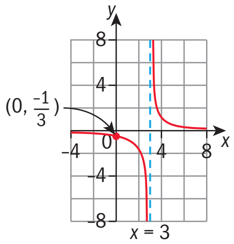
4 Modeling the real world

Answers

Skills check

1



2 $\sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n = \frac{5}{1-\frac{1}{2}} = 10$

Exercise 4A

1 $\lim_{x\to -1} \frac{x^2-1}{x+1} = -2$

2 $\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

3 $\lim_{x\to 2^+} f(x) = \frac{1}{3}$ $\quad \lim_{x\to 2^-} f(x) = 5$

$\therefore \lim_{x\to 2} \begin{cases} 3x-1 & x<2 \\ \frac{1}{x^2-1} & x \ge 2 \end{cases}$ does not exist

4 $\lim_{x\to 0^-} \frac{|x|}{x} = -1 \quad \lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \therefore \lim_{x\to 0} \frac{|x|}{x}$ does not exist

5 $\lim_{x\to 6} (x-6)^{\frac{2}{3}} = 0$

6 $\lim_{x\to 3^-} [x] = 2 \quad \lim_{x\to 3^+} [x] = 3$

$\therefore \lim_{x\to 3} [x]$ does not exist.

Exercise 4B

1 $\lim_{x\to 1^-} f(x) = 0 \quad \lim_{x\to 1^+} f(x) = 2$

$\therefore \lim_{x\to 1} f(x)$ does not exist

$\therefore f$ is not continuous at $x = 1$

2 $\lim_{x\to -2^-} f(x) = 1 \quad \lim_{x\to -2^+} f(x) = 1 \quad \therefore \lim_{x\to -2} f(x) = 1$

Also, $f(-2) = 1$ $\quad \therefore f$ is continuous at $x = -2$

3 $\lim_{x\to 1^-} f(x) = -1 \quad \lim_{x\to 1^+} f(x) = 1$

$\therefore \lim_{x\to 1} f(x)$ does not exist

$\therefore f$ is not continuous at $x = 1$

4 $\lim_{x\to 3^-} f(x) = 8 \quad \therefore f(3) = 8 \quad \therefore 6 = 8 \quad \therefore k = \frac{4}{3}$

5 $\lim_{x\to 3^-} f(x) = 4 \quad \therefore f(3) = 4 \quad \therefore 9a - a = 4$

$8a = 4$

$a = \frac{1}{2}$

6 **a** discontinuous at $x = \pm 1$

b discontinuous at $x = \pm 2$

c continuous

d discontinuous at $x = -4$ and $x = 1$

e discontinuous at $x = 1$

f continuous

Exercise 4C

1 **a** $\lim_{x\to 4} \left(\frac{x+3}{x-3}\right) = 7$

b $\lim_{x\to -2} \left(\frac{x^2+x-2}{x+2}\right) = \lim_{x\to -2} \left(\frac{(x+2)(x-1)}{(x+2)}\right)$

$= \lim_{x\to -2} (x-1) = -3$

c $\lim_{x\to -2} \left(\frac{x^6-64}{x^3-8}\right) = \lim_{x\to -2} \left(\frac{(x^3+8)(x^3-8)}{x^3-8}\right)$

$= \lim_{x\to -2} (x^3+8) = 0$

d $\lim_{x\to 0} \left(\frac{x^2-1}{x^2-x}\right) = \lim_{x\to 0} \left(\frac{(x-1)(x+1)}{x(x-1)}\right)$

$= \lim_{x\to 0} \left(\frac{x+1}{x}\right) = \lim_{x\to 0} \left(1+\frac{1}{x}\right),$

which does not exist

e $\lim_{x\to 1} \frac{x^2-1}{x^2-x} = \lim_{x\to 1} \left(1+\frac{1}{x}\right) = 2$

f $\lim_{x\to 1} \frac{1}{1+\frac{1}{1-x}} = \lim_{x\to 1} \frac{1-x}{2-x} = 0$

g $\lim_{x\to 0} \frac{(2+3x)^2 - 4(1+x)^2}{6x}$

$= \lim_{x\to 0} \frac{4+12x+9x^2-4-8x-4x^2}{6x}$

$= \lim_{x\to 0} \frac{5x^2+4x}{6x} = \lim_{x\to 0} \frac{5x+4}{6} = \frac{2}{3}$

h $\lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a}$

$= \lim_{x\to a} (x+a) = 2a$

2 **a** $\lim_{x\to\infty}\frac{2x}{x+2}=2$

b $\lim_{x\to\infty}\frac{3x^2}{x^2-1}=3$

c $\lim_{x\to\infty}\frac{2x^2+x-1}{3x^2+5x-1}=\frac{2}{3}$

d $\lim_{x\to\infty}\frac{5x^2}{4x^3+2}=0$

e $\lim_{x\to\infty}\frac{x-1}{x^2-3x+5}=0$

f $\lim_{x\to\infty}\frac{\sqrt{4x+3}+2\sqrt{1+x}}{\sqrt{x}}=4$

3 **a** $y=3$ **b** $y=\frac{1}{2}$ **c** $y=0$

d $y=-1$ **e** no horizontal asymptote

Exercise 4D

1 **a** converges **b** converges **c** converges

d diverges **e** converges

2 **a** converges, $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$

b $\sum_{n=1}^{\infty}\left(\frac{\pi}{3.14}\right)^n$ diverges since $\frac{\pi}{3.14}>1$

c converges, $\sum_{n=1}^{\infty}5\left(\frac{1}{3}\right)^n=\frac{\frac{5}{3}}{1-\frac{1}{3}}=\frac{5}{2}$

d converges, $\sum_{n=1}^{\infty}\frac{3}{10^n}=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{1}{3}$

e converges, $\sum_{n=1}^{\infty}\frac{2^n-3^n}{7^n}=\sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n-\sum_{n=1}^{\infty}\left(\frac{3}{7}\right)^n$

$=\frac{\frac{2}{7}}{\frac{5}{7}}-\frac{\frac{3}{7}}{\frac{4}{7}}=\frac{2}{5}-\frac{3}{4}=\frac{-7}{20}$

f converges, $\sum_{n=1}^{\infty}4(-0.6)^{n-1}=\frac{4}{1+0.6}=2.5$

3 $u_1=35$ $r=2^x$

a $-1<2^x<1$ 2^x must be positive

$\therefore 0<2^x<1$ $\therefore x<0$

b $\frac{35}{1-2^x}=40$ $\therefore 1-2^x=\frac{7}{8}$ $\therefore 2^x=\frac{1}{8}$ $\therefore x=-3$

4 $-1<\frac{3x}{x+1}<1$ $\therefore -0.25<x<0.5$

Exercise 4E

1 **a** $y=2x^2-1$ $(x=1)$

$\frac{\Delta y}{\Delta x}=\frac{[2(1+h)^2-1]-[2(1)^2-1]}{(1+h)-1}=\frac{(1+4h+2h^2)-1}{h}$

$=\frac{4h+2h^2}{h}=4+2h$

gradient $=\lim_{h\to0}(4+2h)=4$

b $y=\frac{2}{x}$ $(x=-2)$

$\frac{\Delta y}{\Delta x}=\frac{\frac{2}{-2+h}-\frac{2}{-2}}{(-2+h)-(-2)}=\frac{\frac{2}{-2+h}+1}{h}=\frac{2-2+h}{h(-2+h)}=\frac{1}{-2+h}$

gradient $=\lim_{h\to0}\left(\frac{1}{-2+h}\right)=-\frac{1}{2}$

c $y=x^3$ $(x=1)$

$\frac{\Delta y}{\Delta x}=\frac{(1+h)^3-1^3}{(1+h)-1}=\frac{1+3h+3h^2+h^3-1}{h}=3+3h+h^2$

gradient $=\lim_{h\to0}(3+3h+h^2)=3$

$\therefore$ gradient $=3$

d $y=-x^2$ $(x=1)$

$f'(x)=\lim_{h\to0}\frac{-(x+h)^2-(-x^2)}{h}$

$=\lim_{h\to0}\frac{(-x^2-2xh-h^2+x^2)}{h}$

$=\lim_{h\to0}(-2x-h)$

$=-2x$

$f'(1)=-2(1)=-2$

e $y=\frac{x}{x+1}$

$f'(x)=\lim_{h\to0}\frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h}$

$=\lim_{h\to0}\frac{(x+h)(x+1)-x(x+h+1)}{(x+h+1)(x+1)h}$

$=\lim_{h\to0}\frac{h}{(x+h+1)(x+1)h}=\lim_{h\to0}\frac{1}{(x+h+1)(x+1)}$

$=\frac{1}{(x+1)^2}$

$f'(0)=1$

f $y=\frac{1}{x^2}$ $(x=2)$

$f'(x)=\lim_{h\to0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim_{h\to0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}$

$=\lim_{h\to0}\frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}=\lim_{h\to0}\frac{-2x-h}{x^2(x+h)s'}$

$=\frac{-2x}{x^4}=\frac{-2}{x^3}$

$f'(2)=\frac{-2}{2^3}=-\frac{1}{4}$

2 $f'(x)=-\frac{2}{x^3}$

$f'(x)=2\Rightarrow\frac{-2}{x^3}=2$

$\Rightarrow x^3=-1\Rightarrow x=-1$ $\therefore$ point is $(-1, 1)$

3 $f'(x)=4x-\frac{1}{x^2}$

$f'(x)=3\Rightarrow 4x-\frac{1}{x^2}=3$

$\Rightarrow 4x^3-1=3x^2\Rightarrow 4x^3-3x^2-1=0$

$\Rightarrow (x-1)(4x^2+x+1)=0$

$\Rightarrow x=1$ $\therefore$ point is $(1, 3)$

Exercise 4F

1 **a** $y = x^2 + 2x + 1 \quad f'(x) = 2x + 2 \quad f'(0) = 2$

b $y = x^3 - 1 \quad f'(x) = 3x^2 \quad f'(1) = 3$

c $y = \frac{2}{x} \quad f'(x) = \frac{-2}{x^2} \quad \therefore f'(3) = \frac{-2}{9}$

d $y = \sqrt{x-1} \quad f'(x) = \frac{1}{2\sqrt{x-1}} \quad \therefore f'(2) = \frac{1}{2}$

e $y = \sqrt{x+3}, \quad f'(x) = \frac{1}{2\sqrt{x+3}}, \quad f'(1) = \frac{1}{4}$

f $y = \frac{1}{\sqrt{x}}, \quad f'(x) = -\frac{1}{2\sqrt{x^3}}, \quad f'(4) = -\frac{1}{16}$

