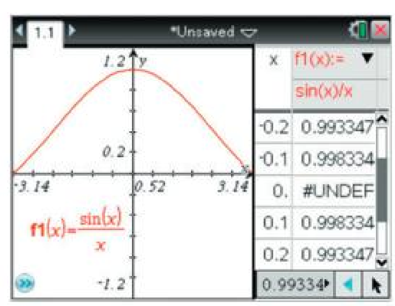
$$\lim_{h\to 0^+}\frac{\sin h}{h} = 1$$

The graph of $\frac{\sin x}{x}$ is even, so the same from both sides.

Therefore since the limit from the left is equal to the limit from the right you can conclude that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

Another confirmation of the result can be obtained numerically by using a calculator.



The graph also confirms that $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

Another limit that will be useful is $\lim_{h \to 0} \frac{\cos h - 1}{h}$. To find its value use these following trigonometric identities.

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2} \Rightarrow \cos\theta - 1 \equiv -2\sin^2\left(\frac{\theta}{2}\right)$$

Using this identity:

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

$$= -\lim_{h \to 0} \left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \sin\left(\frac{h}{2}\right) \right)$$

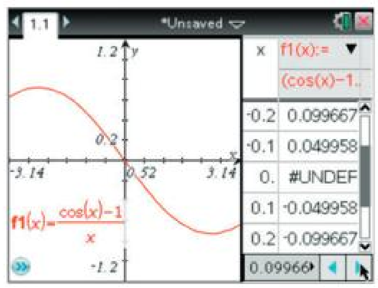
$$= -\lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h \to 0} \left(\sin\left(\frac{h}{2}\right) \right) = -1 \times 0 = 0$$

Rewrite the square and fractions into a desirable form.

Use the results of the known limits.

Notice that when $h \to 0$ then $\frac{h}{2} \to 0$ too.

Again you can verify the value of the limit by looking at the graph of the function $g(x) = \frac{\cos x - 1}{x}$, $x \neq 0$.



Derivatives of trigonometric functions

Now you can differentiate trigonometric functions from first principles.

Example 1

Find the derivative of $f(x) = \sin x$ from first principles.

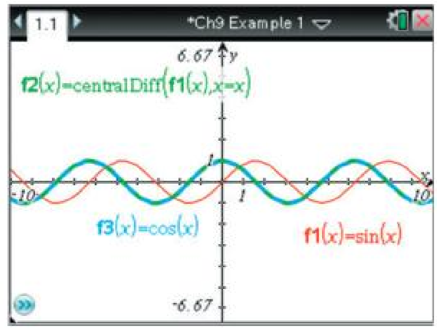
Answer

Solution 1

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	*Use the definition of the derivative.*
$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$	*Use addition formula for sine.*
$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$	
$= \lim_{h \to 0} \left(\cos x \frac{\sin h}{h} + \sin x \frac{\cos h - 1}{h} \right)$	*Rewrite the expression.*
$= \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} + \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h}$	*Use the properties of limits*
$= \cos x \cdot 1 + \sin x \cdot 0 = \cos x$	

Derivatives of trigonometric functions from first principles are not examinable. The derivative of sine is included here so that you can understand the result.

You can use the GDC to confirm the gradient function of $y = \sin x$. First, graph $f\,1(x) = \sin x$, then graph $f\,2(x) = nDeriv\,(\sin x, x = x)$. This is the graph of the derivative which calculates the value of a derivative at all the points in the window range.



Graph $f\,3(x) = \cos(x)$ and change the graphing mode to dashed blue. This will trace over $f\,2(x)$, so the GDC confirms your analytical result.

This graphical method can be used to confirm all the results of differentiation. You need to input the function that you need to differentiate in $f\,1$ and your answer in $f\,3$.

nDeriv and *centralDiff* are equivalent commands.

In a similar way you can find that $\dfrac{d(\cos x)}{dx} = -\sin x$.

You can confirm this in Exercise 9A.

Exercise 9A

1 Use a graphical method to confirm these results.

All of these results can be obtained by finding derivatives from first principles but some of the calculations are challenging.

a $\dfrac{d(\cos x)}{dx} = -\sin x$

b $\dfrac{d\left(\sin \frac{x}{2}\right)}{dx} = \dfrac{1}{2}\cos\dfrac{x}{2}$

c $\dfrac{d(\cos 3x)}{dx} = -3\sin 3x$

d $\dfrac{d(\sin(2x-1))}{dx} = 2\cos(2x-1)$

e $\dfrac{d(\tan x)}{dx} = \sec^2 x$

f $\dfrac{d(\cot x)}{dx} = -\csc^2 x$

→ $\dfrac{d}{dx}(\sin x) = \cos x$

$\dfrac{d}{dx}(\cos x) = -\sin x$

Since you know the derivatives of sine and cosine functions you can find the derivative of the tangent function by using the quotient rule.

Example 2

Find the derivative of $f(x) = \tan x$ by using the quotient rule.

Answer

$$f(x) = \tan x = \frac{\sin x}{\cos x} \Rightarrow$$

Rewrite tangent as quotient of sine and cosine.

$$f'(x) = \frac{\frac{d(\sin x)}{dx} \cdot \cos x - \sin x \cdot \frac{d(\cos x)}{dx}}{(\cos x)^2}$$

Apply the quotient rule.

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Use the derivatives of sin and cos and simplify the expression.

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Use the fundamental trigonometric identity $\cos^2 x + \sin^2 x = 1$.

Example 3

Find the derivative of the function $f(x) = \sin x \cdot \cos x$.

Answer

Solution 1

$$f'(x) = \frac{d(\sin x)}{dx} \cdot \cos x + \sin x \cdot \frac{d(\cos x)}{dx}$$

Use the product rule.

$$= \cos x \cdot \cos x + \sin x \cdot (-\sin x)$$

Use the derivatives of sin and cos.

$$= \cos^2 x - \sin^2 x$$

Simplify the expression.

Solution 2

$$f(x) = \sin x \cdot \cos x$$

$$= \frac{1}{2}\sin 2x \Rightarrow$$

Use $\sin 2\theta = 2 \cdot \sin\theta \cdot \cos\theta$ to rewrite the product.

$$f'(x) = \frac{1}{2} \cdot \frac{d(\sin 2x)}{dx}$$

Use the chain rule.

$$= \frac{1}{2} \cdot \cos 2x \cdot 2 = \cos 2x$$

Simplify the expression.

These two results from the solutions are equivalent since the trigonometric formula for a cosine of double angle is $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

This example looks at the derivative of a reciprocal trigonometric function.

Example 4

Find the derivative of $f(x) = \sec x$.

Answer

$$f(x) = \sec x = (\cos x)^{-1} \Rightarrow$$

Secant is reciprocal cosine.

$$f'(x) = -1 \cdot (\cos x)^{-2} \cdot (-\sin x)$$

Use the chain rule.

$$= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$$

Simplify and rewrite.

Write the reciprocal functions as composite functions, and then apply the chain rule.

Exercise 9B

1 Differentiate with respect to x.

a $y = \cot x$ **b** $y = \csc x$ **c** $y = \sin 3x$

d $y = \tan(5x - 3)$ **e** $y = \cos(8 - 3x)$ **f** $y = \csc\left(\frac{x-3}{4}\right)$

g $y = \cot\left(\frac{7-2x}{13}\right)$

2 Use the chain rule to find $\frac{dy}{dx}$.

a $y = \sin(x^5 - 3)$ **b** $y = \cos(e^x)$

c $y = \csc(x^2 + 11)$ **d** $y = \cot(4x^3 - 2x^2 + 7x + 17)$

e $y = \tan(\ln(2x + 1))$ **f** $y = \sec\left(\sqrt{e^x + 1}\right)$

g $y = \sin(\cos(\tan x))$

Now you can find derivatives of composite functions, products and quotients of trigonometric and other functions.

Example 5

Find the derivatives with respect to x of:

a $y = x^2 \sin 2x$ **b** $y = \frac{e^{3x-1}}{\cos x}$ **c** $y = \ln(x^2 + 1)\tan\frac{x}{2}$

Answers

a $y = x^2 \sin 2x$ — *Use product rule.*

$\Rightarrow y' = 2x\sin 2x + x^2\cos 2x \cdot 2$

$= 2x\sin x^2 + 2x^2\cos 2x$ — *Simplify the expression.*

$[= 2x(\sin 2x + x\cos 2x)]$

b $y = \frac{e^{3x-1}}{\cos x}$ — *Use the quotient rule.*

$$\Rightarrow y' = \frac{e^{3x-1}\cdot 3\cdot\cos x - e^{3x-1}(-\sin x)}{\cos^2 x}$$

$$= \frac{3e^{3x-1}\cos x + e^{3x-1}\sin x}{\cos^2 x}$$

c $y = \ln(x^2 + 1)\tan\frac{x}{2} \Rightarrow$

$$y' = \frac{2x}{x^2+1}\tan\frac{x}{2} + \ln(x^2+1)\sec^2\frac{x}{2}\cdot\frac{1}{2}$$ — *Use product rule.*

$$= \frac{2x\tan\frac{x}{2}}{x^2+1} + \frac{1}{2}\ln(x^2+1)\sec^2\frac{x}{2}$$ — *Simplify the expression.*

Notice that sometimes you can leave answers in a factorized form, especially if you need to do further calculations on the derivatives.

Some derivative expressions are very long and it may not be possible to simplify them.

Gradients of curves

You can use the derivative to determine the gradient of a function at a given point.

See Chapter 4.

Example 6

Find the gradient of the curve $y = 3x\cos(2x)$ at the point $\left(\frac{5\pi}{6}, \frac{5\pi}{4}\right)$.

Answer

$y = 3x\cos(2x)$

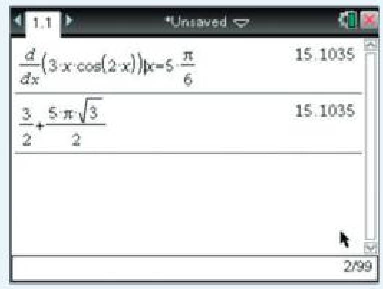
$\Rightarrow y' = 3\cos(2x) + 3x(-\sin(2x)\cdot 2)$

$= 3(\cos(2x) - 2x\sin(2x))$

$$y'\left(\frac{5\pi}{6}\right) = 3\left(\cos\left(2\times\frac{5\pi}{6}\right) - 2\cdot\frac{5\pi}{6}\cdot\sin\left(2\cdot\frac{5\pi}{6}\right)\right)$$

$$= 3\left(\underbrace{\cos\left(\frac{5\pi}{3}\right)}_{\frac{1}{2}} - \frac{5\pi}{3}\cdot\underbrace{\sin\left(\frac{5\pi}{3}\right)}_{-\frac{\sqrt{3}}{2}}\right) = \frac{3}{2} + \frac{5\pi\sqrt{3}}{2}$$

This result can be obtained from the GDC. Notice that the GDC gives a decimal form, so you need to verify our answer.



Sometimes it is easier to first rewrite and simplify the trigonometric expression and then to differentiate it.

Example 7

Find the derivative of $f(x) = (1 + \tan^2 x)\cdot(1 - \sin^2 x)$

Answer

Solution 1 - differentiate first

$$f'(x) = \frac{d(1+\tan^2 x)}{dx}\cdot(1-\sin^2 x) + (1+\tan^2 x)\cdot\frac{d(1-\sin^2 x)}{dx}$$

Use the product rule.

$$= (2\tan x\cdot\sec^2 x)\cdot(1 - \sin^2 x) + (1 + \tan^2 x)\cdot(-2\sin x\cdot\cos x)$$

$$= 2\frac{\sin x}{\cos^3 x}\cdot\cos^2 x + \frac{1}{\cos^2 x}\cdot(-2\sin x\cdot\cos x)$$

$$= 2\frac{\sin x}{\cos x} - 2\frac{\sin x}{\cos x} = 0$$

Use trigonometric identities to simplify.

Solution 2 - simplify first

$f(x) = (1 + \tan^2 x)\cdot(1 - \sin^2 x)$

$= \sec^2 x\cdot\cos^2 x = 1 \Rightarrow$

Use trigonometric identities.

$$f'(x) = \frac{d(1)}{dx} = 0$$

Differentiate the constant.

Exercise 9C

1 Use product and quotient rules to differentiate with respect to x.

a $y = (2x - 1)\cos x$ **b** $y = (3x - x^2)\sin 2x$ **c** $y = e^{1-x}\tan x$

d $y = \dfrac{\sin x}{x}$ **e** $y = \dfrac{2x+3}{\sin 2x}$ **f** $y = \dfrac{\tan x}{\sqrt{2-x}}$

2 Find the gradient of the curve at the given point.

a $y = \sin 2x$, at $x = \dfrac{\pi}{6}$ **b** $y = \cos 3x$, at $x = \dfrac{7\pi}{12}$

c $y = \tan(-x)$, at $x = \dfrac{5\pi}{4}$ **d** $y = (x - 2)\sin x$, at $x = 0$

e $y = -3x\cos x$, at $x = \dfrac{\pi}{2}$ **f** $y = x^2\tan x$, at $x = \dfrac{3\pi}{4}$

g $y = e^x\sec x$, at $x = 0$

3 Find the derivatives of these expressions with respect to the variable indicated.

a $y = \sin^2\alpha + \cos^2\alpha$, α **b** $y = \dfrac{\tan\beta}{\sin\beta}$, β

c $y = \dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$, θ **d** $y = \dfrac{\sin\rho + \sin 2\rho}{\cos\rho + \cos 2\rho}$, ρ

e $y = \dfrac{(\sin\varphi\sin 2\varphi - \cos\varphi)\sec\varphi}{\sin\varphi - \cos\varphi}$, φ

Derivatives of inverse trigonometric functions

To differentiate the inverse trigonometric functions, $y = \arcsin(x)$, $y = \arccos(x)$ and $y = \arctan(x)$ introduced in Chapter 8, you can proceed as follows:

Let $y = \arcsin x$ then $x = \sin y$ so $\dfrac{dx}{dy} = \cos y$

Using $\dfrac{dx}{dy} = \dfrac{1}{\frac{dx}{dy}}$ and $\sin^2 x + \cos^2 x = 1$ gives

$$\frac{dx}{dy} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

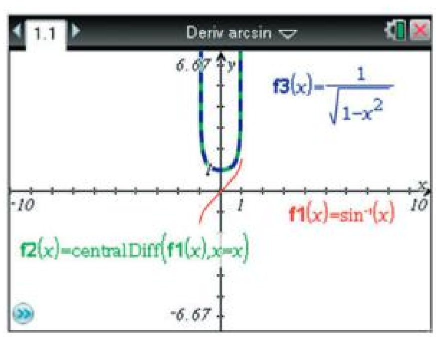
arcsin x:
$[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
In this interval, the cosine value will always be positive so don't consider the negative square root.