2 **a** Average velocity $= \frac{x(a+h) - x(a)}{h}$

$= \frac{12 - 5(a+h)^2 - 12 + 5a^2}{h}$

$= -10a - 5h$

b velocity $= \lim\limits_{h \to 0}$ (average velocity) $= -10a$

Exercise 4G

1 $y = 9 - x^2 \quad \frac{dy}{dx} = -2x$

a When $x = -1$, gradient $= 2$

b When $x = -1$, gradient $= 8$, $\frac{dy}{dx} = 2$

$\therefore$ tangent is $y - 8 = 2(x + 1)$ i.e. $y = 2x + 10$

c Normal is $y - 8 = -\frac{1}{2}(x + 1)$ i.e. $y = -\frac{1}{2}x + \frac{15}{2}$

2 $y = \frac{1}{x-1} \quad \frac{dy}{dx} = \frac{-1}{(x-1)^2} \quad \therefore \frac{-1}{(x-1)^2} = -1$

$\therefore (x - 1)^2 = 1 \quad x - 1 = \pm 1$

$x = 0$ or $2 \quad (0, -1), (2, 1)$

At $(0, -1) \quad y + 1 = -1(x - 0) \quad y = -x - 1$

At $(2, 1) \quad y - 1 = -1(x - 2) \quad y = -x + 3$

3 **a** $y = 4 - 3x - 3x^2 \quad \frac{dy}{dx} = -3 - 6x$

$-3 - 6x = 0 \quad \therefore x = x - \frac{1}{2} \quad \left(\frac{-1}{2}, \frac{19}{4}\right)$

b $y = x^3 + 1 \quad \frac{dy}{dx} = 3x^2$

$3x^2 = 0 \quad \therefore x = 0 \quad (0, 1)$

c $y = \frac{1}{x} \quad \frac{dy}{dx} = \frac{-1}{x^2} \quad \frac{-1}{x^2} \neq 0 \quad \therefore$ no points

d $y = x^2 - 3x \quad \frac{dy}{dx} = 2x - 3$

$2x - 3 = 0 \quad \therefore x = \frac{3}{2} \quad \left(\frac{3}{2}, \frac{-9}{4}\right)$

e $y = \sqrt{x} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \frac{1}{2\sqrt{x}} \neq 0 \quad \therefore$ no points

4 $y = x + \frac{1}{x} \quad \frac{dy}{dx} = 1 - \frac{1}{x^2}$

At $(1, 2) \quad \frac{dy}{dx} = 1 - 1 = 0$

Equation of tangent is $y = 2$

$\therefore$ normal is $x = 1$

Exercise 4H

1 **a** $y = 4 - x - 3x^2 \quad \frac{dy}{dx} = -1 - 6x$

b $y = 2x^4 - 3x + 1 \quad \frac{dy}{dx} = 8x^2 - 3$

c $y = 4x^3 - \frac{1}{x^3} + 2x^2 + \frac{2}{3x^2} = 4x^3 - x^{-3} + 2x^2 + \frac{2}{3}x^{-2}$

$\frac{dy}{dx} = 12x^2 + 3x^{-4} + 4x - \frac{4}{3}x^{-3}$

$= 12x^2 + \frac{3}{x^4} + 4x - \frac{4}{3x^3}$

d $y = \frac{2 - 3x^2 + 5x^4}{x} = 2x^{-1} - 3x + 5x^3$

$\frac{dy}{dx} = -2x^{-2} - 3 + 15x^2 = -\frac{2}{x^2} - 3 + 15x^2$

2 $y = 2(3x^2 - 2x) = 6x^2 - 4x$

$\frac{dy}{dx} = 12x - 4$

At $(1, -2) \quad \frac{dy}{dx} = 8$

Equation of tangent: $y - 2 = 8(x - 1)$

$y = 8x - 6$

3 $y = \frac{x-3}{x} = 1 - 3x^{-1}$

$\frac{dy}{dx} = 3x^{-2} = \frac{3}{x^2}$

At $(-1, 4) \quad \frac{dy}{dx} = 3 \quad \therefore$ gradient of normal $= -\frac{1}{3}$

Equation of normal: $y - 4 = -\frac{1}{3}(x + 1)$

$y = -\frac{1}{3}x + \frac{11}{3}$

Exercise 4I

1 **a** $y = (2x + 3)^5 \quad \frac{dy}{dx} = 5(2x + 3)^4(2) = 10(2x + 3)^4$

b $y = \sqrt{2 - 3x} = (2 - 3x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(2 - 3x)^{\frac{1}{2}}(-3) = \frac{-3}{2\sqrt{2 - 3x}}$

c $y = \frac{2}{x} - 3x + 5x^3$, so $\frac{dy}{dx} = -\frac{2}{x^2} - 3 + 15x^2$

d $y = \frac{-3}{\sqrt{5x^2 + 1}} = -3(5x^2 + 1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}(5x^2 + 1)^{-\frac{3}{2}}(10x) = \frac{15x}{\sqrt{(5x^2 + 1)^3}}$

2 $y = \sqrt{3x^2 - 2x} = (3x^2 - 2x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 2x)^{-\frac{1}{2}}(6x - 2) = \frac{3x - 1}{\sqrt{3x^2 - 2x}}$

At $x = 1$, $\frac{dy}{dx} = 2$ and $y = 1$, so tangent is

$y - 1 = 2(x - 1)$ i.e. $y = 2x - 1$

3 $y = \frac{x-3}{x} = 1 - \frac{3}{x}$

$\frac{dy}{dx} = \frac{3}{x^2}$

At $(1, -2)$, $\frac{dy}{dx} = 3$ so gradient of normal $= -\frac{1}{3}$

$\therefore$ equation of normal is $y + 2 = -\frac{1}{3}(x - 1)$

i.e. $y = -\frac{1}{3}x - \frac{5}{3}$

4 $y = \frac{1}{3x^2 - 6x + 1} = (3x^2 - 6x + 1)^{-1}$

$\frac{dy}{dx} = -(3x^2 - 6x + 1)^{-2}(6x - 6) = -\frac{(6x-6)}{(3x^2 - 6x + 1)^2}$

$6x - 6 = 0 \quad \therefore x = 1 \quad y = \frac{1}{3-6+1} = -\frac{1}{2} \quad \left(1, -\frac{1}{2}\right)$

5 $y = \sqrt{1 - \sqrt{x}} = \left(1 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}\left(1 - x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(-\frac{1}{2}x^{-\frac{1}{2}}\right) = -\frac{1}{4\sqrt{1-\sqrt{x}}\sqrt{x}}$

$\frac{dy}{dx} = \frac{-1}{4\sqrt{\sqrt{x} - x}}$

6 $y = (x-1)^4(3x-2)^{\frac{2}{3}},$

$\frac{dy}{dx} = (x-1)^4\,\frac{2}{3}(3x-2)^{-\frac{1}{3}} \times 3 + 4(x-1)^3(3x-2)^{\frac{2}{3}}$

$= \frac{2(x-1)^4}{(3x-2)^{\frac{1}{3}}} + 4(x-1)^3(3x-2)^{\frac{2}{3}}$

$= \frac{2(x-1)^3(7x-5)}{(3x-2)^{\frac{1}{3}}}$

Exercise 4J

1 $y = (x-1)(x+3)^3$

$u(x) = x - 1 \qquad u'(x) = 1$

$v(x) = (x+3)^3 \qquad v'(x) = 3(x+3)^2$

$\frac{dy}{dx} = (x-1)3(x+3)^2 + (x+3)^3(1)$

$= (x+3)^2(3(x-1) + (x+3))$

$= (x+3)^2(4x)$

$= 4x(x+3)^2$

2 $y = (2x-3)^2(4x+1)^3$

$u(x) = (2x-3)^2 \qquad u'(x) = 4(2x-3)$

$v(x) = (4x+1)^3 \qquad v'(x) = 12(4x+1)^2$

$\frac{dy}{dx} = (2x-3)^2\,12(4x+1)^2 + (4x+1)^3\,4(2x-3)$

$= 4(2x-3)(4x+1)^2[3(2x-3) + (4x+1)]$

$= 4(2x-3)(4x+1)^2(10x-8)$

$= 8(2x-3)(4x+1)^2(5x-4)$

3 $y = \frac{x+1}{x-1} = (x+1)(x-1)^{-1}$

$u(x) = x + 1 \qquad u'(x) = 1$

$v(x) = (x-1)^{-1} \qquad v'(x) = -(x-1)^{-2}$

$\frac{dy}{dx} = -(x+1)(x-1)^{-2} + (x-1)^{-1}$

$\frac{dy}{dx} = (x-1)^{-2}[-(x+1) + (x-1)] = \frac{-2}{(x-1)^2}$

4 $y = x\sqrt{1-2x}$

$u(x) = x \qquad u'(x) = 1$

$v(x) = (1-2x)^{\frac{1}{2}} \qquad v'(x) = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) = -(1-2x)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -x(1-2x)^{-\frac{1}{2}} + (1-2x)^{\frac{1}{2}}$

$= (1-2x)^{-\frac{1}{2}}[-x + (1-2x)] = \frac{1-3x}{\sqrt{1-2x}}$

5 $y = (x^4 - 3x + 1)^{-1}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{(x^4 - 3x + 1)^2}(4x^3 - 3) = \frac{3 - 4x^3}{(x^4 - 3x + 1)^2}$

Exercise 4K

1 a $y = \frac{x^2 - 7}{x^3}$

$u(x) = x^2 - 7 \qquad u'(x) = 2x$

$v(x) = x^3 \qquad v'(x) = 3x^2$

$\frac{dy}{dx} = \frac{x^3(2x) - (x^2 - 7)3x^2}{x^6}$

$= \frac{2x^2 - 3x^2 + 21}{x^4}$

$= \frac{21 - x^2}{x^4}$

b $y = \frac{x}{\sqrt{x^2 + 1}}$

$u(x) = x \qquad u'(x) = 1$

$v(x) = (x^2 + 1)^{\frac{1}{2}} \qquad v'(x) = (x^2 + 1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \left((x^2+1)^{\frac{1}{2}} - x^2(x^2+1)^{-\frac{1}{2}}\right) \div (x^2 + 1)$

$= \frac{(x^2+1) - x^2}{(x^2+1)^{\frac{3}{2}}} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$

c $y = \frac{1}{x^4 - 3x + 1} = (x^4 - 3x + 1)^{-1}$

$\frac{dy}{dx} = -(x^4 - 3x + 1)^{-2}(4x^3 - 3)$

$y = \frac{3 - 4x^3}{(x^4 - 3x + 1)^2}$

d $y = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$

$u(x) = 1 + x^{\frac{1}{2}} \qquad u'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$v(x) = 1 - x^{\frac{1}{2}} \qquad v'(x) = -\frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{\left(1 - x^{\frac{1}{2}}\right)\frac{1}{2}x^{\frac{1}{2}} + \left(1 - x^{\frac{1}{2}}\right)\frac{1}{2}x^{-\frac{1}{2}}}{\left(1 - x^{\frac{1}{2}}\right)^2}$

$= \frac{x^{-\frac{1}{2}}}{\left(1 - x^{\frac{1}{2}}\right)^2} = \frac{1}{\sqrt{x}(1 - \sqrt{x})^2}$

e $y = (\sqrt{x} - x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(\sqrt{x} - x)^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}} - 1\right)$

$= \frac{1}{4}\left(\sqrt{x}-x\right)^{-\frac{1}{2}}\left(x^{-\frac{1}{2}}-2\right)$

$= \frac{x^{-\frac{1}{2}}-2}{4\left(\sqrt{x}-x\right)^{\frac{1}{2}}} = \frac{1-2\sqrt{x}}{4\sqrt{x}\sqrt{(\sqrt{x}-x)}} = \frac{1-2\sqrt{x}}{4\sqrt{x-x\sqrt{x}}}$

f $y = \left(\frac{x}{1-\sqrt{x}}\right)^3 = \frac{x^3}{\left(1-\sqrt{x}\right)^3}$

$u(x) = x^3$ $\quad u'(x) = 3x^2$

$v(x) = \left(1-\sqrt{x}\right)^3$ $\quad v'(x) = 3\left(1-\sqrt{x}\right)^2\left(\frac{-1}{2}x^{\frac{-1}{2}}\right)$

$\frac{dy}{dx} = \frac{3x^2\left(1-\sqrt{x}\right)^3 + \frac{3}{2}x^{\frac{5}{2}}\left(1-\sqrt{x}\right)^2}{\left(1-\sqrt{x}\right)^6}$

$= \frac{3x^2\left(1-\sqrt{x}\right) + \frac{3}{2}x^{\frac{5}{2}}}{\left(1-\sqrt{x}\right)^4}$

$= \frac{6x^2 - 6x^{\frac{5}{2}} + 3x^{\frac{5}{2}}}{2\left(1-\sqrt{x}\right)^4}$

$= \frac{6x^2 + 3x^{\frac{5}{2}}}{2\left(1-\sqrt{x}\right)^4}$

$= \frac{3x^2\left(2-\sqrt{x}\right)}{2\left(1-\sqrt{x}\right)^4}$

2 $y = \frac{4x}{x^2+1}$ $\quad (x = -1)$

$u(x) = 4x$ $\quad u'(x) = 4$

$v(x) = x^2 + 1$ $\quad v'(x) = 2x$

$\frac{dy}{dx} = \frac{4(x^2+1) - 8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2}$

At $x = -1$, gradient = 0

3 $y = \frac{8}{4+x^2} = 8(4 + x^2)^{-1}$ $\quad (x = 1)$

$\frac{dy}{dx} = -8(4 + x^2)^{-2}(2x) = \frac{-16x}{\left(4+x^2\right)^2}$

At $\left(1, \frac{8}{5}\right)$, $\frac{dy}{dx} = \frac{-16}{25}$ gradient of normal $= \frac{25}{16}$

Equation of normal: $y - \frac{8}{5} = \frac{25}{16}(x-1)$, $y = \frac{25}{16}x + \frac{3}{80}$

4 $f(x) = \sqrt[3]{\left(1-\frac{1}{2+x}\right)^2} = \left(1-(2+x)^{-1}\right)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}\left(1-(2+x)^{-1}\right)^{-\frac{1}{3}}(2+x)^{-2}$

$= \frac{2}{3(2+x)^2\sqrt[3]{1-\frac{1}{2+x}}}$

Exercise 4L

1 $f(x) = 4x + 1 + \frac{1}{x}$

$f'(x) = 4 - x^{-2}$ $\quad f''(x) = 2x^{-3} = \frac{2}{x^3}$

2 $f(x) = x^4 - 2x - 1$

$f'(x) = 4x^3 - 2$ $\quad f''(x) = 12x^2$

$f'(0) = -2$ $\quad f''(-1) = 12$

3 $f(x) = x^4 - 4x^3 + 16x - 16$

$f'(x) = 4x^3 - 12x^2 + 16$

$f''(x) = 12x^2 - 24x$

$f(x) = (x + 2)(x - 2)^3 \Rightarrow x = -2$ or $x = 2$

$f(-2) \neq f'(-2) \neq f''(-2)$,
but $f(2) = f'(2) = f''(2) = 0$ so $x = 2$

4 $f(x) = x^4 + rx^2 + sx + t$

$f(-1) = 16 \Rightarrow r - s + t = 15$ (1)

$f'(x) = 4x^3 + 2rx + s$

$f'(-1) = -16 \Rightarrow s - 2r = -12$ (2)

$f''(x) = 12x^2 + 2r$

$f''(-1) = 16 \Rightarrow 12 + 2r = 16$ (3)

Solve equations (1), (2), (3) to find $r = 2$, $s = -8$, $t = 5$.

5 $s(t) = (t - 4)^3(3 - 2t)^2$

a Velocity $= s'(t) = (t - 4)^3 2(3 - 2t)(-2)$
$= 3(t - 4)^2(3 - 2t)^2$

$s'(t) = (t - 4)^2(3 - 2t)[-4(t - 4) + 3(3 - 2t)]$
$= (t - 4)^2(3 - 2t)(25 - 10t)$

$s'(4) = 0$ ms^{-1}

b $s'(t) = (t - 4)^2(75 - 80t + 20t^2)$

$s''(t) = (t - 4)^2(-80 + 40t) + 2(t - 4)(70 - 80t + 20t^2)$

$= (t - 4)[-80t + 40t^2 + 320 - 160t + 150 - 160t + 40t^2]$

$= (t - 4)(80t^2 - 400t + 470)$

acceleration $= s''(4) = 0$ ms^{-2}

c $s''(t) = 80t^3 - 720t^2 + 2070t - 1880$

jerk $= s'''(t) = 240t^2 - 1440t + 2070$

$s'''(1) = 240 - 1440 + 2070 = 870$ ms^{-1}

6 $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$ $\quad f''(x) = 2x^{-3}$ $\quad f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$ $\quad f^{(5)}(x) = -120x^{-6}$

$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

Exercise 4M

1 a $y = x^2 - 3x + 1$

$\frac{dy}{dx} = 2x - 3$

$2x - 3 = 0 \quad \therefore\ x = \frac{3}{2} \qquad y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 1 = -\frac{5}{4}$

x	$x < \frac{3}{2}$	$x > \frac{3}{2}$
$\frac{dy}{dx}$	−	+

$\therefore$ minimum value $= \frac{-5}{4} \quad \left(\text{when } x = \frac{3}{2}\right)$

b $y = -2x^3 + 6x^2 - 3)$

$\frac{dy}{dx} = -6x^2 + 12x$

$-6x^2 + 12x = 0 \quad \therefore\ 6x(-x + 2) = 0$

$x = 0$ or 2 $\quad (0, -3) \quad (2, 5)$

x	$x < 0$	$0 < x < 2$	$x > 2$
$\frac{dy}{dx}$	−	+	−

$\therefore$ minimum value $= -3$ (at $x = 0$)

maximum value $= 5$ (at $x = 2$)

c $y = 3x^4 - 2x^3 - 3x^2 + 4$

$\frac{dy}{dx} = 12x^3 - 6x^2 - 6x$

$12x^3 - 6x^2 - 6x = 0 \quad 6x(2x^2 - x - 1) = 0$

$6x(2x + 1)(x - 1) = 0$

$x = 0, \ \frac{-1}{2}$ or 1 $\quad (0, 4) \quad \left(\frac{-1}{2}, \frac{59}{16}\right) \quad (1, 2)$

x	$x < \frac{-1}{2}$	$\frac{-1}{2} < x < 0$	$0 < x < 1$	$x > 1$
$\frac{dy}{dx}$	−	+	−	+

$\therefore$ minimum values are $\frac{59}{16}$ at $x = -\frac{1}{2}$ and

2 (at $x = 1$) maximum value $= 4$ (at $x = 0$)

d $y = x^4 - 4x^3$

$\frac{dy}{dx} = 4x^3 - 12x^2$

$4x^3 - 12x^2 = 0 \qquad 4x^2(x - 3) = 0$

$x = 0$ or 3 $\quad (0, 0) \quad (3, -27)$

x	$x < 0$	$0 < x < 3$	$x > 3$
$\frac{dy}{dx}$	−	−	+

$\therefore$ horizontal point of inflexion at (0, 0)

minimum value $= -27$ (at $x = 3$)

Exercise 4N

1 a i $x = -1$ or 1 **ii** $]-\infty, -1[\ \cup\ ]1, \infty[$

iii $]-1, 1[$

b i $x = -1, \frac{1}{2}, \frac{3}{2}$ **ii** $]-1, \frac{1}{2}[\ \cup\ ]\frac{3}{2}, \infty[$

iii $]-\infty, -1[\ \cup\ ]\frac{1}{2}, \frac{3}{2}[$

2 a i $x = -\frac{1}{2}$ **ii** $]-\frac{1}{2}, \infty[$

iii $]-\infty, -\frac{1}{2}[$

3 a $y = -3x^2 + 6x - 1$

$\frac{dy}{dx} = -6x + 6 \quad -6x + 6 = 0 \ \Rightarrow x = 1 \quad (1, 2)$

x	$x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	−

maximum at (1, 2)

f is increasing for $-\infty, 1[$

f is decreasing for $1, \infty[$

b $y = x\sqrt{2 - x^2} \qquad \left(-\sqrt{2} \le x \le \sqrt{2}\right)$

$\frac{dy}{dx} = x\,\frac{1}{2}(2 - x^2)^{-\frac{1}{2}}(-2x) + (2 - x^2)^{\frac{1}{2}}$

$= \frac{-x^2}{(2 - x^2)^{\frac{1}{2}}} + (2 - x^2)^{\frac{1}{2}}$

$= \frac{-x^2 + (2 - x^2)}{\sqrt{2 - x^2}} = \frac{2 - 2x^2}{\sqrt{2 - x^2}}$

$\frac{2 - 2x^2}{\sqrt{2 - x^2}} = 0 \quad \Rightarrow x = \pm 1 \ (1, 1) \quad (-1, -1)$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	−	0	+	0	−

minimum at $(-1, -1)$, maximum at (1, 1)

increasing for $]-1, 1[$

decreasing for $\left[-\sqrt{2}, -1[\ \cup\ ]1, \sqrt{2}\right]$

c $y = \frac{x}{x^2 + 1} \quad \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$