Using this result and the chain rule you can find a general formula.

If $y = \arcsin\dfrac{x}{a}$ $\dfrac{dx}{dy} = \dfrac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \dfrac{1}{a} = \dfrac{1}{\sqrt{a^2-x^2}}$

→ If $y = \arcsin x$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

If $y = \arcsin\dfrac{x}{a}$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2-x^2}}$



Example 8

Find the derivative of the function $g(x) = \arctan x$, $x \in \mathbb{R}$.

Answer

$\tan(g(x)) = x$	*Composition of a function and its inverse function gives the identity function.*
$\Rightarrow \sec^2(g(x)) \cdot g'(x) = 1$	*Differentiate with respect to x by using the chain rule.*
$\Rightarrow g'(x) = \cos^2(\arctan(x))$	*Rearrange to make g′(x) the subject.*
$= \dfrac{1}{1+\tan^2(\arctan(x))}$	*Use the trigonometric identity* $\cos^2\theta = \dfrac{1}{1+\tan^2\theta}$ *and simplify.*
$= \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$	

Exercise 9D

1 Find the derivatives of

a $f(x) = \arccos x$ **b** $f(x) = \arcsin 3x$ **c** $f(x) = \arctan(2x+1)$

2 Find $\dfrac{dy}{dx}$.

a $y = 2x \arcsin x$ **b** $y = \dfrac{\arccos x}{x}$ **c** $y = (2x+1)\arctan x$

d $y = \sqrt{1-x^2}\arcsin x$ **e** $y = (4x^2+1)\arctan 2x$

3 Show that these identities are valid and explain why:

a $\dfrac{d(\arcsin x + \arccos x)}{dx} = 0$

b $\dfrac{d(\arctan x + \arctan(-x))}{dx} = 0$

c $\dfrac{d\left(2\arctan x - \arcsin\left(\dfrac{2x}{x^2+1}\right)\right)}{dx} = 0$

4 Differentiate with respect to x these implicitly defined functions.

See Section 4.8

a $x = \sin y$ **b** $x + y = \tan y$

c $x + \sin x = y + \cos y$ **d** $e^{\sin y} = x^2$

e $\cos y = \dfrac{x}{y}$ **f** $\ln(xy) = \tan 2y$

Tangents and normals

As discussed in Section 4.2, equations of a tangent and a normal to the curve $y = f(x)$ at the point (x_1, y_1) are given by

$y = f'(x_1)(x - x_1) + y_1$ and $y = -\frac{1}{f'(x_1)}(x - x_1) + y_1$ respectively.

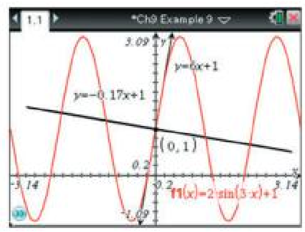
Example 9

Given the function $f(x) = 2\sin(3x) + 1$, $-\pi < x < \pi$, find the equation of:

a the tangent **b** the normal at the point where the graph of the function meets the y-axis.

These results use the point-slope form of a straight line.

Answers	
$x = 0 \Rightarrow f(0) = 2\sin(3 \cdot 0) + 1$ $= 1 \Rightarrow \mathrm{P}(0, 1)$	*Calculate the y-coordinate of the point of intersection.*
$f'(x) = 2\cos(3x) \cdot 3 = 6\cos(3x)$ $x = 0 \Rightarrow f'(0) = 6\cos(3 \cdot 0) = 6$	*Calculate the gradient of the curve at the point.*
a Tangent: $y = f'(0)(x - 0) + 1$ $y = 6x + 1$	*Apply the formula for the equation of a tangent.*
b Normal: $y = -\frac{1}{f'(0)}(x - 0) + 1$ $y = -\frac{1}{6}x + 1$	*Apply the formula for the equation of a normal.* *You can confirm our results on the GDC.*



In this example you use **implicit** differentiation.

Example 10

Find the equation of the normal to the curve $y + 2x = \cos(xy)$ at the point $\mathrm{P}(0, 1)$ in the form $y = mx + c$.

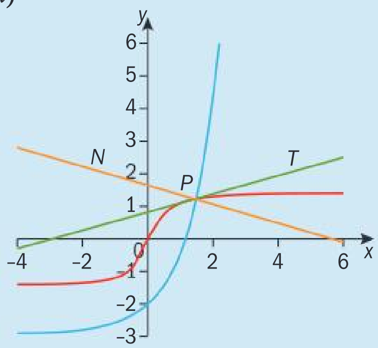
This is a good example of an exam-style question.

Answer	
$y + 2x = \cos(xy)$	
$\frac{\mathrm{d}y}{\mathrm{d}x} + 2 = -\sin(xy)\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right)$	*Differentiate the implicitly defined function with respect to x.*
$m_1 + 2 = -\sin(0 \cdot 1)(1 + 0 \cdot m_1)$ $m_1 + 2 = 0 \Rightarrow m_1 = -2$	*Find the slope of the curve at the given point.*
$m_2 = \frac{1}{2} \Rightarrow N: y = \frac{1}{2}(x - 0) + 1$ $\Rightarrow N: y = \frac{1}{2}x + 1$	*Apply the formula for a normal.*

In this next example you need to use a GDC because the equation cannot be solved using the algebraic methods you have learned so far.

Example 11

Graphs of the functions $f(x) = \arctan(2x)$ and $g(x) = e^x - 3$ are given in the diagram. The point P is the point of intersection between the curves, the line T is the tangent to f at P, and N is the normal to g at P.



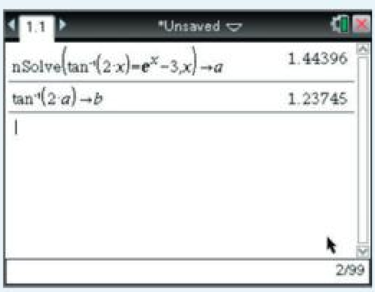
a Find the coordinates of P.

b Find the area of the triangle enclosed by the tangent T, normal N and the x-axis.

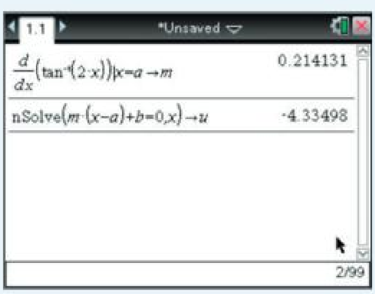
Answers

a P(1.44, 1.24), given correct to 3 sf.

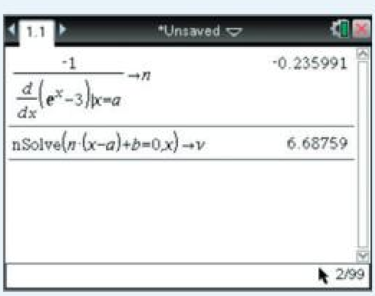
b Area = 6.82



The calculator working is shown here.
(a, b) are the coordinates of P.



The slope of the tangent T is stored in m and the zero of the tangent T is stored in u.



The slope of the normal N is stored in n and the zero of the normal N is stored in v.

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The base of the triangle is calculated by adding the absolute value of u (since u < 0) and v, whilst the height of the triangle is the y-coordinate of the point P.

Exercise 9E

1 Given a function $f(x)$ and the point P, find the equation of the tangent.

a $f(x) = \tan(3x)$, $P(0, 0)$

b $f(x) = \sin(2x) - 1$, $P\left(\frac{\pi}{3}, y\right)$

c $f(x) = 2\cos\left(\frac{x}{2}\right) - e^{2x}$, $P(0, 1)$

d $f(x) = \ln\left(\tan\left(\frac{x}{3}\right)\right) + 2$, $P\left(\frac{3\pi}{4}, y\right)$

2 Given a function $f(x)$ and the point P find the equation of the normal.

a $f(x) = \cos(2x)$, $P(0, 1)$

b $f(x) = \tan(4x)$, $P\left(\frac{\pi}{16}, y\right)$

c $f(x) = 2e^x \sin\left(\frac{x}{2}\right)$, $P(0, y)$

d $f(x) = x\cos(2x) - 3$, $P\left(\frac{\pi}{2}, y\right)$

You may need to use your GDC for some of these equations.

EXAM-STYLE QUESTIONS

3 Given a curve $\ln(x) = \tan y$ find the equation of tangent at the point $P(0, 1)$.

4 Given a curve $y + y^2 = \sin 2x$ find the equation of normal at the point $P(0, -1)$.

5 Consider the curves $y = \cos(x^2)$ and $y = e^{x^2} - 2$.

a Find the point of intersection between the curves that lies in the first quadrant.

b Find the equations of tangents to both curves at the point of intersection.

c Find the angle between the tangents in part **b**.

6 Find the area of a triangle enclosed by the y-axis, the tangent and the normal to the curve $e^y = \sin x + 1$ at the point $P(-\pi, 0)$.

Higher derivatives of trigonometric functions

Higher derivatives were discussed in Chapter 4. You can now investigate them for trigonometric functions.

Example 12

$y = x\tan x$

a Find $\dfrac{d^2y}{dx^2}$

b Calculate the exact value of second derivative at $x = \dfrac{\pi}{3}$.

Answers

a $y = x\tan x \Rightarrow \dfrac{dy}{dx} = \tan x + x \times \sec^2 x$

Find the first derivative using the product rule.

$\Rightarrow \dfrac{d^2y}{dx^2} = \sec^2 x + \sec^2 x + x \times 2\sec x \times \sec x \times \tan x$

Differentiate to find the second derivative.

$= 2\sec^2 x + 2x\sec^2 x\tan x$

$= 2\sec^2 x\,(1 + x\tan x)$

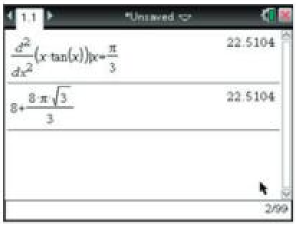
Simplify.

b $\dfrac{dy^2}{dx^2}\left(\dfrac{\pi}{3}\right) = 2\sec^2\left(\dfrac{\pi}{3}\right)\left(1 + \dfrac{\pi}{3}\tan\left(\dfrac{\pi}{3}\right)\right)$

Substitute $x = \dfrac{\pi}{3}$

$= 8 \times \left(1 + \dfrac{\pi\sqrt{3}}{3}\right) = 8 + \dfrac{8\pi\sqrt{3}}{3}$

You can check this on a GDC



This example shows an interesting connection between the trigonometric functions sine and cosine.

You may already have noticed that their graphs are similar.

Example 13

Find the pattern that emerges in higher derivatives of the function $f(x) = \sin x$.

Answer

$f(x) = \sin x$

$\Rightarrow f'(x) = \cos x \quad \Rightarrow f^{(3)}(x) = -\cos x \quad \Rightarrow f^{(5)}(x) = \cos x$

$\Rightarrow f''(x) = -\sin x \quad \Rightarrow f^{(4)}(x) = \sin x$

Notice that you completed a cycle and began the same cycle again.

$$f^{(n)}(x) = \begin{cases} \cos x, n = 4k-3 \\ -\sin x, n = 4k-2 \\ -\cos x, n = 4k-1 \\ \sin x, n = 4k \end{cases}, \quad k \in \mathbb{Z}^+$$

The graph of the cosine function is related to the graph of the sine function by a horizontal translation of $\dfrac{\pi}{2}$ units:

$$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), \ n = 0, 1, 2, \ldots,$$

where the 0th derivative is the original function itself.

You can prove this formula using **mathematical induction**.

This is very similar to the emerging pattern of the powers of the imaginary unit i.
Later on in Chapter 12 you will use the polar form of a complex number to explain this emerging pattern.

Exercise 9F

1 Find the exact value of the second derivative for these functions at the given value of x.

a $f(x) = \tan x,\ x = \frac{\pi}{3}$

b $f(x) = x \sin x,\ x = 0$

c $f(x) = (x^2 + 1)\cos x,\ x = 0$

d $f(x) = \sqrt{x}\ \cos\frac{x}{2},\ x = 1$

e $f(x) = e^x \sin 2x,\ x = \frac{\pi}{4}$

f $f(x) = 2x \sec x,\ x = \pi$

Check all your answers by using a calculator.

2 Describe any emerging patterns when successively differentiating these functions:

a $f(x) = \cos x$

b $g(x) = \sin 3x$

c $h(x) = \cos(ax + b),\ a, b \in \mathbb{R},\ a \neq 0.$

3 A function $f(x) = \sin 2x$ defines a sequence in such a way that the general term of the sequence is defined by the formula

$$a_n = f^{(n-1)}\left(\frac{\pi}{8}\right),\ n = 1, 2, 3, \ldots,$$

where the 0th derivative is the original function itself.

a Write the first four terms of the sequence.

b Find the sum of the first 10 terms of the sequence.

EXAM-STYLE QUESTION

4 Prove the following statements by mathematical induction:

a $f(x) = \sin x \Rightarrow f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right),\ n = 0, 1, 2, \ldots$

b $g(x) = \cos x \Rightarrow g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right),\ n = 0, 1, 2, \ldots$

where the 0th derivative is the original function itself.

You should now be able to differentiate a variety of trigonometric functions. These results will be useful when doing further integrals.

Example 14

Find the derivatives of

a $f(x)=\ln\left(\frac{1+\sin x}{\cos x}\right)+c$

b $f(x)=\ln\left(\frac{\cos x}{1-\sin x}\right)$

c $f(x)=\ln(\tan x+\sec x)$

Plot and compare the graphs of the three functions. What do you notice?