$\frac{dy}{dx} = 0$ when $x = \pm 1 \quad \left(1, \frac{1}{2}\right) \quad \left(-1, \frac{-1}{2}\right)$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	−	0	+	0	−

minimum at $\left(-1, -\frac{1}{2}\right)$, maximum at $\left(1, \frac{1}{2}\right)$

increasing for $]-1, 1[$

decreasing for $]-\infty, -1[\ \cup\ ]1, \infty[$

d $y = x^{\frac{1}{3}}(x-2) = x^{\frac{4}{3}} - 2x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}}, \quad \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}} = 0$

$\therefore 4x - 2 = 0 \quad \therefore x = \frac{1}{2} \quad \left(\frac{1}{2}, \frac{-3}{2(2)^{\frac{1}{3}}}\right)$

x	$x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\frac{dy}{dx}$	−	0	+

minimum at $\left(\frac{1}{2}, \frac{3}{2^{\frac{4}{3}}}\right)$

increasing for $\left]\frac{1}{2}, \infty\right[$, decreasing for $\left]-\infty, \frac{1}{2}\right[$

e $y = x^2\sqrt{2-x^2} \qquad \left(-\sqrt{2} \le x \le \sqrt{2}\right)$

$\frac{dy}{dx} = x^2 \frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x) + 2x(2-x^2)^{\frac{1}{2}}$

$= \frac{-x^3 + 2x(2-x^2)}{(2-x^2)^{\frac{1}{2}}} = \frac{4x - 3x^3}{\sqrt{2-x^2}}$

$\frac{dy}{dx} = 0 \Rightarrow 4x - 3x^3 = 0,\ x(4 - 3x^2) = 0$

$x = 0, \pm\frac{2}{\sqrt{3}} \quad (0, 0), \left(\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right), \left(-\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right)$

x	$x < -\frac{2}{\sqrt{3}}$	$x = -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < 0$	$x = 0$
$\frac{dy}{dx}$	+	0	−	0

x	$0 < x < \frac{2}{\sqrt{3}}$	$x = \frac{2}{\sqrt{3}}$	$x > \frac{2}{\sqrt{3}}$
$\frac{dy}{dx}$	+	0	−

maxima at $\left(\frac{-2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right)$,

minimum at (0, 0),

increasing for $\left]-\sqrt{2}, -\frac{2}{\sqrt{3}}\right[\cup \left]0, \frac{2}{\sqrt{3}}\right[$

decreasing for $\left]-\frac{2}{\sqrt{3}}, 0\right[\cup \left]\frac{2}{\sqrt{3}}, \sqrt{2}\right]$

Exercise 40

1 a $y = 2x^3 + 3x^2 - 12x - 3$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$6(x^2 + x - 2) = 0$

$6(x + 2)(x - 1) = 0$

$x = -2$ or 1

$\frac{d^2y}{dx^2} = 12x + 6$

$f''(-2) = -18 < 0 \quad \therefore f$ has a maximum at $(-2, 17)$

$f''(1) = 18 > 0 \quad \therefore f$ has a minimum at $(1, -10)$

b $y = -x^4 + 2x - 1$

$\frac{dy}{dx} = -4x^3 + 2 \quad -4x^3 + 2 = 0 \therefore x^3 = \frac{1}{2} \quad x = \frac{1}{\sqrt[3]{2}}$

$= 0.794$

$\frac{d^2y}{dx^2} = -12x^2 < 0 \therefore$ maximum at (0.794, 0.191)

c $y = x^5 - 5x$

$\frac{dy}{dx} = 5x^4 - 5 \quad 5x^4 - 5 = 0, \quad x^4 = 1, \quad x = \pm 1$

$\frac{d^2y}{dx^2} = 20x^3$

$f''(-1) < 0 \quad \therefore$ maximum at $(-1, 4)$

$f''(1) > 0 \quad \therefore$ minimum at $(1, -4)$

d $y = \frac{12}{x^2 + 2x - 3} = 12(x^2 + 2x - 3)^{-1}$

$\frac{dy}{dx} = -12(x^2 + 2x - 3)^{-2}(2x + 2) = \frac{-24(x+1)}{\left(x^2 + 2x - 3\right)^2}$

$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = -1$

x	$x < -1$	$x = -1$	$x > -1$
$\frac{dy}{dx}$	+	0	−

maximum at $(-1, -3)$

e $y = \frac{3x + 3}{x(3 - x)} = \frac{3x + 3}{3x - x^2}$

$\frac{dy}{dx} = \frac{(3x - x^2)3 - (3x + 3)(3 - 2x)}{\left(3x - x^2\right)^2}$

$= \frac{9x - 3x^2 - 9x + 6x^2 - 9 + 6x}{\left(3x - x^2\right)^2}$

$= \frac{3x^2 + 6x - 9}{x^2(3 - x)^2} = \frac{3(x^2 + 2x - 3)}{x^2(3 - x)^2} = \frac{3(x + 3)(x - 1)}{x^2(3 - x)^2}$

$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = -3$ or 1

x	$x < -3$	$x = -3$	$-3 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	−	0	+

maximum at $\left(-3, \frac{1}{3}\right)$, minimum at $(1, 3)$

2 b $y = \frac{1 - x}{x^2 + 8}$

i $\frac{dy}{dx^2} = \frac{(-x^2 + 8) - 2x(1 - x)}{(x^2 + 8)^2} = \frac{x^2 - 2x - 8}{(x^2 + 8)^2}$

$\frac{dy}{dx} = 0 \Rightarrow (x - 4)(x + 2) = 0$

$x = -2$ or 4

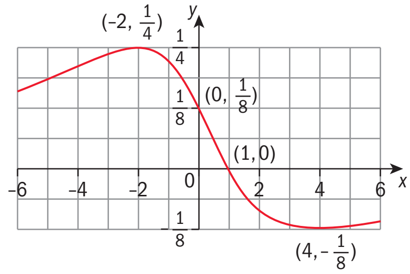
x	$x < -2$	$x = -2$	$-2 < x < 4$	$x = 4$	$x > 4$
$\frac{dy}{dx}$	+	0	−	0	+

maximum at $\left(-2, \frac{1}{4}\right)$ minimum at $\left(4, \frac{-1}{8}\right)$

ii f is increasing for $]-\infty, -2[\, \cup\,]4, \infty[$

f is decreasing for $]-2, 4[$

iii



Exercise 4P

1 a $y = x^3 - x$

i $\frac{dy}{dx} = 3x^2 - 1$ $\quad \frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad x = 0$

x	$x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	–	0	+

point of inflexion at $(0, 0)$

ii concave up for $]0, \infty[$

concave down for $]-\infty, 0[$

b $y = x^4 - 3x + 2$

i $\frac{dy}{dx} = 4x^3 - 3$ $\quad \frac{d^2y}{dx^2} = 12x^2$

$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad x = 0$

x	$x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	+	0	+

no point of inflexion

ii concave up for $]-\infty, 0[\, \cup\,]0, \infty[$

c $y = \sqrt{4x - x^2}$ where $x \in [0, 4]$

i $\frac{dy}{dx} = \frac{1}{2}\left(4x - x^2\right)^{-\frac{1}{2}}(4 - 2x) = \frac{2 - x}{\left(4x - x^2\right)^{\frac{1}{2}}}$

$$\frac{d^2y}{dx^2} = \frac{-\left(4x - x^2\right)^{\frac{1}{2}} - (2 - x)\frac{1}{2}\left(4x - x^2\right)^{-\frac{1}{2}}\left(4 - 2x\right)}{\left(4x - x^2\right)}$$

$$= \frac{-\left(4x - x^2\right) - (2 - x)^2}{\left(4x - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4x + x^2 - 4 + 4x - x^2}{\left(4x - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4x}{\left(4x - x^2\right)^{\frac{3}{2}}}$$

$\frac{d^2y}{dx^2} < 0 \quad \therefore$ no points of inflexion

ii concave down for $[0, 4]$

d $y = (x - 1)^{\frac{2}{3}}$

i $\frac{dy}{dx} = \frac{2}{3}(x - 1)^{-\frac{1}{3}}$ $\quad \frac{d^2y}{dx^2} = -\frac{2}{9}(x - 1)^{-\frac{4}{3}}$

$\frac{d^2y}{dx^2} = \frac{-2}{9(x - 1)^{\frac{4}{3}}}$ $\quad \frac{d^2y}{dx^2} \neq 0$

$\therefore$ no points of inflexion

ii

x	$x < 1$	$x > 1$
$\frac{d^2y}{dx^2}$	–	0

concave down for $]-\infty, 1[\cup]1, \infty[$

e $y = \frac{3x^2}{x - 1}$

i $\frac{dy}{dx} = \frac{6x(x - 1) - 3x^2}{(x - 1)^2} = \frac{3x^2 - 6x}{(x - 1)^2}$

$$\frac{d^2y}{dx^2} = \frac{(x - 1)^2(6x - 6) - (3x^2 - 6x)2(x - 1)}{(x - 1)^4}$$

$$= \frac{(x - 1)(6x - 6) - (3x^2 - 6x)2}{(x - 1)^3}$$

$$= \frac{6x^2 - 12x + 6 - 6x^2 + 12x}{(x - 1)^3}$$

$$= \frac{6}{(x - 1)^3}$$

$\frac{d^2y}{dx^2} \neq 0 \quad \therefore$ no points of inflexion

ii

x	$x < 1$	$x > 1$
$\frac{d^2y}{dx^2}$	–	+

concave up for $]1, \infty[$, concave down for $]-\infty, 1[$

Exercise 4Q

1 $s(t) = -5t^2 + 5t + 10$

a $s(0) = 10$ m

b $-5t^2 + 5t + 10 = 0$

$t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$

$\therefore t = 2$ seconds

c $v(t) = -10t + 5$ $\quad a(t) = -10$

$v(2) = -15\text{ ms}^{-1}$ $\quad a(2) = -10\text{ ms}^{-2}$

The diver is moving downwards and speeding up when he hits the water.

2 $s = 50t - 15t^2$

a $v = 50 - 30t = 0$ when $t = \frac{5}{3}$

maximum height $= 5\left(\frac{5}{3}\right) = 41\frac{2}{3}$ m

b $20 = 50t - 15t^2$

$3t^2 - 10t + 4 = 0$

$t = 0.4648$ or 2.8685

$v = 50 - 30t \quad v(0.4648) = 36.1\ \text{ms}^{-1}$

$v(2.8685) = -36.1\ \text{ms}^{-1}$

speed = 36.1 ms^{-1} upwards (when $t = 0.4648$) and downwards (when $t = 2.8685$)

c $a = -30\ \text{ms}^{-2}$

d $50t - 15t^2 = 0$

$5t(10 - 3t) = 0$

$\therefore$ rock hits the ground again when $t = \frac{10}{3}$ s

3 $s = 7t + 5t^2 - 2t^3$

a $v = 7 + 10t - 6t^2 \qquad a = 10 - 12t$

$v(0) = 7\ \text{ms}^{-1} \qquad a(0) = -10\ \text{ms}^{-2}$

Initially the particle is moving in a positive direction and is slowing down.

b $v(2) = 3\ \text{ms}^{-1} \qquad a(2) = -14\ \text{ms}^{-2}$

The particle is moving in a positive direction and slowing down.

4 $s = 10t^2 - t^3$

a $s(3) = 63\text{m} \quad \therefore$ average velocity $= \frac{63}{3} = 21\ \text{ms}^{-1}$

b $v = 20t - 3t^2 \qquad a = 20 - 6t$

$v(3) = 33\ \text{ms}^{-1} \qquad a(3) = 2\ \text{ms}^{-2}$

c Speeding up.

d $20t - 3t^2 = 0$

$t(20 - 3t) = 0$

$t = 0$ or $\frac{20}{3}$ direction changes when $t = \frac{20}{3}$ s

5 $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t$

a $v(t) = t^2 - 6t + 8 \qquad a(t) = 2t - 6$

b **i** $t^2 - 6t + 8 = 0$

$(t - 2)(t - 4) = 0$

$t = 2$ s or 4 s

ii

t	0 < t < 2	2 < t < 4	t > 4
v	+	−	+
t	0 < t < 3	t > 3	
a	−	+	

v and a have the same sign for $2 < t < 3$ and $t > 4$ $\therefore$ the particle is speeding up at these times.

iii the particle is slowing down for $0 < t < 2$ and $3 < t < 4$

c $a(2) = -2\ \text{ms}^{-1} \qquad a(4) = 2\ \text{ms}^{-2}$

the particle changes direction from positive to negative when $t = 2s$ and from negative to positive when $t = 4s$.

d $t = 2s$ and $t = 4s$.

e $0 - 2s$ distance $= s(2) - s(0) = 6\frac{2}{3} - 0 = 6\frac{2}{3}$

$2 - 4s$ distance $= s(4) - s(2) = 5\frac{1}{3} - 6\frac{2}{3} = -1\frac{1}{3}$

$4 - 5s$ distance $= s(5) - s(4) = 6\frac{2}{3} - 5\frac{1}{3} = 1\frac{1}{3}$

total distance $= 6\frac{2}{3} + 1\frac{1}{3} + 1\frac{1}{3} = 9\frac{1}{3}$ m

Exercise 4R

1 $c(x) = 20\,000 + 180x - 0.1x^2$

a $c'(x) = 180 - 0.2x$

b $c'(100) = 180 - 0.2 \times 100$

$= 160$ euros / tank

c $c(101) - c(100) = 159.9 \Rightarrow$ cost of producing 1 extra tank is nearly the same as the marginal cost function.

2 **a** **i** $p(x)$ must be > 0 so $0.002x < 7$ i.e. $x < 3500$

$\therefore$ domain is $0 < x < 3500$

ii $c'(x) = 3$ euros / unit $\Rightarrow$ it will always cost 3 euros to make an extra memory stick

iii $r(x) = x(7 - 0.002x)$

b Break-even points: $r(x) = c(x)$ when

$7x = 0.002x^2 = 500 + 3x$

i.e. $0.002x^2 - 4x + 500 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4}}{0.004}$

$= 134$ or 1870 (3 sf).

For profit, need to make x memory sticks where $134 < x < 1870$.

3 Average cost per 100 units $= \frac{c(x)}{x} = 500 + \frac{1000}{x}$

To minimize cost, need $\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$

$\Rightarrow 500 - \frac{1000}{x^2} = 0$

$\Rightarrow x^2 = 2 \Rightarrow x = 1.41$ (3 sf)

$\therefore$ costs are minimised by making 141 units.

4 **b** $r(x) = 35x - 3$ and $p(x) = r(x) - c(x)$

so $p(x) = 35x - 3 - 400 - 20x + 0.2x^2 - 0.0004x^3$

$= 15x - 403 + 0.2x^2 - 0.0004x^3$

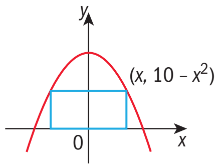
$p'(x) = 15 + 0.4x - 0.0012x^2 = 0$

$\Rightarrow x = 367$, so 367 jackets must be made to maximise profit.

c Minimising costs will not necessarily maximise profits.

Exercise 4S

1



$A = 2x(10 - x^2)$

$\quad = 20x - 2x^3$

$A' = 20 - 6x^2$

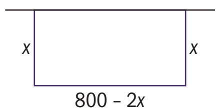
$20 - 6x^2 = 0 \Rightarrow x = \sqrt{\frac{10}{3}}$

$A'' = -12x$

If $x = \sqrt{\frac{10}{3}}$, $\quad A'' < 0 \quad \therefore$ max.

base $= 2\sqrt{\frac{10}{3}}$ $\quad$ height $= 10 - \frac{10}{3} = \frac{20}{3}$ (3.65 by 6.67)

2



$A = x(800 - 2x)$

$\quad = 800x - 2x^2$

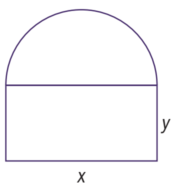
$A' = 800 - 4x$

$A' = 0 \quad \Rightarrow \quad x = 200$

$A'' = -4 < 0 \quad \therefore$ maximum

maximum area $= 200 \times 400 = 80\,000\ \text{m}^2$

3



$x + 2y + \frac{\pi x}{2} = 4$

$2x + 4y + \pi x = 8$

$y = \frac{8 - 2x - \pi x}{4}$

$A = xy = \frac{1}{4}(8x - 2x^2 - \pi x^2)$

$A' = \frac{1}{4}(8 - 4x - 2\pi x)$

$A = 0 \Rightarrow x(4 + 2\pi)\ 8$

$x = \frac{8}{4 + 2\pi} = 0.778 \quad y = 1$

$A'' = \frac{1}{4}(-4 - 2\pi) < 0 \quad \therefore$ maximum

dimensions: $\frac{8}{4 + 2\pi}$ m by 1m

Exercise 4T

1 **a** $3y^2 + x^2 = 4$

$6y\frac{dy}{dx} + 2x = 0 \quad \therefore \frac{dy}{dx} = \frac{-x}{3y}$

b $y^4 = x^3 + 1$

$4y^3\frac{dy}{dx} = 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2}{4y^3}$

c $x^2 + y^2 - 3x + 4y = 2$

$2x + 2y\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3 - 2x}{2y + 4}$

d $2x^2 - 3x^2y^2 + y^2 = 9$

$4x - 3(x^2\, 2y\frac{dy}{dx} + 2xy^2) + 2y\frac{dy}{dx} = 0$

$4x - 6x^2y\frac{dy}{dx} - 6xy^2 + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3xy^2 - 2x}{y - 3x^2y}$

e $(x + y)^2 = 5 - 2x$

$2(x + y)\left(1 + \frac{dy}{dx}\right) = -2$

$1 + \frac{dy}{dx} = -\frac{1}{x + y}$

$\frac{dy}{dx} = -\frac{1}{x + y} - 1 = -\frac{(1 + x + y)}{x + y}$

f $x^2 = \frac{x - y}{x + y} \quad x^3 + x^2y = x - y$

$3x^2 + x^2\frac{dy}{dx} + 2xy = 1 - \frac{dy}{dx}$

$\frac{dy}{dx}(x^2 + 1) = 1 - 3x^2 - 2xy$

$\frac{dy}{dx} = \frac{1 - 3x^2 - 2xy}{x^2 + 1}$

2 $x^2 - y^2 = 9 \qquad (5, 4)$

$2x - 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{x}{y} = \frac{5}{4}$

$y - 4 = \frac{5}{4}(x - 5)$

$y = \frac{5}{4}x - \frac{9}{4}$

3 $y^2 = 3x + 1 \qquad (1, -2)$

$2y\frac{dy}{dx} = 3 \quad \frac{dy}{dx} = \frac{3}{2y} = \frac{-3}{4}$

$y + 2 = \frac{4}{3}(x - 1)$

$y = \frac{4}{3}x - \frac{10}{3}$

4 $x^2 - \sqrt{3}xy + 2y^2 = 5 \qquad (\sqrt{3}, 2)$

$2x - \sqrt{3}\left(x\frac{dy}{dx} + y\right) + 4y\frac{dy}{dx} = 0$

$-\sqrt{3}\frac{dy}{dx} + 4y\frac{dy}{dx} = \sqrt{3}y - 2x$

$\frac{dy}{dx} = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x} = 0$

Tangent: $y = 2$ $\qquad$ Normal: $x = \sqrt{3}$

5 $x^2 + y^2 - 6x - 8y = 0$

$2x + 2y\frac{dy}{dx} - 6 - 8\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3 - x}{y - 4} \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 3$

$9 + y^2 - 18 - 8y = 0$

$y^2 - 8y - 9 = 0$
$(y - 9)(y + 1) = 0$
$y = 9$ or -1
stationary points $(3, 9)$, $(3, -1)$

6 $3x^2 + 2xy + y^2 = 3 \qquad (1, -2)$

$6x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{(3x+y)}{x+y} = -\frac{(3-2)}{-1} = 1$

$3x + x\frac{dy}{dx} + y + y\frac{dy}{dx} = 0$

$3 + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

$3 + \frac{d^2y}{dx^2} + 2 - 2\frac{d^2y}{dx^2} + 1 = 0$

$\frac{d^2y}{dx^2} = 6$

7 $x^2 + xy + y^2 = 3$

$y = 0 \quad x^2 = 3 \quad x = \pm\sqrt{3}$

$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{(2x+y)}{x+2y}$

At $(\sqrt{3}, 0)$ $\frac{dy}{dx} = -2$ At $(-\sqrt{3}, 0)$ $\frac{dy}{dx} = -2$

$\therefore$ tangents are parallel

9 $x + y = x^2 - 2xy + y^2$

a $1 + \frac{dy}{dx} = 2x - 2\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx}$

$1 + \frac{dy}{dx} = 2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx}$

$\frac{dy}{dx} + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - 2y - 1$

$\frac{dy}{dx} = \frac{2x-2y-1}{2x-2y+1}$

b $1 - \frac{dy}{dx} = 1 - \frac{(2x-2y-1)}{(2x-2y+1)}$

$= \frac{2x-2y+1-2x+2y+1}{2x-2y+1}$

$1 - \frac{dy}{dx} = \frac{2}{2x-2y+1}$

c $\frac{d^2y}{dx^2} = \frac{(2x-2y+1)\left(2-2\frac{dy}{dx}\right) - (2x-2y-1)\left(2-2\frac{dy}{dx}\right)}{(2x-2y+1)^2}$

$= \frac{4 - 4\frac{dy}{dx}}{(2x-2y+1)^2}$

$= \frac{4 - 4\frac{(2x-2y-1)}{2x-2y+1}}{(2x-2y+1)^2}$

$= \frac{4(2x-2y+1) - 4(2x-2y-1)}{(2x-2y+1)^3}$

$= \frac{8}{(2x-2y+1)^3} = \left(\frac{2}{2x-2y+1}\right)^3$

$\therefore \frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3$ (from **b**)

Exercise 4U

1 $A = \pi r^2 \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

2 $A = 2\pi r^2 + 2\pi rh$

$\frac{dA}{dt} = 4\pi r\frac{dr}{dt} + 2\pi r\frac{dh}{dt} + 2\pi r\frac{dr}{dt}$

$\frac{dA}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$

3 Let x = diagonal of the box.

$x^2 = l^2 + w^2 + h^2$

$2x\frac{dx}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}$

$\frac{dx}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt} + h\frac{dh}{dt}}{(l^2 + w^2 + h^2)^{\frac{1}{2}}}$

4 $\frac{dl}{dt} = 2\text{ cm s}^{-1} \quad \frac{dw}{dt} = -2\text{ cm s}^{-1}\ l = 12\text{ cm}, w = 5\text{ cm}$

a $A = lw$

$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}$

$= 12(-2) + 5(2)$

$= -14\text{ cm}^2\text{s}^{-1}$

b $p = 2l + 2w$

$\frac{dp}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$

$= 2(2) + 2(-2)$

$= 0\text{ cms}^{-1}$

c Let x = diagonal of the rectangle.