Answers

a $f'(x)=\frac{1}{\frac{1+\sin x}{\cos x}}\cdot\frac{-\cos x\cdot\cos x-(1+\sin x)\cdot(-\sin x)}{\cos^2 x}$

$=\frac{\cos x}{1+\sin x}\times\frac{\cos^2 x+\sin x+\sin^2 x}{\cos^2 x}$

$=\frac{1}{1+\sin x}\cdot\frac{1+\sin x}{\cos x}$

$=\sec x$

Use $\frac{d}{dx}(\ln x)=\frac{1}{x}$ and product rule.

b $f'(x)=\frac{1}{\frac{\cos x}{1-\sin x}}\cdot\frac{-\sin x\cdot(1-\sin x)-\cos x\cdot(-\cos x)}{(1-\sin x)^2}$

$=\frac{1-\sin x}{\cos x}\cdot\frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2}$

$=\frac{1}{\cos x}\cdot\frac{1-\sin x}{1-\sin x}$

$=\sec x$

$\frac{1}{\cos x}=\sec x$

c $f'(x)=\frac{1}{\tan x+\sec x}\cdot(\sec^2 x+\sec x\tan x)$

$=\frac{\sec x\cdot(\sec x+\tan x)}{\tan x+\sec x}$

$=\sec x$

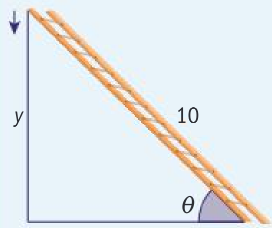
9.2 Related rates of change with trigonometric expressions

The derivative of a function, $y = f(x)$, measures the rate of change of the independent variable, y, with respect to a change in the dependent variable, x.

Example 15

A 10 m long industrial ladder is leaning against a wall on a building construction site. It starts to slip down the wall at a rate of 0.5 ms^{-1}. How fast is the angle between the ladder and the ground changing when the vertical height of the ladder is 8 m?

Answer

	*Sketch a diagram representing the given information.*
$\frac{dy}{dt} = -0.5$ and $y = 8$ You need to find $\frac{d\theta}{dt}$.	*Write down the given information, and what you are asked to find.*
$\sin\theta = \frac{y}{10}$	*Identify the relationship between the height of the ladder and the angle.*
$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}$	*Differentiate implicitly with respect to time.*
$y = 8 \Rightarrow \sin\theta = \frac{8}{10} = \frac{4}{5}$	*Evaluate sin θ at y = 8.*
$\Rightarrow \cos\theta = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$	*Use* $\cos\theta = \sqrt{1 - \sin^2\theta}$.
$\frac{3}{5} \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \left(-\frac{1}{2}\right) \Rightarrow \frac{d\theta}{dt}$	*Substitute and solve.*
$= -\frac{1}{12}$ cs^{-1}	*Notation °s^{-1} denotes radians per second.*
So the angle is decreasing at a rate of $\frac{1}{12}$ cs^{-1} or 4.77° s^{-1}.	*Interpret your answer in the context of the problem.* *°g^{-1} denotes degrees per second*

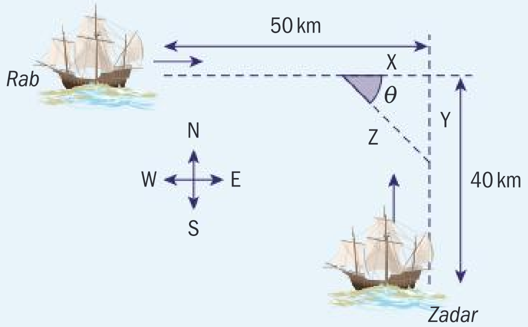
Example 16

There are two ships at sea, *Zadar* and *Rab*. At a given moment *Zadar* is 40 km south and 50 km east of *Rab*. *Zadar* sails north at a rate of 12 kmh^{-1}, whilst *Rab* sails east at a rate of 15 kmh^{-1}.

a How fast are the two ships approaching each other after 2 hours?

b How fast is the bearing of *Zadar* from *Rab* changing after 2 hours?

Answers



Sketch a diagram representing the given information.

a Given that $x = 50 - 15t \Rightarrow \frac{dx}{dt} = -15$

and $y = 40 - 12t \Rightarrow \frac{dy}{dt} = -12$

you need to find $\frac{dz}{dt}$.

Write down the given information, and what we are asked to find.

$z^2 = x^2 + y^2$

Identify the relationship between the variables using Pythagoras' theorem.

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

Differentiate as an implicit function with respect to time.

$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

Simplify.

$x = 50 - 15 \cdot 2 = 20$

$y = 40 - 12 \cdot 2 = 16$

$z = \sqrt{20^2 + 16^2} = 25.612\ldots$

Calculate x, y and z when t = 2.

$25.6\frac{dz}{dt} = 20 \cdot (-15) + 16 \cdot (-12)$

Substitute and solve.

$\frac{dz}{dt} = \frac{-492}{25.6} = -19.2$ kmh^{-1}, correct to 3 sf.

Interpret your answer in the context of the problem.

So the distance between the two ships is decreasing at a rate of 19.2 kmh^{-1}.

Notice that the bearing of Zadar from Rab is 90° + θ, therefore the bearing is changing at the same rate as the angle θ itself.

b Given that $\frac{dx}{dt} = -15$ and $\frac{dy}{dt} = -12$

you need to find $\frac{d\theta}{dt}$.

Write down the given information, and what you are asked to find.

Continued on next page

$\tan\theta = \frac{y}{x}$	*Identify the relationship between the variables.*
$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{\frac{dy}{dt}\cdot x - y\cdot\frac{dx}{dt}}{x^2}$	*Differentiate as an implicit function with respect to time.*
$x = 50 - 15.2 = 20$ $y = 40 - 12.2 = 16$	*Calculate x, y and z when t = 2.*
$\tan\theta = \frac{16}{20} = \frac{4}{5} \Rightarrow \sec^2\theta = 1 + \left(\frac{4}{5}\right)^2 = \frac{41}{25}$ $\frac{41}{25}\cdot\frac{d\theta}{dt} = \frac{-12\cdot 20 - 16\cdot(-15)}{20^2}$	*Substitute and solve.*
$\frac{d\theta}{dt} = 0°\,h^{-1}$ So the bearing is not changing at all.	*Notice that the bearing from one ship to another is not changing since the ratio of the initial positions of the ships is equal to the ratio of their corresponding velocities.*

Example 17

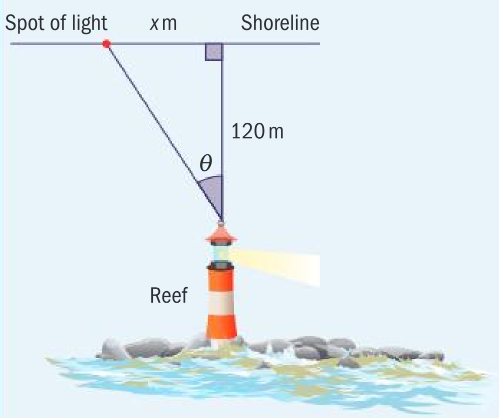
A reef 120 m from a straight shoreline is marked by a beacon which rotates six times per minute.

a How fast is the beam moving along the shoreline at the moment when the light beam and the shoreline are at right angles?

b How fast is that beam moving along the shoreline when the beam hits the shoreline 50 m from the point on the shoreline closest to the lighthouse?

c What is happening to the velocity of the light beam when the ray is parallel to the shoreline?

Answer



Sketch a diagram representing the given information.

Assume the beacon is at the same height as the shoreline.

Continued on next page

$\frac{d\theta}{dt} = \frac{6 \cdot 2\pi}{60} = \frac{\pi}{5}$ cs^{-1}, from the speed of the beam and you need to find $\frac{dx}{dt}$.	*Write down the given information, and what we are asked to find.*
$\tan\theta = \frac{x}{120}$	*Identify the relationship between the variables.*
$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{120} \cdot \frac{dx}{dt}$	*Differentiate as an implicit function with respect to time.*
$\theta = 0$	*When the beam is at 90° to the shoreline, $\theta = 0$.*
$\sec^2 0 \cdot \frac{\pi}{5} = \frac{1}{120} \cdot \frac{dx}{dt}$	*Substitute and solve.*
$\Rightarrow \frac{dx}{dt} = 24\pi = 75.4$ ms^{-1}, (3 sf)	
The beam is moving along the shoreline at 75.4 ms^{-1}	*Interpret your answer in the context of the problem.*
b $\sec^2\theta \frac{d\theta}{dt} = \frac{1}{120} \cdot \frac{dx}{dt}$	*Start with the derivative again.*
$\tan\theta = \frac{50}{120} = \frac{5}{12}$	
$\Rightarrow \sec^2\theta = 1 + \left(\frac{5}{12}\right)^2 = \frac{169}{144}$	*Use $\sec^2\theta = 1 + \tan^2\theta$*
$\frac{169}{144} \cdot \frac{\pi}{5} = \frac{1}{120} \cdot \frac{dx}{dt}$	*Substitute and solve.*
$\Rightarrow \frac{dx}{dt} = \frac{169}{6}\pi = 88.5$ ms^{-1} (3 sf)	
The beam is moving along the shoreline at 88.5 ms^{-1}	*Interpret your answer in the context of the problem.*
c As the light ray approaches the position parallel to the shoreline, then angle $\theta \to \frac{\pi}{2} \Rightarrow \sec^2\theta \to \infty$ $\Rightarrow$ velocity $\to \infty$ according to the model.	

Exercise 9G

1 A 2.5 m long ladder is leaning against a wall on a building construction site. It starts to slip horizontally along the ground at a rate of 4 cms^{-1}. How fast is the angle between the ladder and the ground changing when the bottom of the ladder is 1 m away from the wall?

2 Two planes A and B are flying to their destinations. At a given moment plane A is 25 km north and 18 km east of plane B. Plane A flies west at a speed of 200 ms^{-1}, whilst plane B flies direction north at a speed of 160 ms^{-1}.

- **a** How fast are the two planes approaching each other after 0.5 minutes?
- **b** How fast is the bearing of plane B from plane A changing after 1 minute?

3 A professional cameraman on a safari is at a spot 30 metres from a tree, following birds that are moving at a speed of 95 kmh^{-1}. The birds are moving perpendicularly to the line joining the tree and the spot. How fast does he need to turn the camera when filming a bird:

- **a** that is directly in front of the camera
- **b** one second later?

4 An isosceles triangle with the sides 6, 5 and 5 cm is going through a transformation where the longest side is decreasing at a rate of 0.1 cms^{-1}.

- **a** Find the rate of change of the angle opposite to the decreasing side at the start.
- **b** Find the rate of change of the angle opposite to the decreasing side when the triangle is equilateral.

5 A balloon has a spherical shape. There is a hole in the balloon and the air is leaking at 2 $cm^3\,min^{-1}$.

- **a** Find the rate at which the radius is decreasing when $r = 12$ cm.
- **b** Find the rate at which the surface area is decreasing when $r = 4$ cm.

6 A scientist is pointing with a laser to a flying object whose trajectory is vertically above her. The object is flying at a constant height of 10 000 m and maintaining a speed of 1025 kmh^{-1}. Find the rate in degrees per second of the rotating laser when

- **a** the horizontal distance of the object is 8 km from the scientist,
- **b** the object is directly above the scientist.

7 A train is moving along a straight track at 75kmh^{-1} due east. A camera positioned 2 km from the track west of the train is focused on the train.

- **a** Find the rate of change of the distance between the camera and the train when the train is 4 km from the camera.
- **b** At what rate is the camera rotating when the train is 4 km from the camera? Give your answer in degrees per second correct to the nearest tenth of a degree.

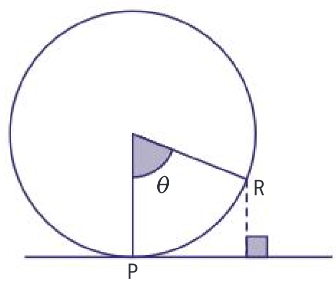
8 An observer is watching a fireworks rocket from a distance of 10 metres. He uses a laser to measure the distance to the rocket which is changing at a rate of 5 ms^{-1}. At a particular moment the distance measured to the rocket is 20 metres.

a Find the rate of increase of the angle of elevation at that moment.

b Find the speed of the rocket at that moment.

EXAM-STYLE QUESTION

9 A Ferris wheel 15 metres in diameter makes two revolutions per minute. Assume that the wheel is tangential to the ground and let P be the point of tangency.



At what rate is the distance between P and a rider R changing, when she is 5 metres above the ground and going up?

9.3 Integration of trigonometric functions

Basic integrals of trigonometric functions

Since integration is a process of finding the antiderivative of the integrand function you can deduce some standard integrals:

→ $\int \cos x \, dx = \sin x + c, c \in \mathbb{R}$ since $\frac{d(\sin x)}{dx} = \cos x$

$\int \sin x \, dx = -\cos x + c$ since $\frac{d(-\cos x)}{dx} = \sin x$

$\int \sec^2 x \, dx = \tan x + c$ since $\frac{d(\tan x)}{dx} = \sec^2 x$

More integrals of the trigonometric functions will be found later using methods of substitution and integration by parts.

Chapter 7 introduces the compound formula, and here it can be used to obtain other antiderivatives.

→ $\int f(ax + b) dx = \frac{1}{a} F(ax + b) + c$

You can find all the integrals of the form $\int f(ax + b) \, dx$ where f can be any of the three functions mentioned above.

Example 18

Find these integrals.

a $\int \cos 5x \, dx$ **b** $\int 2\sin(5-3x)\,dx$ **c** $\int \frac{1}{2}\sec^2\frac{x}{4}\,dx$

Answers

a $\int \cos 5x\,dx = \frac{1}{5}\sin 5x + c$	*Use compound formula.*
b $\int 2\sin(5-3x)\,dx$	
$= -\frac{2}{3}(-\cos(5-3x)) + c$	*Use compound formula.*
$= \frac{2}{3}\cos(5-3x) + c$	*Simplify the expression.*
c $\int \frac{1}{2}\sec^2\frac{x}{4}\,dx = \frac{\frac{1}{2}}{\frac{1}{4}}\tan\frac{x}{4} + c$	*Use compound formula.*
$= 2\tan\frac{x}{4} + c$	*Simplify the expression.*

There are some more complicated integrals that can be determined using the trigonometric identities from Chapter 8.

Example 19

Use trigonometric identities to find these integrals.

a $\int 2\sin x\cos x\,dx$ **b** $\int (2\cos^2 3x - 1)\,dx$ **c** $\int \left(\tan^2\frac{x}{3} + 1\right)dx$

Recap all the trigonometric identities from Chapter 8.

Answers

a $\int 2\sin x\cos x\,dx = \int \sin 2x\,dx$	*Use: $\sin 2\theta = 2\sin\theta\cos\theta$.*
$= -\frac{1}{2}\cos 2x + c$	*Use compound formula.*
b $\int (2\cos^2 3x - 1)\,dx$	*Use: $\cos 2\theta = 2\cos^2\theta - 1$.*
$= \int \cos 6x\,dx$	
$= \frac{1}{6}\sin 6x + c$	*Use compound formula.*
c $\int \left(\tan^2\frac{x}{3} + 1\right)dx$	*Use $\tan^2\theta + 1 = \sec^2\theta$.*
$= \int \sec^2\frac{x}{3}\,dx$	*Use compound formula.*
$= 3\tan\frac{x}{3} + c$	

Exercise 9H

1 Find these integrals.

a $\int \sin 3x \, dx$ **b** $\int \cos(2x+1)\, dx$ **c** $\int \sec^2 3x \, dx$

d $\int \sec^2(1-x)\, dx$ **e** $\int \sin\left(\frac{5x-1}{3}\right) dx$ **f** $\int \cos\left(\frac{3x+2}{7}\right) dx$

2 Solve these integrals using trigonometric identities.

a $\int (1-2\cos^2 x)dx$ **b** $\int (1+\tan^2 x)dx$

c $\int \sin^2 x \, dx$ **d** $\int \cos^2 x \, dx$

e $\int (1-2\sin^2(2x))\, dx$ **f** $\int (2+2\tan^2(5x))\, dx$

g $\int (1+\tan^2 x)(1-\sin^2 x)\, dx$ **h** $\int 4\sin^2 x \cos^2 x \, dx$

When you integrate a linear combination of functions you get a linear combination of the integrals.

Recap properties of integrals in Chapter 4.

Example 20

Find these integrals.

a $\int (4-x^3+5\cos 2x)\, dx$

b $\int (7e^x-3x^2+1-2\sin 2x)\, dx$

Answers

a $\int (4-x^3+5\cos 2x)\, dx$ — *Integrate.*

$= 4x - \frac{x^4}{4} + 5 \cdot \frac{1}{2}\sin 2x + c$ — *Simplify. Don't forget the constant.*

$= 4x - \frac{x^4}{4} + \frac{5}{2}\sin 2x + c$

b $\int (7e^x-3x^2+1-2\sin 2x)\, dx$ — *Integrate.*

$= 7e^x - 3 \cdot \frac{x^3}{3} + x - 2\frac{1}{2}(-\cos 2x) + c$ — *Simplify.*

$= 7e^x - x^3 + x + \cos 2x + c$

Exercise 9I

1 Integrate these functions.

a $f(x) = 2\sin x - 3\cos x$

b $f(x) = x^2 - 7\sin x$

c $f(x) = 4e^x - \frac{1}{3}\sec^2 x$

d $f(x) = 1 - \sqrt{2x} + 7\sin 3x$

e $f(x) = \frac{5}{2x} + \sec^2\left(\frac{x}{3}\right)$

f $f(x) = \frac{x}{x+1} - \sin\left(\frac{3x}{4}\right)$

g $f(x) = 2^x + 5\sin\frac{x}{2} - \cos\frac{2x}{3}$

h $f(x) = 3^{-2x} - 11\sec^2(11x)$

Finding a particular antiderivative

In Chapter 7, you found that there is no unique antiderivative function, but a family of functions that are distinguished by a constant. In order to find a particular function you need to be given a certain initial condition that the function must satisfy.

Example 21

Given that $f'(x) = 2 - 3\sin x$ find the function f such that $f(0) = -2$.

Answer

$f(x) = \int (2 - 3\sin x)\,dx$	
$= 2x - 3\cdot(-\cos x) + c$	
$= 2x + 3\sin x + c$	*Simplify.*
$f(0) = -2x$	*Use the given condition.*
$\Rightarrow 2\cdot 0 + 3\underbrace{\cos 0}_{1} + c = -2$	
$c = -5 \Rightarrow f(x) = 2x + 3\cos x - 5$	*Solve for the constant and write the function.*

When higher derivatives are involved, you need as many initial conditions as the order of the derivative given.

Example 22

Find the function g that satisfies these conditions
$g''(x) = \cos x - e^x$, $g'(0) = 2$ and $g(0) = 3$.

Answer

$g''(x) = \cos x - e^x \Rightarrow$ $g'(x) = \int(\cos x - e^x)\,dx$ $= \sin x - e^x + c_1$ $\Rightarrow g'(0) = 1 \Rightarrow \sin 0 - e^0 + c_1 = 1$	*Integrate the second derivative function to find the first derivative. Use the condition for the first derivative.*
$c_1 = 2 \Rightarrow g'(x) = \sin x - e^x + 2 \Rightarrow$	*Substitute for c_1 in $g'(x)$.*
$g(x) = \int(\sin x - e^x + 2)\,dx$ $= -\cos x - e^x + 2x + c_2$ $\Rightarrow g(0) = 3$	*Integrate the first derivative function to find $g(x)$.*
$\Rightarrow -\cos 0 - e^0 + 2\cdot 0 + c_2 = 3$ $c_2 = 5$	*Use the condition for $g(x)$.*
$\Rightarrow g(x) = -\cos x - e^x + 2x + 5$	*Substitute for c_2 in $g(x)$.*

Exercise 9J

1 Find $f(x)$ given these conditions:

a $f'(x) = 5 - 2\cos x$, $f(0) = 0$

b $f'(x) = 4x - 6\sin(2x)$, $f(0) = 1$

c $f'(x) = 3\cos x - 2\sec^2 x$, $f\left(\frac{\pi}{6}\right) = -\frac{2\sqrt{3}}{3}$

d $f'(x) = 3x^2 - 2e^x + \cos 4x$, $f(0) = -5$

e $f'(x) = \frac{3}{x} + \cos(3x) - 4$, $f(1) = \frac{\sin(1)}{3}$

f $f'(x) = \frac{7}{3-4x} - 8x + 4e^{2x-1}$, $f\left(\frac{1}{2}\right) = -1$

2 Find $f(x)$ given these conditions:

a $f''(x) = 4\sin x$, $f'\left(\frac{\pi}{3}\right) = 0$, $f(0) = 1$

b $f''(x) = 1 + \cos x$, $f'(0) = 3$, $f(1) = -\cos(1)$

c $f''(x) = e^{1-x} + \sin(1-x)$, $f'(1) = 2$, $f(1) = 2$

d $f''(x) = e^{2x} + \sin(2x) + x^3 - 2x + 1$, $f'(0) = 2$, $f(0) = 2$

Definite integrals

→ To evaluate definite integrals, apply the fundamental theorem of calculus.

$$\int f(x)\,dx = F(x) + c \Rightarrow \int_a^b f(x)dx = F(b) - F(a)$$

The fundamental theorem of calculus (FTC) was introduced in Section 7.3.

Example 23

Evaluate these integrals.

a $\displaystyle\int_0^{\pi}(x+\sin 2x)\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{2}}(e^{2x}+\cos(3x))\,dx$ **c** $\displaystyle\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}}\left(1-\frac{1}{5}\sec^2\frac{x}{10}\right)dx$

Answers

a $\displaystyle\int_0^{\pi}(x+\sin 2x)\,dx=\left[\frac{x^2}{2}-\frac{1}{2}\cos 2x\right]_0^{\pi}$

Integrate and apply the FTC.

$$=\left(\frac{\pi^2}{2}-\frac{1}{2}\underbrace{\cos 2\pi}_{1}\right)-\left(\frac{0^2}{2}-\frac{1}{2}\underbrace{\cos 0}_{1}\right)$$

$$=\frac{\pi^2}{2}-\frac{1}{2}+\frac{1}{2}=\frac{\pi^2}{2}$$

Simplify.
Check on GDC.

b $\displaystyle\int_0^{\frac{\pi}{2}}(e^{2x}+\cos(3x))\,dx$

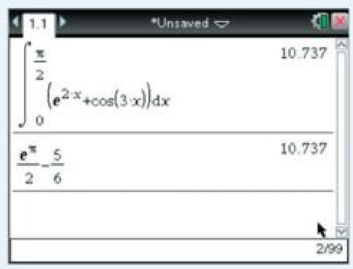
$$=\left[\frac{1}{2}e^{2x}+\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{2}}$$

Apply the FTC.

$$=\left(\frac{1}{2}e^{2\cdot\frac{\pi}{2}}+\frac{1}{3}\underbrace{\sin\left(3\cdot\frac{\pi}{2}\right)}_{-1}\right)-\left(\frac{1}{2}e^0+\frac{1}{3}\underbrace{\sin 0}_{0}\right)$$

$$=\frac{1}{2}e^{\pi}-\frac{1}{3}-\frac{1}{2}=\frac{e^{\pi}}{2}-\frac{5}{6}$$

Simplify.
Check on GDC.



c $\displaystyle\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}}\left(1-\frac{1}{5}\sec^2\frac{x}{10}\right)dx=\left[x-\frac{1}{5}\cdot\frac{1}{\frac{1}{10}}\tan\frac{x}{10}\right]_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}}$

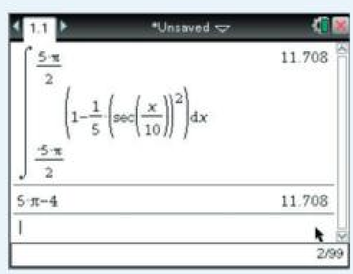
$$=\left[x-2\tan\frac{x}{10}\right]_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}}$$

Integrate and apply the FTC.
Simplify the expression before evaluating.
Apply the formula.

$$=\left(\frac{5\pi}{2}-2\tan\frac{\frac{5\pi}{2}}{10}\right)-\left(-\frac{5\pi}{2}-2\tan\left(\frac{-\frac{5\pi}{2}}{10}\right)\right)$$

$$\frac{5\pi}{2}-2\underbrace{\tan\frac{\pi}{4}}_{1}+\frac{5\pi}{2}-2\underbrace{\tan\frac{\pi}{4}}_{1}=5\pi-4$$

Simplify.
Check on GDC.



Exercise 9K

1 Evaluate these integrals. Check each solution using a GDC

a $\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} (2x - \sin x)\,dx$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (5 + \cos x)\,dx$

c $\int_{0}^{\frac{\pi}{4}} (2\sec^2 x + 1)\,dx$

d $\int_{0}^{\frac{\pi}{3}} (e^x + 2\sin x)\,dx$

e $\int_{-2\pi}^{2\pi} \left(3^{-x} + \frac{\cos\frac{x}{4}}{4}\right) dx$

f $\int_{0}^{\frac{\pi}{2}} \left(\frac{e^{3x}}{3} - \frac{2\sin 2x}{5}\right) dx$

g $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(1 - \frac{x}{2} + 2\sin 2x\right) dx$

h $\int_{0}^{\frac{\pi}{12}} (2^x + 3\cos 6x)\,dx$

i $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (x^2 + 2\sec^2 2x)\,dx$

j $\int_{0}^{\pi} (16e^{8x} + 9\sin 3x)\,dx$

9.4 Integration by substitution

This section introduces the method of **substitution**. It comes from the chain rule (composite function rule)

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

To find the integral $\int (2x + 3)^6\,dx$ it would be easier to have a single variable to the power of 6 rather than the expansion of the binomial expression $(2x + 3^6)$.

Once you write the substitution equation you need to differentiate both sides with respect to x.

$$t = 2x + 3$$

Let $\frac{dt}{dx} = 2 \Rightarrow dx = \frac{1}{2}dt$

Use the substitution to obtain the new simpler integral for t.

So the **new integral** must be in terms of the **new variable only**. Take care not to mix the variables.

$$\Rightarrow \int (2x + 3)^6\,dx = \int t^6 \times \frac{1}{2}dt = \frac{1}{2}\int t^6\,dt$$

Solve the new integral for t.

$$= \frac{1}{2} \times \frac{t^7}{7} + c$$

Substitute for t to obtain the final answer in terms of x.

$$= \frac{(2x + 3)^7}{14} + c,\ c \in \mathbb{R}$$

Example 24

Find the integral $\int (2x+1)e^{x^2+x}\,dx$.

> The initial integral is in terms of x, so the final answer must also be in terms of x.

Answer

Let $x^2 + x = u$ $2x+1 = \frac{du}{dx} \Rightarrow (2x+1)dx = du$ $\Bigg\} \Rightarrow$	*Notice that $2x + 1$ is the derivative of $x^2 + x$ so if you define it as a new variable u you will have its derivative du too.*
$\int (2x+1)e^{x^2+x}\,dx = \int e^u\,du$ $= e^u + c = e^{x^2+x} + c$	*Use the substitution and solve the new integral for u.* *Leave your answer in terms of x.*

Example 25

Find the integral $\int \cot x\,dx$.

Answer

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$ Let $\sin x = v$ $\cos x = \frac{dv}{dx} \Rightarrow \cos x\,dx = dv$ $\Bigg\}$	*Rewrite in terms of sine and cosine. Notice that $\cos x$ is the derivative of $\sin x$ so you have the new variable v and its derivative dv.*
$\Rightarrow \int \frac{\cos x}{\sin x}dx = \int \frac{dv}{v}$ $= \ln\lvert v\rvert + c = \ln\lvert \sin x\rvert + c$	*Use the substitution and integrate with respect to v.* *Give your answer in terms of x.*

Example 26

Use an appropriate substitution to find $\int x^2 \sin(x^3 - 2)\,dx$.