$x^2 = l^2 + w^2$

$2x\frac{dx}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} \quad (l = 12, w = 5, x = 13)$

$13\frac{dx}{dt} = 12(2) + 5(-2)$

$\frac{dx}{dt} = \frac{14}{13}\text{cms}^{-1}$

5 $\frac{dv}{dt} = 1.5\text{ m}^3\text{s}^{-1} \quad v = 81\text{ m}^3 \quad x = \sqrt[3]{81}\text{ m}$

Let x = side length of cube

$V = x^3 \qquad A = 6x^2$

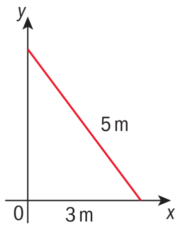
$\frac{dA}{dt} = \frac{dv}{dt} \times \frac{dx}{dv} \times \frac{dA}{dx}$

$= 1.5 \times \frac{1}{3x^2} \times 12x$

$= \frac{6}{x}$

$\frac{dA}{dt} = \frac{6}{\sqrt[3]{81}} = 1.39\text{ m}^2\text{s}^{-1}$

6



When $x = 3\,\text{m}$, $\frac{dx}{dt} = 0.5\ \text{ms}^{-1}$

a $x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$3(0.5) + 4\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -0.375$

$\therefore$ 0.375 ms^{-1} down the wall

b $A = \frac{1}{2}xy \quad \frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}y\frac{dx}{dt}$

$= \frac{1}{2}(3)(-0.375) + \frac{1}{2}(4)0.5$

$= 0.4375\ \text{m}^2\text{s}^{-1}$

7 $\frac{dA}{dt} = 2\ \text{cm}^2\text{s}^{-1}$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \qquad (r = 5)$

$2 = 10\pi\frac{dr}{dt} \qquad \frac{dr}{dt} = \frac{1}{5\pi} = 0.0637\ \text{cms}^{-1}$

10 $\frac{dr_1}{dt} = 1.2\ \text{ms}^{-1} \qquad \frac{dr_2}{dt} = 1.5\ \text{ms}^{-1}$

$A = \pi r_1^2 - \pi r_2^2$

$\frac{dA}{dt} = 2\pi r_1\frac{dr_1}{dt} - 2\pi r_2\frac{dr_2}{dt}$

$= 2\pi(9 \times 1.2 - 1 \times 1.5)$

$= 18.6\pi = 58.4\ \text{m}^2\text{s}^{-1}$

Review exercise

1 **a** $\lim_{x\to 1}\frac{x^3-3}{x+1} = -1$

b $\lim_{x\to 0}\frac{\sqrt{x^2-1}}{x}$ does not exist since the domain is $]-\infty, -1] \cup [1, \infty[$

c $\lim_{x\to 0}\frac{3^x-1}{x} = 1.10$ (3sf)

d $\lim_{x\to 0}\frac{3x^2+x^2}{x^2} = 4$

e $\lim_{x\to\infty}\frac{5x^2}{2x^3+1} = 0$

f $\lim_{x\to -\infty}\frac{7}{x^3+1} = 0$

2 $y = \begin{cases} x^2+2x & x \le 2 \\ x^3-6x & x > 2 \end{cases}$

$f(2) = 8 \qquad \lim_{x\to 2^-} f(x) = -4$

$f(x)$ is not continuous at $x = 2$

3 $a_n = \frac{2n^2-3}{n^3-2}$ sequence converges since $a_n \to 0$ as $n \to \infty$.

4 $\sum_{n=0}^{\infty} 3\left(\frac{(-1)^n}{5^n}\right)$ is a geometric series with $r = -\frac{1}{5}$ hence it converges.

$\sum_{n=0}^{\infty} 3\left(\frac{(-1)^n}{5^n}\right) = \frac{3}{1-\left(-\frac{1}{5}\right)} = 2.5$

5 Geometric series with, $r = \frac{1}{1+a^2}$

$\frac{1}{1+a^2} < 1$ provided $a \ne 0$

sum $= \frac{a^2}{1-\frac{1}{1+a^2}} = \frac{a^2(1+a^2)}{1+a^2-1} = 1+a^2$

6 $y = \frac{x^3-2x^2+5}{x^2-x^3}$

a $y = -1$

b $\frac{x^3-2x^2+5}{x^2-x^3} = -1$

$x^3-2x^2+5 = -x^2+x^3$

$5 = x^2$

$x = \pm\sqrt{5}$

$(\sqrt{5}, -1) \qquad (-\sqrt{5}, -1)$

7 $y = \frac{2x+1}{x^2+1} \qquad (0, 1)$

$\frac{dy}{dx} = \frac{2(x^2+1)-2x(2x+1)}{(x^2+1)^2}$

If $x = 0$, $\frac{dy}{dx} = 2$

Tangent: $y - 1 = 2x \qquad y = 2x + 1$

Normal: $y - 1 = -\frac{1}{2}x \qquad y = -\frac{1}{2}x + 1$

9 $x + y = -3 \qquad$ gradient $= -1$

$y = x\sqrt{x+1}$

$\frac{dy}{dx} = x\frac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}}$

$= \frac{x+2(x+1)}{2(x+1)^{\frac{1}{2}}}$

$= \frac{3x+2}{2(x+1)^{\frac{1}{2}}}$

$\frac{3x+2}{2(x+1)^{\frac{1}{2}}} = -1 \quad \Rightarrow 3x+2 = -2(x+1)^{\frac{1}{2}}$

$(3x+2)^2 = 4(x+1)$

$9x^2 + 12x + 4 = 4x + 4$

$9x^2 + 8x = 0$

$x(9x+8) = 0$

$x = 0 \text{ or } -\frac{8}{9}$

If $x = 0, \frac{dy}{dx} = 1$ If $x = -\frac{8}{9}, \frac{dy}{dx} = -1$

$\therefore\ y = x\sqrt{x+1}$ is parallel to the line $x + y = -3$ at $\left(-\frac{8}{9}, -\frac{8}{27}\right)$

10 $\frac{dy}{dx} = \frac{1}{2}(8x^3 - 15x^2 - 10x + 3) = -7$

normal: $y + \frac{5}{2} = \frac{1}{7}(x-1)$

$y = \frac{1}{7}x - \frac{37}{14}$

$\frac{1}{2}(2x^4 - 5x^3 - 5x^2 + 3) = \frac{1}{7}x - \frac{37}{14}$

$14x^4 - 35x^3 - 35x^2 + 21x = 2x - 37$

$14x^4 - 35x^3 - 35x^2 + 19x + 37 = 0$

$x = 3.0782 \quad y = -2.2031 \quad (3.08, -2.20)$

11 $f(x) = [g(x)]^3 \quad \therefore f(0) = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8} \quad \left(0, -\frac{1}{8}\right)$

$f'(x) = 3[g(x)]^2 g'(x)$

$f'(0) = 3\left(-\frac{1}{2}\right)^2\left(\frac{8}{3}\right) = 2$

$y + \frac{1}{8} = 2x$ or $y = 2x - \frac{1}{8}$

12 a $y = (1-3x)^7(3x+5)^3$

$\frac{dy}{dx} = (1-3x)^7\, 3(3x+5)^2\, 3 + (3x+5)^3$
$7(1-3x)^6(-3)$

$= 9(1-3x)^7(3x+5)^2 - 21(1-3x)^6(3x+5)^3$

$= 3(1-3x)^6(3x+5)^2[3(1-3x) - 7(3x+5)]$

$= 3(1-3x)^6(3x+5)^2(-30x - 32)$

$= -6(1-3x)^6(3x+5)^2(15x+16)$

b $y = \sqrt{(4x^2 - 3x + 1)^5}$

$\frac{dy}{dx} = \frac{5}{2}(4x^2 - 3x + 1)^{\frac{3}{2}}(8x - 3)$

c $y = \frac{x^2 - 3}{\sqrt{x+1}} \quad x \neq -1$

$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}}2x - \frac{1}{2}(x+1)^{-\frac{1}{2}}(x^2-3)}{(x+1)}$

$= \frac{4x(x+1) - (x^2-3)}{2(x+1)^{\frac{3}{2}}}$

$= \frac{3x^2 + 4x + 3}{2(x+1)^{\frac{3}{2}}}$

d $y = \sqrt{x + \sqrt{x^2+1}} \quad = \left(x + (x^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}\left(x + (x^2+1)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cancel{2}x\right)$

$= \frac{1 + x(x^2+1)^{-\frac{1}{2}}}{2\left(x + (x^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$

$= \frac{(x^2+1)^{\frac{1}{2}} + x}{2\left(x^2+1\right)^{\frac{1}{2}}\left(x + (x^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$

$= \frac{\left[(x^2+1)^{\frac{1}{2}} + x\right]^{\frac{1}{2}}}{2\left(x^2+1\right)^{\frac{1}{2}}}$