Answer

Let $x^3 - 2 = t$ $3x^2 = \frac{dt}{dx} \Rightarrow x^2dx = \frac{1}{3}dt$ $\Bigg\} \Rightarrow$	*Notice that x^2 is not exact derivative of x^3, but $\frac{1}{3}$ of it.*
$\int x^2 \sin(x^3 - 2)\,dx$	*Use the substitution.*
$= \int \sin t \times \frac{1}{3}dt = \frac{1}{3}\int \sin t\,dt$	*Simplify and integrate with respect to t.*
$= -\frac{1}{3}\cos t + c = -\frac{1}{3}\cos\left(x^3 - 2\right) + c$	*Give your answer in terms of x.*

Exercise 9L

1 Find these integrals by the method of substitution.

a $\int 2x\sin x^2\,dx$ **b** $\int 3x^2\sqrt{x^3+3}\,dx$

c $\int (3-4x)e^{1+3x-2x^2}\,dx$ **d** $\int \tan x\,dx$

e $\int 2\cos 2x\,e^{\sin 2x}\,dx$ **f** $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$

g $\int 2^x\ln 2\sin(2^x)\,dx$ **h** $\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

i $\int \frac{2\arctan 2x}{1+4x^2}\,dx$

An appropriate substitution for **1a** would be $u = x^2$. For **1b** it would be $u = x^3 + 3$

2 Use an appropriate substitution to find these integrals.

a $\int x\cos x^2\,dx$ **b** $\int x^5\sqrt[3]{x^6-1}\,dx$

c $\int (x+2)e^{3x^2+12x-7}\,dx$ **d** $\int \frac{\tan(5x+4)}{5}\,dx$

e $\int \sin 3x\cdot 3^{\cos 3x}\,dx$ **f** $\int \frac{\sin\sqrt[4]{x}}{\sqrt[4]{x^3}}\,dx$

g $\int 5^x\cos(5^x)\,dx$ **h** $\int \frac{e^{2x}+e^{-2x}}{e^{-2x}-e^{2x}}\,dx$

i $\int \frac{\sqrt{\arctan\frac{x}{3}}}{9+x^2}\,dx$ **j** $\int (x^2+x)\cos\left(x^3+\frac{3}{2}x^2\right)dx$

k $\int \frac{\arcsin^2(2x+1)}{\sqrt{-x-x^2}}\,dx$

Definite integrals and integration by substitution

When solving definite integrals there are two methods.

Method I Solve the original integral by using substitution and then just apply The Fundamental Theorem of Calculus on the given boundaries to the solution.

Method II When substituting a new variable in the process to obtain a simpler integral we use the substitution to change the boundaries. Use new boundaries and apply the Fundamental Theorem to the new integral.

Example 27

Evaluate these integrals.

a $\displaystyle\int_1^7 2x\sqrt{x^2-1}\,dx$ **b** $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x \sin x\, dx$

Answers

a Method I

$$\text{Let } \left.\begin{aligned} x^2-1&=t \\ 2x\mathrm{d}x&=\mathrm{d}t \end{aligned}\right\} \Rightarrow$$

Identify the substitution.

$$\int 2x\sqrt{x^2-1}\,\mathrm{d}x = \int \sqrt{t}\,\mathrm{d}t$$

Use the substitution and as $2x\,dx = 2dt$ you can simplify.

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\left(x^2-1\right)^{\frac{3}{2}} + c$$

$$\int_1^7 2x\sqrt{x^2-1}\,\mathrm{d}x = \left[\frac{2}{3}\left(x^2-1\right)^{\frac{3}{2}}\right]_1^7$$

Solve the integral and give the answer in terms of the original variable.

$$= \frac{2}{3}\left(\left(7^2-1\right)^{\frac{3}{2}} - \underbrace{\left(1^2-1\right)^{\frac{3}{2}}}_{0}\right) = \frac{2}{3}\cdot 48^{\frac{3}{2}}$$

Use the FTC for definite integral.

$$= \frac{2}{3}\cdot 64\cdot 3\sqrt{3} = 128\sqrt{3}$$

Calculate and simplify.

Method II

$$\text{Let } \left.\begin{aligned} x^2-1&=t \quad & 1^2-1&=0 \\ 2x\mathrm{d}x&=\mathrm{d}t \quad & 7^2-1&=48 \end{aligned}\right\} \Rightarrow$$

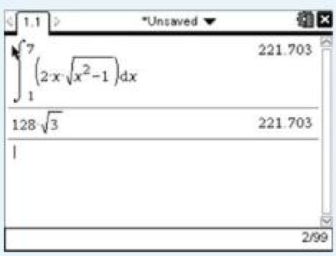
Identify the substitution and find new boundaries.

$$\int_1^7 2x\sqrt{x^2-1}\,\mathrm{d}x = \int_0^{48} \sqrt{t}\,\mathrm{d}t$$

Use the substitution and apply the new boundaries.

$$= \left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^{48} = \frac{2}{3}\cdot 48^{\frac{3}{2}} = 128\sqrt{3}$$

Use the FTC and calculate the answer.
Check on GDC.



b Method I

$$\text{Let } \left.\begin{aligned} \sin x&=t \\ \cos x\,\mathrm{d}x&=\mathrm{d}t \end{aligned}\right\} \Rightarrow$$

Identify the substitution and $dx = \dfrac{dt}{\cos x}$

$$\int \cos x \sin x\,\mathrm{d}x$$

$$= \int t\,\mathrm{d}t = \frac{t^2}{2} + c = \frac{\sin^2 x}{2} + c$$

Solve the integral and give the answer in terms of x.

Continued on next page

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x \sin x \,dx = \left[\frac{\sin^2 x}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}}$$

Use the FTC for definite integral.

$$= \frac{\left(\sin\left(\frac{\pi}{4}\right)\right)^2}{2} - \frac{\left(\sin\left(-\frac{\pi}{6}\right)\right)^2}{2}$$

Evaluate the definite integral.

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} - \frac{\left(-\frac{1}{2}\right)^2}{2} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

Calculate and simplify.

Method II

$$\text{Let} \quad \begin{matrix} \sin x = t \\ \cos x \,dx = dt \end{matrix} \quad \left.\begin{matrix} \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \\ \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \end{matrix}\right\} \Rightarrow$$

Identify the substitution and find new boundaries.

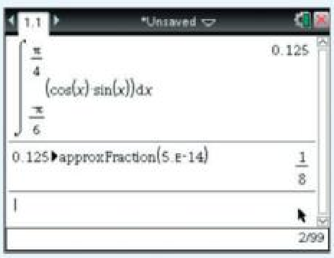
$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x \sin x \,dx = \int_{-\frac{1}{2}}^{\frac{\sqrt{2}}{2}} t \,dt = \left[\frac{t^2}{2}\right]_{-\frac{1}{2}}^{\frac{\sqrt{2}}{2}}$$

Use the substitution and apply the new boundaries.

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} - \frac{\left(-\frac{1}{2}\right)^2}{2}$$

$$= \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

Evaluate the definite integral.
Notice that you could have used $\cos x = t$; the boundaries would be changed but the final result will remain the same.
Check on GDC.



The work in both methods is similar, but it is slightly simpler in Method II, so you could use this method this in paper 1 to gain some time. In paper 2, unless otherwise stated in the question simply use a GDC.

Exercise 9M

Find the **exact values** of these integrals:

1 $\int_0^1 3x^2(x^3-1)^4 \,dx$ **2** $\int_0^3 \frac{2x}{x^2+1} dx$ **3** $\int_0^{\frac{\pi}{6}} \cos x\sqrt{\sin x} \,dx$

4 $\int_1^{e^3} \frac{\ln x}{x} dx$ **5** $\int_0^{\ln 2} \frac{e^x}{e^x+1} dx$ **6** $\int_0^{\frac{\pi}{6}} 2\tan(2x) \,dx$

7 $\int_0^1 (x^2+x)\cos\left(x^3+\frac{3}{2}x^2\right) dx$ **8** $\int_0^3 2^x\sqrt{2^x+1} \,dx$

9 Repeat questions 1 to 8 using a GDC.

9.5 Integration by parts

Integration by parts is related to the product rule for differentiation.

$$\frac{\mathrm{d}(u.v)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}.v + u.\frac{\mathrm{d}v}{\mathrm{d}x}$$

Integrating this identity with respect to x you find:

$$\int \frac{\mathrm{d}(u \cdot v)}{\mathrm{d}x}\mathrm{d}x = \int \frac{\mathrm{d}u}{\mathrm{d}x} \cdot v\,\mathrm{d}x + \int u \cdot \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x \Rightarrow uv = \int v\mathrm{d}u + u\mathrm{d}v$$

This gives:

> → $$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$
> where u and v are functions of x.

These examples give typical integrals that can be calculated using integration by parts.

Integration by parts allows you to convert an integral into another one that is simpler. The method is a kind of **reduction formula**.

Example 28

Find the integral $\int 2x\mathrm{e}^x\,\mathrm{d}x$.

Answer

Let

$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \Rightarrow \mathrm{d}u = 2\,\mathrm{d}x,$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \int \mathrm{e}^x\mathrm{d}x = \mathrm{e}^x$

In these cases, always differentiate the polynomial and integrate the exponential.
You could try it the other way around.

$\int 2x\mathrm{e}^x\mathrm{d}x = 2x\mathrm{e}^x - \int \mathrm{e}^x \cdot 2\,\mathrm{d}x$

$= 2x\mathrm{e}^x - 2\int \mathrm{e}^x\,\mathrm{d}x$

Choose the variables and apply the formula.
Use integral properties to simplify it.

$= 2x\mathrm{e}^x - 2\mathrm{e}^x + c$

$= (2x - 2)\mathrm{e}^x + c,\ c \in \mathbb{R}$

$= 2\mathrm{e}^x(x - 1) + c$

Simplify the final answer.
Notice that in the process of integrating dv you do not add a constant at the end, but only add the constant to the final answer.

Normally, let $\frac{\mathrm{d}v}{\mathrm{d}x}$ be the more complicated function that is still integrable. Considering $y = 2x$ and $y = \mathrm{e}^x$, $y = \mathrm{e}^x$ is the more complicated function of the two whose integral you can still find.

Exercise 9N

Find these integrals using integration by parts.

1 $\int xe^x\,dx$ **2** $\int (2x+9)\cos x\,dx$ **3** $\int (2-5x)\sin x\,dx$

4 $\int (3x-1)e^{3x}\,dx$ **5** $\int (4x-7)e^{(4x-1)}\,dx$ **6** $\int \frac{x+3}{2}\sin(2x+3)\,dx$

7 $\int \frac{3-x}{4}\cos\left(\frac{x}{4}\right)dx$ **8** $\int x\cdot 2^x\,dx$ **9** $\int (1-x)\cdot 5^x\,dx$

10 $\int \frac{(2-x)}{7\cdot 3^x}\,dx$ **11** $\int \frac{4x\cdot 3^x}{5^x}\,dx$

Example 29

Integrate the expression $y = (4x+5)\ln x$ with respect to x.

Answer

Let

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$ *Choose the variables.*

$\frac{dv}{dx} = (4x+5) \Rightarrow v = \int (4x+5)dx$

$= 2x^2 + 5x$

$\int (4x+5)\ln x\,dx$ *Apply the formula and simplify the integral.*

$= (2x^2+5x)\ln x - \int (2x^2+5x)\frac{1}{x}\,dx$

$= (2x^2+5x)\ln x - \int (2x+5)dx$

$= (2x^2+5x)\ln x - (x^2+5x) + c$ *Simplify, if possible.*

$= x^2(2\ln x - 1) + 5x(\ln x - 1) + c$

Notice that in this case you differentiate ln x and integrate $(4x + 5)$ even though polynomials are considered simple than logarithms. This is the only such case.

Exercise 9O

Find these integrals using integration by parts.

1 $\int x\ln x\,dx$ **2** $\int (3x+2)\ln x\,dx$ **3** $\int (1-x)\ln x\,dx$

4 $\int x\ln(4x)dx$ **5** $\int (3x-2)\ln\left(\frac{x}{5}\right)dx$ **6** $\int (3+4x)\ln(3+4x)\,dx$

7 $\int (5+7x)\ln(4-11x)\,dx$ **8** $\int x^2\ln x\,dx$ **9** $\int (2-x+x^2)\ln(3x)\,dx$

In this section you will use integration by parts to find some special integrals.

Example 30

Integrate these functions.

a $f(x) = \ln x$ **b** $f(x) = \arcsin x$

Answers

a Let

$u = \ln x \Rightarrow du = \frac{1}{x}dx$ — *$\ln x = 1 \times \ln x$. Choose the variables.*

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}dx$ — *Apply the formula.*

$= x\ln x - \int 1 \cdot dx = x\ln x - x + c$ — *Simplify and integrate.*

$= x(\ln x - 1) + c$

b Let $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$ — *Choose the variables.*

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\int \arcsin x\,dx$

$= x\arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}}dx$ — *Apply the formula.*

$= x\arcsin x + \int \frac{-x}{\sqrt{1-x^2}}dx$ — *Simplify the integral.*

$t = 1 - x^2 \Rightarrow \frac{dt}{dx} = -2x$ — *Use substitution to solve the new integral.*

$\Rightarrow \frac{1}{2}dt = -x\,dx$

$\int \frac{-x}{\sqrt{1-x^2}}dx$

$= \frac{1}{2}\int t^{-\frac{1}{2}}dt = \left(\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + c = \sqrt{1-x^2} + c$

$\int \arcsin x\,dx$

$= x\arcsin x + \sqrt{1-x^2} + c$ — *Apply the result and find the final answer.*

Exercise 9P

Find these integrals.

1 $\int \log x \, dx$ **2** $\int \log_a x \, dx$ **3** $\int \arctan x \, dx$

4 $\int \arccos x \, dx$ **5** $\int 2x \arctan x \, dx$ **6** $\int x^2 \arcsin x \, dx$

Sometimes you have to apply integration by parts more than once until we reach a simple integral. That often occurs with polynomials of a higher degree.

Example 31

Find $\int (3x^2 - x + 1)\sin x \, dx$.