$= \frac{1}{2}\sqrt{\frac{(x^2+1)^{\frac{1}{2}} + x}{\left(x^2+1\right)}}$

e $y = (x + 2 + (x-3)^8)^3$

$\frac{dy}{dx} = 3(x + 2 + (x-3)^8)^2(1 + 8(x-3)^7)$

13 $f(x) = ax^3 + 6x^2 - bx$

$f'(x) = 3ax^2 + 12x - b$

$f''(x) = 6ax + 12$

$f''(1) = 0 \quad \therefore 6a + 12 = 0 \quad \therefore a = -2$

$f'(-1) = 0 \quad \therefore -6 - 12 - b = 0 \quad \therefore b = -18$

14 $y = x - 3\sqrt[3]{x} = x - 3x^{\frac{1}{3}}$

a $(0, 0) \quad x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3) = 0$

$x = 0 \text{ or } x^{\frac{2}{3}} = 3$

$x = \pm 27 \quad (\sqrt{27}, 0) \quad (-\sqrt{27}, 0)$

b $\frac{dy}{dx} = (1 - x^{\frac{-2}{3}})$

$1 - x^{\frac{-2}{3}} = 0 \quad \therefore 1 = \frac{1}{x^{\frac{2}{3}}} \quad \therefore x^{\frac{2}{3}} = 1 \quad x = \pm 1$

$\frac{d^2y}{dx^2} = \frac{2}{3}x^{\frac{-5}{3}} = \frac{2}{3\sqrt[3]{x^5}}$

If $x = 1$, $\frac{d^2y}{dx^2} > 0 \quad \therefore$ minimum at $(1, -2)$

If $x = -1$, $\frac{d^2y}{dx^2} < 0 \quad \therefore$ maximum at $(-1, 2)$

c $\frac{d^2y}{dx^2} \neq 0 \quad \therefore$ no points of inflexion

d

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	–	0	+

i function increases for $]-\infty, -1[\cup]1, \infty[$

ii function decreases for $]-1, 1[$

15 $y = \frac{2x}{x^2 - 1}$

a $x = 1, x = -1, y = 0$

b $f(-x) = \frac{-2x}{(-x)^2 - 1} = \frac{-2x}{x^2 - 1} = -f(x)$

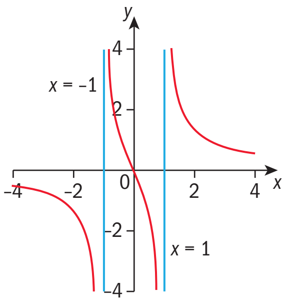
$\therefore$ function is odd

c $\frac{dy}{dx} = \frac{(x^2 - 1)2 - 2x(2x)}{(x^2 - 1)^2}$

$= \frac{-2x^2 - 2}{(x^2 - 1)^2} < 0$

$\therefore \frac{dy}{dx} < 0$

d



16 $f(x) = \frac{(x - 3)^2}{x^2 - 3}$

a $(0, -3) \quad (3, 0) \quad x = \pm\sqrt{3} \quad y = 1$

b $f'(x) = \frac{(x^2 - 3)2(x - 3) - 2x(x - 3)^2}{(x^2 - 3)^2}$

$= \frac{2(x - 3)[x^2 - 3 - x(x - 3)]}{(x^2 - 3)^2}$

$= \frac{2(x - 3)(3x - 3)}{(x^2 - 3)^2} = \frac{6(x - 3)(x - 1)}{(x^2 - 3)^2}$

$f'(x) = 0 \quad \Rightarrow \quad x = 3 \text{ or } 1$

$f'(x) = \frac{6x^2 - 24x + 18}{(x^2 - 3)^2}$

$f''(x) = \frac{(x^2 - 3)^2(12x - 24) - 2(x^2 - 3)2x(6x^2 - 24x + 18)}{(x^2 - 3)^4}$

$= \frac{(x^2 - 3)(12x - 24) - 4x(6x^2 - 24x + 18)}{(x^2 - 3)^3}$

$= \frac{-12x^3 + 72x^2 - 108x + 72}{(x^2 - 3)^3}$

If $x = 3, f''(x) > 0 \quad \therefore$ minimum at $(3, 0)$

If $x = 1, f''(x) < 0 \quad \therefore$ maximum at $(1, -2)$

c $f''(x) = 0 \quad \Rightarrow \quad x = 4.1958$

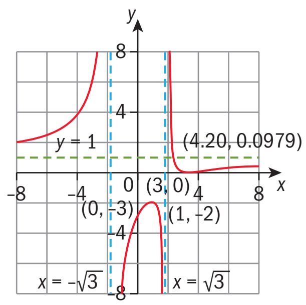
x	$x < 4.1958$	$x > 4.1958$
$f''(x)$	+	–

$\therefore$ point of inflexion at $(4.20, 0.0979)$

d **i** increasing for $]-\infty, -\sqrt{3}[\cup]-\sqrt{3}, 1[\cup]3, \infty[$

ii decreasing for $]1, \sqrt{3}[\cup]\sqrt{3}, 3[$

e



17 $x = y^5 - y \quad 0 = y(y^4 - 1)$

$y = 0, \pm 1 \ (0, 0) \ (0, 1) \ (0, -1)$

$\frac{dx}{dy} = 5y^4 - 1 \quad \therefore \frac{dy}{dx} = \frac{1}{5y^4 - 1}$

At $(0, 0)$ $\frac{dy}{dx} = -1$

At $(0, 1)$ $\frac{dy}{dx} = \frac{1}{4}$

Review exercise

1 $(1.5, 0) \quad (x, \sqrt{x}) \quad$ Let l = distance

$l^2 = (x - 1.5)^2 + x$

$= x^2 - 2x + 2.25$

$l = (x^2 - 2x + 2.25)^{\frac{1}{2}}$

$\frac{dl}{dx} = \frac{1}{2}(x^2 - 2x + 2.25)^{\frac{1}{2}}(2x - 2)$

$= \frac{x - 1}{(x^2 - 2x + 2.25)^{\frac{1}{2}}}$

$\frac{dl}{dx} = 0 \Rightarrow x = 1$

x	$x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	–	0	+

$\therefore$ minimum distance when $x = 1$

minimum distance $= \sqrt{1.25}$

2 Let r = radius, x = side of square

$4\pi r + 4x = 80$

$x = 20 - \pi r$

$A = 2\pi r^2 + x^2$

$= 2\pi r^2 + (20 - \pi r)^2$

$= 2\pi r^2 + 400 - 40\pi r + \pi^2 r^2$

$\frac{dA}{dr} = 4\pi r - 40\pi + 2\pi^2 r$

$\frac{dA}{dr} = 0 \Rightarrow 4\pi r + 2\pi^2 r = 40\pi$

$2r + \pi r = 20$

$r = \frac{20}{2 + \pi}$

$\frac{d^2A}{dr^2} = 4\pi + 2\pi^2 > 0 \therefore$ minimum

$r = \frac{20}{2 + \pi}$

3 $\frac{dr}{dt} = 3$ cm min^{-1} $\quad \frac{dh}{dt} = -4$ cm min^{-1}

$v = \pi r^2 h$

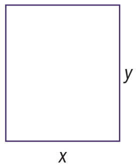
$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh \frac{dr}{dt}$

$= \pi(81)(-4) + 2\pi(9)(12)(3)$

$= 324\pi$ cm^3 min^{-1}

increasing at a rate of 324π cm^3 m^{-1}

4



$xy = 180 \quad y = \frac{180}{x}$

printing area, $A = (x - 2)(y - 3)$

$A = xy - 3x - 2y + 6$

$= 180 - 3x - \frac{360}{x} + 6$

$\frac{dA}{dx} = -3 + \frac{360}{x^2}$

$\frac{dA}{dx} = 0 \Rightarrow 3x^2 = 360$

$x^2 = 120$

$x = \sqrt{120} = 10.95 \quad y = 16.43$

$\frac{d^2A}{dx^2} = \frac{-720}{x^3} < 0 \therefore$ maximum

$\therefore$ dimensions are 11.0 cm by 16.4 cm

5 $\frac{dx}{dt} = \frac{1}{1 + 2x} = (1 + 2x)^{-1}$

acceleration $= \frac{d^2x}{dt^2} = -(1 + 2x)^{-2}\, 2\frac{dx}{dt}$

$= \frac{-2}{(1 + 2x)^3}$

At $x = 2$, acceleration $= \frac{-2}{125}$

7 The evolution of calculus

Answers

Skills check

1 **a** $y = x\ln x \quad \frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$

b $y = \frac{e^{2x-3}}{\sqrt{2-x}}$

$$\frac{dy}{dx} = \frac{\left((2-x)^{\frac{1}{2}}2e^{2x-3} + e^{2x-3}\frac{1}{2}(2-x)^{-\frac{1}{2}}\right)}{(2-x)}$$

$$= \frac{4(2-x)e^{2x-3} + e^{2x-3}}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{e^{2x-3}(9-4x)}{2(2-x)^{\frac{3}{2}}}$$

c $y = x^4 - \frac{1}{x^4}$

$\frac{dy}{dx} = 4x^3 + 4x^{-5} = 4x^3 + \frac{4}{x^5}$

2 **a** $y = 3x - 2 \quad y^2 = x^2 - 2x + 4$

$x^2 - 2x + 4 = 3x - 2$

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2$ or 3 $\quad$ $(2, 4)$ or $(3, 7)$

b $y = 1 - x \quad y = \sqrt{2x+1}$

$1 - x = \sqrt{2x+1} \quad (1)$

$(1-x)^2 = 2x + 1$

$1 - 2x + x^2 = 2x + 1$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0$ or 4

Check in (1)

if $x = 0$, LHS = 1 $\quad$ RHS = 1 ✓

if $x = 4$, LHS = −3 $\quad$ RHS = 3 ✗

$\therefore x = 0 \; (0, 1)$

c $y = \frac{6}{x} + 3x \qquad y = x^3 - 5x$

$\frac{6}{x} + 3x = x^3 - 5x$

$6 + 3x^2 = x^4 - 5x^2$

$x^4 - 8x^2 - 6 = 0$

$x = -2.948$ or 2.948

$(-2.95, -10.9)$ or $(2.95, 10.9)$

3 $s(t) = 3t^4 - t^3 + t$

$v(t) = s'(t) = 12t^3 - 3t^2 + 1$

$a(t) = v'(t) = 36t^2 - 6t$

Exercise 7A

1 $\int -2x\,dx = -x^2 + c$

2 $\int 3x^8\,dx = \frac{x^9}{3} + c$

3 $\int -5x^4\,dx = -x^5 + c$

4 $\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{-x^{-4}}{4} + c = \frac{-1}{4x^4} + c$

5 $\int \sqrt{x^3}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + c$

6 $\int \frac{1}{\sqrt{x^3}}\,dx = \int \frac{1}{x^{\frac{3}{2}}}\,dx = \int x^{\frac{-3}{2}}\,dx = -2x^{\frac{-1}{2}} + c = \frac{-2}{\sqrt{x}} + c$