Answer	
$u = 3x^2 - x + 1 \Rightarrow \frac{du}{dx} = 6x - 1$ $\Rightarrow du = (6x-1)dx$ $\frac{dv}{dx} = \sin x \Rightarrow v = \int \sin x \, dx = -\cos x$	*Choose the variables.*
$\int (3x^2 - x + 1)\sin x \, dx$	*Apply the formula.*
$= (3x^2 - x + 1)\cdot(-\cos x) - \int (6x-1)\cdot(-\cos x)\, dx$ $= -(3x^2 - x + 1)\cdot \cos x + \int (6x-1)\cdot \cos x \, dx$	*Simplify the integral and identify the new integral.*
$u = 6x - 1 \Rightarrow \frac{du}{dx} = 6 \Rightarrow du = 6\,dx$ $\frac{dv}{dx} = \cos x \Rightarrow v = \int \cos x \, dx = \sin x$	*Choose the variables for the new integral to be solved by parts.*
$\int (6x-1)\cdot \cos x \, dx = (6x-1)\sin x - \int 6 \cdot \sin x \, dx$	*Apply the formula.*
$= (6x-1)\sin x - 6(-\cos x)$ $= (6x-1)\sin x + 6\cos x$	*Simplify the integral.*
$\int (3x^2 - x + 1)\sin x \, dx =$ $-(3x^2 - x + 1)\cdot \cos x + \int (6x-1)\cdot \cos x \, dx$ $= -(3x^2 - x + 1)\cdot \cos x + (6x-1)\sin x + 6\cos x + c$ $= (-3x^2 + x + 5)\cdot \cos x + (6x-1)\sin x + c$	*Continue with integration of the original integral.* *Use the result.* *Simplify.*

Exercise 9Q

Integrate:

1 $\int x^2 e^x \, dx$

2 $\int (x^2 + 1)\sin x \, dx$

3 $\int (2x - x^2)\cos x \, dx$

4 $\int (1 + x - x^2) e^{2x} \, dx$

5 $\int (2x^2 + x + 3)\cos(2x) \, dx$

6 $\int x^2 \sin(1 - 2x) \, dx$

7 $\int x^2 3^x \, dx$

8 $\int (1+x^3) e^{\frac{x}{2}} \, dx$

9 $\int (x^3 + x^2)\sin(5x) \, dx$

10 $\int x^4 \cos x \, dx$

11 $\int x^5 e^{2x} \, dx$

Multiple applications of the method will occur in a product of an exponential and sine or cosine function.

Example 32

Find $\int e^x \cos x \, dx$.

Answer

$u = e^x \Rightarrow \frac{du}{dx} = e^x \cdot 3 \Rightarrow du = e^x dx$ $\frac{dv}{dx} = \cos x \Rightarrow v = \int \cos x \, dx = \sin x$	*Choose the variables.*
$\int e^x \cos x \, dx = e^x \sin x - \int \sin x \cdot e^x \, dx$	*Apply the formula.*
$u = e^x \Rightarrow \frac{du}{dx} = e^x \cdot \Rightarrow du = e^x dx$ $\frac{dv}{dx} = \sin x \Rightarrow v = \int \sin x \, dx = -\cos x$	*Simplify the integral and identify the new integral.* *Choose the variables for the new integral to be solved by parts.*

Exercise 9R

Find these integrals.

1 $\int \sin x\, e^{x}\, dx$

2 $\int e^{2x} \cos x\, dx$

3 $\int \cos 3x e^{4x}\, dx$

4 $\int \frac{\sin(2x)}{e^{x}}\, dx$

5 $\int \sin x\, e^{x} dx$

See further questions on the CD.

This process of multiple application of integration by parts can be shown in a table. The expression for u is successively differentiated in a column, whilst dv is successively integrated in the other column as many times as needed.

Example 33

Find the integral $\int x^3 e^{3x} dx$.

Answer

	$dv = e^{3x} d^x$	sign
$u = x^3$	$v = \frac{1}{3}e^{3x}$	+
$3x^2$	$\frac{1}{9}e^{3x}$	−
$6x$	$\frac{1}{27}e^{3x}$	+
6	$\frac{1}{81}e^{3x}$	−

So the result is

$$\int x^3 e^{3x}\, dx$$

$$= x^3 \cdot \frac{1}{3}e^{3x} - 3x^2 \cdot \frac{1}{9}e^{3x} + 6x \cdot \frac{1}{27}e^{3x} - 6 \cdot \frac{1}{81}e^{3x} + c$$

$$= \frac{x^3}{3}e^{3x} - \frac{x^2}{3}e^{3x} + \frac{2x}{9}e^{3x} - \frac{2}{27}e^{3x} + c$$

$$= \frac{e^{3x}}{27}\left(9x^3 - 9x^2 + 6x - 2\right) + c$$

Notice that the second integral in the integration by parts formula has a minus in front of it, so the sequence of the products of derivatives and integrals signs alternate.

You can use the method of Example 33 to verify your solutions to Exercise 9R.

9.6 Special substitutions

In this section you are going to study some special substitutions that are not immediately obvious.

Substitution in radical expressions

To simplify a radical linear expression, substitute the entire radical expression with a new variable and then integrate with respect to that new variable.

Example 34

Find the integral $\int x\sqrt{2x+1}\,dx$.

Answer

Let $\sqrt{2x+1} = t \Rightarrow 2x+1 = t^2$ $\Rightarrow x = \frac{1}{2}(t^2-1)$	*Express x in terms of t.*
$dx = \frac{1}{2}\cdot 2t\,dt \Rightarrow dx = t\,dt$	*Find dx in terms of t and simplify it.*
$\int x\sqrt{2x+1}\,dx = \int \frac{t^2-1}{2}\cdot t\cdot t\,dt$	*Use the substitutions to obtain the integral in terms of t only.*
$= \frac{1}{2}\int (t^4-t^2)\,dt = \frac{1}{2}\left(\frac{t^5}{5}-\frac{t^3}{3}\right)$	*Simplify the new integral and integrate it with respect to t.*
$= \frac{(2x+1)^{\frac{5}{2}}}{10} - \frac{(2x+1)^{\frac{3}{2}}}{6} + c$	*Use the substitution to return the old variable x.*
$= \frac{(2x+1)^{\frac{3}{2}}(3(2x+1)-5)}{30} + c$	*Simplify.*
$= \frac{(2x+1)^{\frac{3}{2}}(3x-1)}{15} + c,\ c \in \mathbb{R}$	

You can also take only the radicand for the new variable and then integrate with respect to it.

Example 35

Find the integral $\int 3x^2 \sqrt[3]{2-3x}\,\mathrm{d}x$.

Answer

Let $2-3x=u \Rightarrow 2-u=3x$ $\Rightarrow x=\frac{1}{3}(2-u)$	*Express x in terms of u.*
$\mathrm{d}x=-\frac{1}{3}\mathrm{d}u$	*Find dx in terms of the variable u.*
$x^2=\frac{1}{9}(2-u)^2 \Rightarrow x^2=\frac{1}{9}(4-4u+u^2)$	*Express the remaining factor in the integral x^2 in terms of u.*
$\int 3x^2\sqrt[3]{2-3x}\,\mathrm{d}x = \int 3\cdot\frac{1}{9}(4-4u+u^2)\cdot u^{\frac{1}{3}}\left(-\frac{1}{3}\right)\mathrm{d}u$	*Use all the substitutions to obtain the integral in terms of u.*
$=-\frac{1}{9}\int\left(4u^{\frac{1}{3}}-4u^{\frac{4}{3}}+u^{\frac{7}{3}}\right)\mathrm{d}t$ $=-\frac{1}{9}\left(\frac{4u^{\frac{4}{3}}}{\frac{4}{3}}-\frac{4u^{\frac{7}{3}}}{\frac{7}{3}}+\frac{u^{\frac{10}{3}}}{\frac{10}{3}}\right)+c$ $=-\frac{u^{\frac{4}{3}}}{3}+\frac{4u^{\frac{7}{3}}}{21}-\frac{u^{\frac{10}{3}}}{30}+c$	*Simplify the new integral and integrate it with respect to u.*
$=-\frac{u^{\frac{4}{3}}(70-40u+u^2)}{210}+c$	*Factorize and simplify.*
$=-\frac{(2-3x)^{\frac{4}{3}}\left(70-40(2-3x)+(2-3x)^2\right)}{210}+c$	*Substitute for x.*
$=-\frac{(2-3x)^{\frac{4}{3}}(70-80+120x+28-84x+63x^2)}{210}+c$	*Expand and simplify.*
$=-\frac{(2-3x)^{\frac{4}{3}}(18+36x+63x^2)}{210}+c$	*Factorize and simplify.*
$=-\frac{(2-3x)^{\frac{4}{3}}9\cdot(2+4x+7x^2)}{210}+c$	
$=-\frac{3(2-3x)^{\frac{4}{3}}(7x^2+4x+2)}{70}+c$	

Exercise 9S

Find these integrals.

1 $\int x\sqrt{x+2}\,dx$

2 $\int 3x\sqrt{1-2x}\,dx$

3 $\int 5x^2\sqrt{3+4x}\,dx$

4 $\int x\sqrt[3]{x+3}\,dx$

5 $\int x^2\sqrt[4]{x+1}\,dx$

6 $\int x^3\sqrt[5]{1-x}\,dx$

You have used trigonometric identities to solve the integrals of squares of the sine and cosine functions. Here you can find out what happens with higher powers.

Notice that in Examples 38 and 39 you always factorized the trigonometric function raised to an odd power and express everything in terms of the trigonometric function raised to an even power.

Example 36

Find the integral $\int \sin^3 x\,dx$.

Answer

$\int \sin^3 x\,dx = \int \sin^2 x \sin x \cdot dx$ $= \int (1-\cos^2 x)\cdot \sin x\,dx$	*Rewrite sine in terms of cosine so that you have a new variable cos x and its differential −sin x.*
Let $v = \cos x \Rightarrow \frac{dv}{dx} = -\sin x$ $\Rightarrow -dv = \sin x\,dx$	
$\int (1-v^2)\cdot(-dv)$ $= \int (v^2-1)dv$ $= \frac{v^3}{3} - v + c$	*Use the substitution.*
$= \frac{\cos^3 x}{3} - \cos x + c$	*Return the original variable x.*

Example 37

Find the integral $\int \sin^4 x\,dx$.

Answer

$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$ — *Use double angle formula.*

$= \int \frac{1-2\cos 2x+\cos^2 2x}{4}\,dx$ — *Expand the expression.*

$= \int \left(\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cdot\frac{1+\cos 4x}{2}\right)dx$ — *Again use double angle formula.*

$= \frac{1}{4}x-\frac{1}{2}\cdot\frac{\sin 2x}{2}+\frac{1}{8}\left(x+\frac{\sin 4x}{4}\right)+c$ — *Simplify.*

$= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c$

Exercise 9T

Find these integrals.

1 $\int \cos^3 x\,dx$ **2** $\int \cos^4 x\,dx$ **3** $\int \sin^5\left(\frac{x}{5}\right)dx$ **4** $\int 48\cos^6(2x)\,dx$

Investigation – recursive formula

Use integration by parts to find a **recursive** formula for the integrals of the forms:

1 $\int \sin^n x\,dx$, where n is a positive integer

2 $\int \cos^n x\,dx$, where n is a positive integer.

Example 38

Find the integral $\int \sin^3 x\cos^2 x\,dx$.

Answer

$\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cdot\sin x\cdot\cos^2 x\,dx = \int (1-\cos^2 x)\cdot\sin x\cdot\cos^2 x\,dx$

$= \int (\cos^2 x-\cos^4 x)\cdot\sin x\,dx$

Let $u=\cos x \Rightarrow \frac{du}{dx}=-\sin x \Rightarrow -du=\sin x\,dx$

$= \int (u^2-u^4)\cdot(-du) = -\frac{u^3}{3}+\frac{u^5}{5}+c = \frac{\cos^5 x}{5}-\frac{\cos^3 x}{3}+c$

Example 39

Find the integral $\int \sin^4 x \cos^7 x \, dx$.

Answer

$\int \sin^4 x \cos^7 x \, dx = \int \sin^4 x \cdot \cos^6 x \cdot \cos x \, dx$

$= \int \sin^4 x \cdot (1 - \sin^2 x)^3 \cdot \cos x \, dx$ — *Use $\cos^2 x = 1 - \sin^2 x$*

Let $t = \sin x \Rightarrow \frac{dt}{dx} = \cos x \Rightarrow dt = \cos x \, dx$

$= \int t^4 \cdot (1 - t^2)^3 \, dt$

$= \int t^4 \cdot (1 - 3t^2 + 3t^4 - t^6) \, dt$

$= \int t^4 - 3t^6 + 3t^8 - t^{10}) \, dt$

$= \frac{t^5}{5} - \frac{3t^7}{7} + \frac{3t^9}{9} - \frac{t^{11}}{11} + c$

$= \frac{\sin^5 x}{5} - \frac{3\sin^7 x}{7} + \frac{3\sin^9 x}{9} - \frac{\sin^{11} x}{11} + c$ — *Substitute back in terms of x.*

Notice that in Examples 38 and 39 you always factorized the trigonometric function raised to an odd power and expressed everything in terms of the trigonometric function raised to an even power.

Investigation – more recursive formula

Find recursive formulae for the integrals of the form $\int \sin^n x \cos^m x \, dx$, where n and m are positive integers.

Trigonometric substitutions

→ When the integrand contains a quadratic radical expression use one of these trigonometric substitutions to transform the integral.

1. If the form is $\sqrt{a^2 - x^2}$ use the substitution $x = a \sin\theta$.
2. If the form is $\sqrt{x^2 - a^2}$ use the substitution $x = a \sec\theta$.
3. If the form is $\sqrt{x^2 + a^2}$ use the substitution $x = a \tan\theta$.

Example 40

Use an appropriate trigonometric substitution to find $\int \sqrt{25-x^2}\,dx$.

Answer

Let $\sqrt{25-x^2} = \sqrt{5^2-x^2} \Rightarrow$

Identify the substitution $x = 5\sin\theta$.

$$\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta}$$
$$= 5\sqrt{1-\sin^2\theta} = 5\cos\theta$$

Express radical expression in terms of θ.

Let $x = 5\sin\theta \Rightarrow dx = 5\cos\theta\,d\theta$

Find dx in terms of the variable θ.

$$\int\sqrt{25-x^2}\,dx = \int 5\cos\theta\cdot 5\cos\theta\,d\theta$$

Use the substitutions to obtain the integral in terms of the variable θ.

$$= 25\int\cos^2\theta\,d\theta = 25\int\frac{1+\cos 2\theta}{2}\,d\theta$$

Use double angle formula to simplify the integral.

$$= \frac{25}{2}\theta + \frac{25}{4}\sin 2\theta + c$$

Integrate with respect to θ.

$$= \frac{25}{2}\theta + \frac{25}{4}\cdot 2\sin\theta\cos\theta + c$$

Use double angle formula to simplify the primitive function.

In order to return the variable x express θ in terms of x.

$$x = 5\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{5}\right)$$

Now proceed in substituting θ in term of x.

$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{25}{4}\cdot 2\cos\left(\arcsin\left(\frac{x}{5}\right)\right) + c$$

Return to x.

$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{5x}{2}\cdot\sqrt{1-\sin^2\left(\arcsin\left(\frac{x}{5}\right)\right)} + c$$

Express cosine in terms of sine.

$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{5x}{2}\sqrt{1-\left(\frac{x}{5}\right)^2} + c$$

Simplify the trigonometric expressions.

$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{x}{2}\sqrt{25-x^2} + c,\ c\in\mathbb{R}$$

Simplify the radical expression.

In this problem you could also use the substitution $x = a\cos\theta$. The radical expression would be equal to $\sqrt{a^2-x^2} = a\sin\theta$ and the differential $dx = a\cos\theta\,d\theta$.

Example 41

Use an appropriate trigonometric substitution to find $\int \frac{1}{\sqrt{3x^2-48}}\,dx$

Answer

$\int \frac{1}{\sqrt{3x^2-48}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2-16}}\,dx$	*Identify the substitution $x = 4\sec\theta$.*
Let $\sqrt{x^2-16} = \sqrt{16\sec^2\theta - 16}$	*Express x in terms of θ.*
$= 4\sqrt{\sec^2\theta - 1} = 4\tan\theta$	*Use the formula $\sec^2\theta - 1 = \tan^2\theta$.*
$dx = 4\cdot\left(-\frac{\sin\theta}{\cos^2\theta}\right)d\theta$	*Find dx in terms of the variable θ.*
$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2-16}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{4\tan\theta}\cdot\left(-4\frac{\sin\theta}{\cos^2\theta}\right)d\theta$	*Now proceed to solving the original integral.*
$= -\frac{1}{\sqrt{3}}\int \sec\theta\, d\theta$	
$= -\frac{1}{\sqrt{3}}\left(\ln\left(\frac{\sin x + 1}{\cos\theta}\right)\right) + c$	*Use the substitutions to obtain the integral in terms of the variable θ.*
$x = 4\sec\theta \Rightarrow \cos\theta = \left(\frac{4}{x}\right) \Rightarrow \theta = \arccos\left(\frac{4}{x}\right)$	*Now substitute θ in terms of x.*
$= -\frac{1}{\sqrt{3}}\left(\ln\left(\frac{\sin\left(\arccos\left(\frac{4}{x}\right)\right)+1}{\frac{4}{x}}\right)\right) + c$	*Return to the variable x.*
$= -\frac{1}{\sqrt{3}}\left(\ln\left(\frac{\sqrt{1-\left(\cos\left(\arccos\left(\frac{4}{x}\right)\right)\right)^2}+1}{\frac{4}{x}}\right)\right) + c$	*Express sine in terms of cosine and apply the inverse function.*
$= -\frac{1}{\sqrt{3}}\left(\ln\left(\frac{\sqrt{1-\left(\frac{4}{x}\right)^2}+1}{\frac{4}{x}}\right)\right) + c = -\frac{1}{\sqrt{3}}\left(\ln\left(\frac{\frac{\sqrt{x^2-16}}{x}+1}{\frac{4}{x}}\right)\right) + c$	*Simplify the radical expressions.* *Simplify the fraction.*
$= -\frac{1}{\sqrt{3}}\cdot\ln\left(\frac{\sqrt{x^2-16}+x}{4}\right) + c$	
$= -\frac{1}{\sqrt{3}}\cdot\left(\ln\left(\sqrt{x^2-16}+x\right) - \ln 4\right) + c$	
$= -\frac{1}{\sqrt{3}}\cdot\ln\left(\sqrt{x^2-16}+x\right) \underbrace{-\frac{\ln 4}{\sqrt{3}} + c}_{k\in\mathbb{R}}$	*Simplify the constant.*
$= -\frac{1}{\sqrt{3}}\cdot\ln\left(\sqrt{x^2-16}+x\right) + k$	

Example 42

Use an appropriate trigonometric substitution to find $\int \frac{1}{\sqrt{x^2+2}}\,dx$

Answer

Notice that 2 is not a perfect square, but you can write it as

Identify the substitution $x = \sqrt{2}\tan\theta$.

$x^2 + 2 = x^2 + \left(\sqrt{2}\right)^2$

Let $\sqrt{x^2 + 2} = \sqrt{2\tan^2\theta + 2}$

$= \sqrt{2}\cdot\sqrt{\tan^2\theta + 1} = \sqrt{2}\sec\theta$

Use the formula $\tan^2\theta + 1 = \sec^2\theta$.

$x = \sqrt{2}\tan\theta \Rightarrow dx = \sqrt{2}\sec^2\theta\, d\theta$

Find dx in terms of the variable θ. Now solve the original integral.
Use the substitutions to obtain the integral in terms of the variable θ and simplify it.

$$\int \frac{1}{\sqrt{x^2+2}}\,dx = \int \frac{1}{\sqrt{2}\sec\theta}\cdot\sqrt{2}\sec^2\theta\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$x = \sqrt{2}\tan\theta \Rightarrow \theta = \arctan\left(\frac{x}{\sqrt{2}}\right)$

In order to return the variable x express θ in terms of x.

$\sec\theta = \sqrt{1+\tan^2\theta}$

Also express sec in terms of tan.

$$\int \frac{1}{\sqrt{x^2+2}}\,dx = \ln\left(\frac{x}{\sqrt{2}} + \sqrt{1+\left(\frac{x}{\sqrt{2}}\right)^2}\right) + c$$

Now substitute θ in terms of x.

$$= \ln\left(\frac{x}{\sqrt{2}} + \sqrt{\frac{2+x^2}{2}}\right) + c$$

$$= \ln\left(\frac{x+\sqrt{2+x^2}}{\sqrt{2}}\right) + c$$

$$= \ln\left(x+\sqrt{2+x^2}\right) \underbrace{-\ln\sqrt{2} + c}_{k}$$

$$= \ln\left(x+\sqrt{2+x^2}\right) + k$$

Exercise 9U

Use an appropriate trigonometric substitution to find these integrals:

1 $\int \sqrt{4-x^2}\,dx$ **2** $\int \frac{1}{\sqrt{x^2-1}}\,dx$ **3** $\int \sqrt{x^2+9}\,dx$

4 $\int \frac{3}{\sqrt{36-x^2}}\,dx$ **5** $\int 3\sqrt{x^2-16}\,dx$ **6** $\int \frac{5}{\sqrt{x^2+121}}\,dx$

7 $\int \frac{2}{\sqrt{81-4x^2}}\,dx$ **8** $\int \sqrt{3x^2-75}\,dx$ **9** $\int \frac{7}{\sqrt{7x^2+28}}\,dx$

Further examples of trigonometric substitutions can be found on the CD.

9.7 Applications and modelling

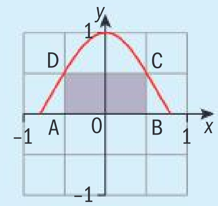
You can now revisit some of the applications of calculus studied in the previous sections of this chapter.

Minima and maxima problems

Example 43

A rectangle ABCD is inscribed under a curve $y = \cos 2x,\ -\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

Two vertices, A and B on the x-axis and two vertices, C and D on the curve, are shown in the diagram.



a Given that the coordinates of vertex C are (x, cos $2x$), $x > 0$ find the area of the rectangle in terms of x.

b Find the value of x for which the area is maximum. What is the maximum area?

Answers

a $A(x) = \underbrace{2x}_{\text{length}} \cdot \underbrace{\cos 2x}_{\text{width}}$

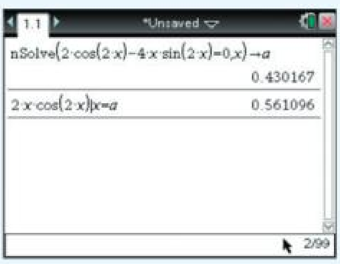
b **Method I**

$A'(x) = 2\cos 2x + 2x(-\sin 2x \cdot 2)$
$= 2\cos 2x - 4x\sin 2x$

Find the derivative and simplify the expression.

$A'(x) = 0 \Rightarrow 2\cos 2x - 4x\sin 2x = 0$

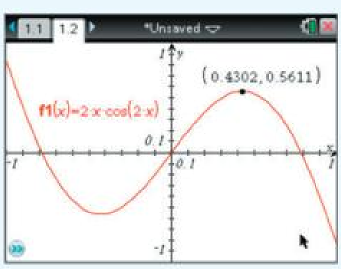
Find the x-coordinate of the maximum point.
Make A′(x) = 0 for maximum (and minimum) points.
Since you cannot solve this equation algebraically, use a GDC to find the values of x and the corresponding A(x).



So the area will reach a maximum for $x = 0.430$ and its maximum value will be $A_{max} = 0.561$, both given correct to 3 sf.

Method II

Since there are no demands on showing workings in this case, simply use the graphical method on a GDC, apply it to the area function and find the maximum.



Notice that the coordinates give both the answers.
$x = 0.430$ and $A_{max} = 0.561$

Example 44

Find the minimum distance between the point A(0,2) and the curve $y = \sin 2x$.

Answer

Method 1

Let P(x, sin $2x$) be any point on the curve.

$$AP = \sqrt{(x-0)^2 + (\sin 2x - 2)^2}$$

Use the distance formula to find the length of AP.

$$AP^2 = x^2 + \sin^2 2x - 4\sin 2x + 4$$

Square the equation to obtain an expression for a simpler calculation.

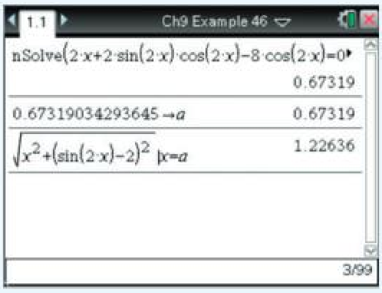
$$\frac{d}{dx}(AP^2) = 2x + 4\sin 2x\cos 2x - 8\cos 2x$$

Differentiate it with respect to x.

$$2x + 2\sin 2x\cos 2x - 8\cos 2x = 0$$

Since you cannot solve this equation algebraically use a GDC to find the value of x and the corresponding distance.

Find the x-coordinate of the minimum point.



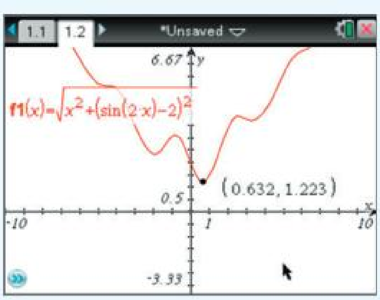
So the minimum distance between the point A and the curve $y = \sin 2x$ is 1.22, correct to 3 sf. The distance is obtained at the point P(0.632, 0.953) where the coordinates are also given correct to 3 sf.

Notice that squaring the distance had no effect on the minimum x-value.

Method 2

Since there are no demands on showing working in this case, use the graphical method on a GDC, apply it to the distance function and find the minimum.

Notice that in this method you can immediately read out the answer $AP_{min} = 1.22$.
To find the actual point, read out the x-coordinate but you still need to find the y-coordinate.



Exercise 9V

In this exercise use a GDC where appropriate.

1 A rectangle is inscribed under a curve $y = \sin x$, $0 \le x \le \pi$ in such a way that one side is on the x-axis. Find the dimensions of such a rectangle that has a maximum area.

2 Find the minimum distance between the point A(1, −1) and the curve $y = \cos x$.

3 A circle has a radius of 10 cm. A tangent is drawn through a point A on the circumference. Chord BC is parallel to the tangent.

a Show that the area of the triangle ABC is given by the formula $A(\alpha) = 100(1 + \cos\alpha)\sin\alpha$.

b Find the value of α for which the area is maximum.

4 A roller coaster track will have a gap of 15 metres between the first two posts. The part of the track is straight and going downhill. The formula to calculate the time needed for a car to pass that part of the track is $t = 2\sqrt{\frac{15}{g \cdot \sin 2\theta}}$, where g is the Earth's acceleration and θ is the angle of inclination to the horizontal position.

a What is the value of the angle θ such that the speed of the car would be maximum?

b What is the length of the track is that case? Give your answer correct to the nearest millimetre.

5 An object's displacement, d metres, from a fixed point F is given by the formula $d(t) = \sin\left(\frac{\pi t}{6}\right) + \cos\left(\frac{\pi t}{6}\right)$, $t > 0$ at time t seconds.

a Show that the acceleration of the object is proportional to the displacement.

b Find the maximum speed of the object and the first time when is achieved.

6 A metal chain is hanging between two poles that are 3 metres apart. The height (in metres) of the chain is given by the formula

$$h(x) = e^{-\frac{x}{2}} + e^{\frac{x}{5}},\ 0 \le x \le 3$$

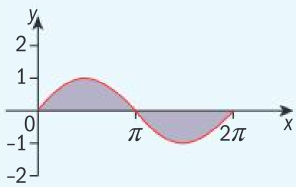
where x is the distance to the first pole. What is the minimum height of the chain and which pole is closer to the point of minimum height? Justify your answer.

Areas and volumes of revolution

Example 45

Shade the region between the curve $y = \sin x$, $0 \le x \le 2\pi$ and the x-axis and find its area.

Answer



Shade the region required, and notice that the region above the x-axis is symmetrical to the region below the x-axis.

$$A = \int_0^{2\pi} |\sin x|\, dx$$

$$= 2 \cdot \int_0^{\pi} \sin x\, dx = 2 \cdot [-\cos x]_0^{\pi}$$

$$= 2 \cdot (-\cos \pi - (-\cos 0))$$

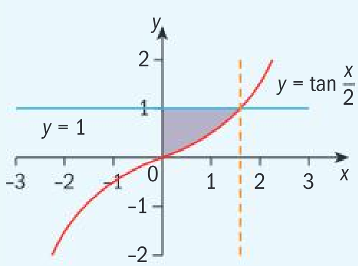
$$= 2(1 + 1) = 4$$

So the required area is twice the area of the curve between 0 and π.
Evaluate.

Example 46

Find the area of the region bounded by the curve $y = \tan\left(\frac{x}{2}\right)$, the line $y = 1$ and the y-axis.

Answer



Sketch the graph and shade the area required.

$$\tan\left(\frac{x}{2}\right) = 1 \Rightarrow \frac{x}{2} = \frac{\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{2} \Rightarrow \left(\frac{\pi}{2}, 1\right)$$

Put tan $\left(\frac{x}{2}\right)$ = 1 to find the point of intersection of the curve and y = 1.

$$\text{Area} = \underbrace{\frac{\pi}{2} \cdot 1}_{\text{Rectangle}} - \underbrace{\int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx}_{\text{Area under curve}}$$

$$= \frac{\pi}{2} - \left[2 \cdot \ln\left|\sec\frac{x}{2}\right|\right]_0^{\frac{\pi}{2}}$$

First find the area of the rectangle OABC.
Subtract the area under the curve from this.

▶ Continued on next page

$$= \frac{\pi}{2} - 2 \cdot \left(\ln\left|\sec\frac{\pi}{4}\right| - \ln\left|\sec 0\right| \right)$$

Calculate trigonometric values.

$$= \frac{\pi}{2} - 2 \cdot \left(\ln\sqrt{2} - \ln 1 \right)$$

$$= \frac{\pi}{2} - 2 \cdot \left(\frac{1}{2}\ln 2 \right) = \frac{\pi}{2} - \ln 2$$

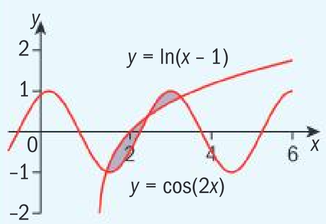
Simplify logarithmic expressions.

Note that you could alternatively integrate with respect to y. Try it and you should get the same answer.

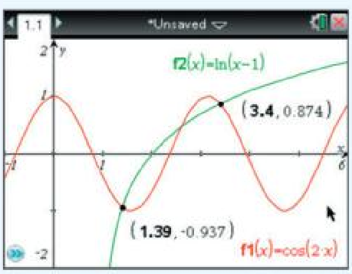
Example 47

Shade the finite regions enclosed by the curves $y = \cos(2x)$ and $y = \ln(x - 1)$. Find the area of the shaded region.

Answer



To find points of intersections between these two curves, solve the equation cos(2x) = ln(x −1). It cannot be solved algebraically, so use a GDC this time.



Store the x-coordinates of the points of intersections in the variables a and b.

You don't need to find the middle point of intersection since you are using the absolute value of the difference of two functions.

Exercise 9X

1 Shade the region enclosed by each of these curves and find its area.

a $y=\cos x,\ y=0,\ x=-\pi,\ x=\frac{\pi}{3}$

b $y=\sec^2 x,\ y=0,\ x=-\frac{\pi}{4},\ x=\frac{\pi}{4}$

c $y=2\sin 2x,\ y=0,\ x=-\frac{\pi}{3},\ x=\frac{\pi}{6}$

d $y=\frac{1}{3}\cos 3x,\ y=0,\ x=\frac{5\pi}{18},\ x=\frac{2\pi}{3}$

e $y=4\tan\frac{x}{2},\ y=0,\ x=-\frac{\pi}{2},\ x=-\frac{\pi}{3}$

2 Given that x is positive, shade the region enclosed by each of these curves and find its area.

a $y=\cos x, x=0, y=\frac{1}{2}$

b $y=\tan 2x, x=0, y=\sqrt{3}$

c $y=3\sin\frac{x}{2},\ x=0,\ y=-\frac{3\sqrt{3}}{2}$

d $y=2\cos\left(2x-\frac{\pi}{6}\right),\ x=0,\ y=\sqrt{2}$, given x is non-negative

e $y=\tan\frac{x}{3},\ x=0,\ y=\frac{1}{\sqrt{3}}$

3 Shade the first region enclosed by the curves $y=\cos\frac{x}{2}$ and $y=\cos 2x, x>0$. Find the area of the shaded region.

4 Shade the finite region enclosed by the curve $y=\tan x,\ 0\le x\le\frac{\pi}{2}$, its tangent at a point where $x=\frac{\pi}{4}$ and the x-axis. Find the area of the shaded region.

5 Shade the regions enclosed by the curves $y=2\sin x,\ x>0$ and $y=e^{\frac{x}{2}-4}+1$. Find the area of the shaded region.

6 Consider the curves $y=\frac{8}{4+x^2}$ and $y=\frac{x^2}{4}$.

a Find the points of intersection.

b Write down the integral that represents the area of the region enclosed by the curves.

c Calculate the area of the region.

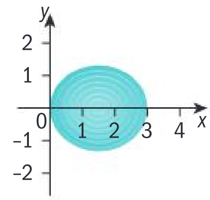
The volume V of a solid formed by a curve $y = f(x)$, between $x = a$ and $x = b$ rotated through 2π radians about the x-axis is given by

$$V = \pi \int_a^b y^2 \, dx$$

You met this formula in Section 7.3

Example 48

Find the volume of a solid obtained by rotating the curve $y = \sqrt{\sin x}$, $0 \le x \le \pi$ through 2π radians about the x-axis.



Answer

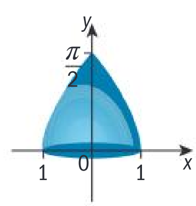
$V = \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx$	*Use the formula* $V = \pi \int_a^b y^2 \, dx$
$= \pi \int_0^{\pi} \sin x \, dx = \pi \left[-\cos x\right]_0^{\pi}$	*Simplify.*
$= \pi(-\cos \pi - \cos 0) = 2\pi$ units^3	*Evaluate.*

The volume of a solid formed by rotating a curve, $y = f(x)$, through 2π radians about the y-axis is given by $V = \pi \int_a^b x^2 \, dy$.

To use this formula directly you must first find $x = f^{-1}(y)$.

Example 49

Find the volume of a solid that is obtained by rotating the curve $y = \arccos x$, $0 \le x \le 1$ through 2π radians about the y-axis.



Answer

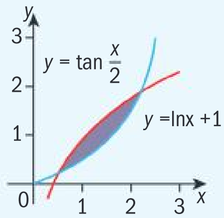
$y = \arccos x$, $0 \le x \le 1$ $\Rightarrow x = \cos y$, $0 \le y \le \frac{\pi}{2}$	*Express x in terms of y and find the domain of y values.*
$V = \pi \int_0^{\frac{\pi}{2}} (\cos y)^2 \, dy$	*Use the volume formula.*
$= \pi \int_0^{\pi} \frac{1 - \cos 2y}{2} \, dy$	*Use the half angle formula.*
$= \pi \left[\frac{y}{2} - \frac{\sin 2y}{4}\right]_0^{\pi}$	
$= \pi\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \frac{\pi^2}{2}$ units^3	*Evaluate.*

You can subtract volumes of two different curves.

Example 50

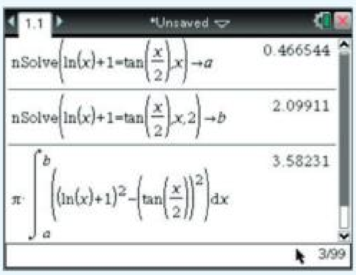
Find the volume of a solid that is obtained by rotating the finite region enclosed by the curves $y = \ln x + 1$ and $y = \tan\frac{x}{2}$ through 2π radians about the x-axis.

Answer



Sketch the graph to identify the finite region and the points of intersection.

The equation $\ln x = \tan\frac{x}{2}$ cannot be solved algebraically so use a GDC.



Solve the equation $\ln x = \tan\frac{x}{2}$ and store the solutions as the variables a and b. Apply the formula for the volume of rotating region between two curves.

Vol. = 3.58 units^3

Exercise 9X

1 Find the volume of a solid generated by rotating the region bounded by the given curves through 2π radians about the x-axis.

a $y = \sqrt{\cos x},\ x = 0,\ x = \frac{\pi}{2},\ y = 0$

b $y = \sec x,\ x = 0,\ x = \frac{\pi}{4},\ y = 0$

c $y = \cos x,\ x = \frac{\pi}{6},\ x = \frac{5\pi}{6},\ y = 0$

d $y = \sin x,\ x = \frac{\pi}{3},\ x = \frac{2\pi}{3},\ y = 0$

2 Find the volume of a solid generated by rotating the region bounded by the given curves through 2π radians about the y-axis.

a $y = \arcsin x,\ x = 0,\ y = 1$

b $y = \arcsin x,\ x = 0,\ x = 1,\ y = 0$

c $y = \tan x,\ x = 0,\ x = \frac{\pi}{4},\ y = 0$

d $y = \tan x,\ x = 0,\ y = 1$

3 Find the volume of a solid generated by rotating the region enclosed by the curves

a $y = \sin x$, $y = \cos x$, $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$ through 2π radians about the x-axis.

b $y = \sin 2x$, $y = \sin x$, $0 \le x \le \pi$ through 2π radians about the x-axis.

c $y = e^{\frac{x}{3}} - 1$ and $y = \arctan x$ through 2π radians about the x-axis.

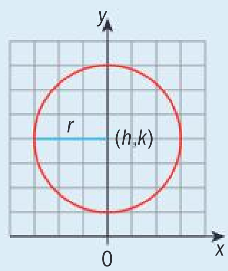
d $y = e^{\frac{x}{3}} - 1$ and $y = \arctan x$ through 2π radians about the y-axis.

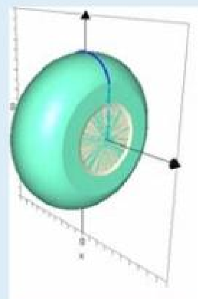
See further questions on the CD.

Investigation – volume of a torus

Find the volume of a torus that is obtained by rotating a circle with the centre at (h, k) and a radius r, k, $r > 0$ and $k > r$, about the x-axis.

Look back at the chapter introduction on page 441.





Exercise 9Y

1 Use volume of revolution to find the volume of a torus obtained by rotating the circle $(x - 4)^2 + (y + 3)^2 = 4$ through 2π radians about the x-axis.

2 Use volume of revolution to find the volume of a torus obtained by rotating the circle $(x - 4)^2 + (y + 3)^2 = 4$ through 2π radians about the y-axis.

3 Use volume of revolution to find the volume of a sphere obtained by rotating the circle $x^2 + y^2 = 9$ through 2π radians about the x-axis.

4 Find the volume of a solid obtained by rotating the ellipse $4x^2 + 9y^2 = 36$ through 2π radians about the x-axis.

5 Find the volume of a solid obtained by rotating the ellipse $4x^2 + 9y^2 = 36$ through 2π radians about the y-axis.

6 Find the volume of a solid obtained by rotating the ellipse $(5 - x)^2 + 9y^2 = 36$ through 2π radians about the x-axis.

Review exercise

1 Differentiate with respect to x:

a $f(x) = (2x + 3)\sin x$

b $g(x) = e^x \cos 3x$

c $h(x) = \dfrac{\tan x}{2x^2}$

2 Find the equation of a tangent to the curve $\sin y + e^{2x} = 1$ at the origin.

3 Find the value of m that satisfies this equation

$$\int_{\frac{\pi}{4}}^{m} \sec^2 x \, dx = 2\left(\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\right).$$

4 Use the method of integration by parts to solve:

a $\int (2x - 5)e^{2x}\, dx$;

b $\int (x^2 - 5x)\cos x\, dx$;

c $\int e^x \cos 3x\, dx$.

5 The diagonal of a square is increasing at a rate of 0.2 cm s^{-1}. Find the rate of change of the area of the square when the side has a length of 5 cm.

6 The curve $y = e^{2x-1}$ is given.

a Find the equation of the tangent to the curve that passes through the origin.

b Find the area, in terms of e, of the region bounded by the curve, the tangent and the y-axis.

c Find the volume of the revolution, in terms of π, obtained by rotating the region in part **b** about the x-axis.

7 Use the substitution $x = 3\cos\theta$ to find $\int \sqrt{9 - x^2}\, dx$.

8 The region bounded by the curve $y = \ln(2x)$, the vertical line $x = 1$ and the x-axis is rotated through 2π radians about the y-axis.

a Sketch the region in the coordinate system.

b Find the exact value of the volume of revolution obtained by this rotation.

9 The velocity, v, of an object, at a time t, is given by $v = 5e^{-\frac{2t}{3}}$, where t is in seconds and v is in m s^{-1}.

a Find the distance travelled in the first k seconds, $k > 0$.

b What is the total distance travelled by the object?

10 Find the equation of the normal to the curve $x^2y^3 = \cos(\pi x)$ at the point $(1, -1)$.

Review exercise

1 Find the points of inflection of the curve $y = x^2 \sin 2x$, $-1 \le x \le 1$.

2 Given the curve $y^3 = \cos x$, find the equation of the tangent at the point where $x = 1$.

3 Find the value of a, $0 < a < 1$, such that $\int_{a^2}^{0} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = 0.2709$

4 An airplane is flying at a constant speed at a constant altitude of 10 km in a straight line directly over an observer. At a given moment the observer notes that the angle of elevation θ to the plane is 54° and is increasing at 1° per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.

5 The region in the first quadrant bounded by the curves $y = \cos x$ and $y = e^x - 1$ is rotated by the x-axis by 2π radians. Find the volume of revolution of the solid generated.

CHAPTER 9 SUMMARY

Derivatives of trigonometric functions

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

Derivatives of inverse trigonometric functions

If $y = \arcsin x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$

If $y = \arcsin \frac{x}{a}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{a^2-x^2}}$

Basic integrals of trigonometric functions

$\int \cos x\,\mathrm{d}x = \sin x + c,\ c \in \mathbb{R}$ since $\frac{\mathrm{d}(\sin x)}{\mathrm{d}x} = \cos x$

$\int \sin x\,\mathrm{d}x = -\cos x + c$ since $\frac{\mathrm{d}(-\cos x)}{\mathrm{d}x} = \sin x$

$\int \sec^2 x\,\mathrm{d}x = \tan x + c$ since $\frac{\mathrm{d}(\tan x)}{\mathrm{d}x} = \sec^2 x$

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + c$$

Continued on next page

Definite integrals

$$\int f(x)\,\mathrm{d}x = F(x) + c \Rightarrow \int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Integration by parts

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Trigonometric substitutions

If an integral contains a quadratic radical expression use one of the following substitutions.

If the form is $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$.

If the form is $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec\theta$.

If the form is $\sqrt{x^2 + a^2}$ use the substitution $x = a\tan\theta$.