7 $\int \frac{2x}{\sqrt{x}}\,dx = \int 2x^{\frac{1}{2}}\,dx = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{4}{3}x^{\frac{3}{2}} + c$

8 $\int \frac{-\sqrt[4]{x^5}}{7x^3}\,dx = \int -\frac{1}{7}x^{\frac{-7}{4}}\,dx = -\frac{1}{7}x^{\frac{-3}{4}}\left(\frac{-4}{3}\right) + c$

$$= \frac{4}{21}x^{\frac{-3}{4}} + c$$

Exercise 7B

1 **a** $\int \left(5x^2 - \frac{1}{5x^2}\right)dx = \int \left(5x^2 - \frac{1}{5}x^{-2}\right)dx$

$$= \frac{5x^3}{3} + \frac{1}{5x} + c$$

b $\int (x+3)(2x-1)dx = \int (2x^2 + 5x - 3)dx$

$$= \frac{2x^3}{3} + \frac{5x^2}{2} - 3x + c$$

c $\int \frac{x^2 - 1}{x^4}\,dx = \int (x^{-2} - x^{-4})dx$

$$= -x^{-1} + \frac{x^{-3}}{3} + c = -\frac{1}{x} + \frac{1}{3x^3} + c$$

d $\int \left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2})dx$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

e $\int \frac{(x+3)(x-4)}{x^5}\,dx = \int (x^{-3} - x^{-4} - 12x^{-5})dx$

$$= \frac{-x^{-2}}{2} + \frac{x^{-3}}{3} + \frac{12x^{-4}}{4} + c$$

$$= -\frac{1}{2x^2} + \frac{1}{3x^3} + \frac{3}{x^4} + c$$

f $\int\left(\sqrt{x}-\frac{5}{\sqrt[3]{x}}\right)\mathrm{d}x=\int\left(x^{\frac{1}{2}}-5x^{\frac{-1}{3}}\right)\mathrm{d}x$

$=\frac{2}{3}x^{\frac{3}{2}}-5x^{\frac{2}{3}}\left(\frac{3}{2}\right)+c$

$=\frac{2}{3}x^{\frac{3}{2}}-\frac{15}{2}x^{\frac{2}{3}}+c$

2 $\frac{\mathrm{d}y}{\mathrm{d}x}=(3x^2-4)\quad(2,-1)$

$y=x^3-4x+c$

$-1=8-8+c\quad\therefore c=-1$

$y=x^3-4x-1$

3 $f'(t)=t+3-\frac{1}{t^2}\qquad\left(1,\frac{-1}{2}\right)$

$f(t)=\frac{t^2}{2}+3t+\frac{1}{t}+c$

$-\frac{1}{2}=\frac{1}{2}+3+1+c\quad\therefore c=-5$

$f(t)=\frac{t^2}{2}+3t+\frac{1}{t}-5$

4 $\frac{\mathrm{d}y}{\mathrm{d}x}=(2x+3)^3=8x^3+3(2x)^2(3)+3(2x)3^2+3^3$

$=8x^3+36x^2+54x+27$

$y=2x^4+12x^3+27x^2+27x+c$

$z=2-12+27-27+c\quad\therefore c=12$

$y=2x^4+12x^3+27x^2+27x+12=\frac{(2x+3)^4+15}{8}$

5 $\frac{\mathrm{d}A}{\mathrm{d}x}=(2x+1)(x^2-1)=2x^3+x^2-2x-1$

$A=\frac{x^4}{2}+\frac{x^3}{3}-x^2-x+c$

$0=\frac{1}{2}+\frac{1}{3}-1-1+c\quad\therefore c=\frac{7}{6}$

$A=\frac{x^4}{2}+\frac{x^3}{3}-x^2-x+\frac{7}{6}$

6 $\frac{\mathrm{d}s}{\mathrm{d}t}=3t-\frac{8}{t^2}$

$s=\frac{3t^2}{2}+\frac{8}{t}+c$

$1.5=1.5+8+c\quad\therefore c=-8$

$s=\frac{3t^2}{2}+\frac{8}{t}-8$

7 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x-1\quad\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2-x+c$

$4=12-2+c\quad\therefore c=-6$

$\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2-x-6$

$y=x^3-\frac{x^2}{2}-6x+c$

$0=8-2-12+c\quad\therefore c=6$

$y=x^3-\frac{x^2}{2}-6x+6$

8 $a(t)=6t+1$

$v(t)=3t^2+t+c$

$2=c\quad\therefore v(t)=3t^2+t+2$

$s(t)=t^3+\frac{t^2}{2}+2t+c$

$1=c\quad\therefore s(t)=t^3+\frac{t^2}{2}+2t+1$

Exercise 7C

1 $\int(3x-1)^7\,\mathrm{d}x=\frac{(3x-1)^8}{24}+c$

2 $\int-2\sqrt{2x+1}\,\mathrm{d}x=\int-2(2x+1)^{\frac{1}{2}}\,\mathrm{d}x$

$=\frac{-2(2x+1)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}+c=\frac{-2(2x+1)^{\frac{3}{2}}}{3}+c$

3 $\int\frac{1}{(4x-1)^5}\,\mathrm{d}x=\int(4x-1)^{-5}\,\mathrm{d}x$

$=\frac{(4x-1)^{-4}}{4(-4)}+c=\frac{-1}{16(4x-1)^4}+c$

4 $\int\frac{2}{\sqrt[4]{3-x}}\,\mathrm{d}x=\int 2(3-x)^{-\frac{1}{4}}\,\mathrm{d}x$

$=\frac{2(3-x)^{\frac{3}{4}}}{\frac{-3}{4}}+c=\frac{-8(3-x)^{\frac{3}{4}}}{3}+c$

5 $\int\left(\frac{2}{(2-5x)^{\frac{1}{3}}}+\sqrt[3]{1-x}\right)\mathrm{d}x=\int\left(2(2-5x)^{\frac{-1}{3}}+(1-x)^{\frac{1}{3}}\right)\mathrm{d}x$

$=\frac{2(2-5x)^{\frac{2}{3}}}{-5\left(\frac{2}{3}\right)}+\frac{(1-x)^{\frac{4}{3}}}{\frac{-4}{3}}+c$

$=\frac{-3(2-5x)^{\frac{2}{3}}}{5}-\frac{3(1-x)^{\frac{4}{3}}}{4}+c$

6 $\int\left(4\sqrt{2-3x}-6(3x+2)^{\frac{2}{3}}\right)\mathrm{d}x$

$=\int\left(4(2-3x)^{\frac{1}{2}}-6(3x+2)^{\frac{2}{3}}\right)\mathrm{d}x$

$=\frac{4(2-3x)^{\frac{3}{2}}}{-3\left(\frac{3}{2}\right)}-\frac{6(3x+2)^{\frac{5}{3}}}{3\left(\frac{5}{3}\right)}+c$

$=\frac{-8(2-3x)^{\frac{3}{2}}}{9}-\frac{6(3x+2)^{\frac{5}{3}}}{5}+c$

Exercise 7D

1 $\int-5\mathrm{e}^{-2x}\,\mathrm{d}x=\frac{5\mathrm{e}^{-2x}}{2}+c$

2 $\int\frac{1}{\mathrm{e}^{3x+2}}\,\mathrm{d}x=\int\mathrm{e}^{-3x-2}\,\mathrm{d}x$

$=\frac{-1}{3}\mathrm{e}^{-3x-2}+c$

3 $\int\left(\sqrt[3]{e^x}-\frac{2}{e\sqrt{e^{2x}}}\right)dx=\int(e^{\frac{x}{3}}-2e^{-x-1})dx$

$=3e^{\frac{x}{3}}+2e^{-x-1}+c$

4 $\int 3^x\,dx=\frac{3^x}{\ln 3}+c$

5 $\int\frac{1}{3^{2x}}\,dx=\int 3^{-2x}\,dx=-\frac{3^{-2x}}{2\ln 3}+c$

6 $\int 4^{1-x}\,dx=-\frac{4^{1-x}}{\ln 4}+c$

7 $\int m^{ax+b}\,dx$ Let $u=ax+b$

$\frac{du}{dx}=a\quad\therefore dx=\frac{du}{a}$

$\int m^{ax+b}\,dx=\int m^u\frac{du}{a}=\frac{1}{a}\int m^u\,du$

$=\frac{1}{a}\frac{m^u}{\ln(m)}+c$

$=\frac{1}{a\ln(m)}m^{ax+b}+c$

Exercise 7E

1 $\int\frac{1}{3x}\,dx=\frac{1}{3}\ln|x|+c$

2 $\int-\frac{6}{x}dx=-6\ln|x|+c$

3 $\int\frac{1}{2-3x}\,dx=-\frac{1}{3}\ln|2-3x|+c$

4 $\int\frac{5}{3-5x}\,dx=-\ln|3-5x|+c$

5 $\int-2(4+3x)^{-1}\,dx=-\frac{2}{3}\ln|4+3x|+c$

Exercise 7F

1 $\int_1^3\left(3x+\frac{1}{x^2}\right)dx=\left[\frac{3x^2}{2}-\frac{1}{x}\right]_1^3=\frac{27}{2}-\frac{1}{3}-\frac{3}{2}+1$

$=12-\frac{1}{3}+1$

$=\frac{38}{3}$

2 $\int_0^2 3\sqrt{4x+1}\,dx=3=\left[\frac{(4x+1)^{\frac{3}{2}}}{4\left(\frac{3}{2}\right)}\right]_0^2$

$=\frac{1}{2}\left[(4x+1)^{\frac{3}{2}}\right]_0^2$

$=\frac{1}{2}(27-1)=13$

3 $\int_{-1}^2-2e^{1-3x}\,dx=\frac{2}{3}\left[e^{1-3x}\right]_{-1}^2$

$=\frac{2}{3}(e^{-5}-e^4)=\frac{2(1-e^9)}{3e^5}$

4 $\int_1^3 3.2^{x+1}dx=\frac{3}{\ln 2}\left[2^{x+1}\right]_1^3$

$=\frac{3}{\ln 2}(16-4)=\frac{36}{\ln 2}$

5 $\int_{-2}^0 2(1-3x)^5dx=\frac{2}{-3(6)}\left[(1-3x)^6\right]_{-2}^0$

$=-\frac{1}{9}\left[1-7^6\right]$

$=13072$

6 $\int_1^4\frac{1-\sqrt{x}}{\sqrt{x}}dx=\int_1^4(x^{-\frac{1}{2}}-1)dx$

$=\left[2x^{\frac{1}{2}}-x\right]_1^4$

$=(4-4)-(2-1)=-1$

Exercise 7G

1 $\int_{-1}^0(2r-1)^4dr=\left[\frac{(2r-1)^5}{10}\right]_{-1}^0$

$=\frac{1}{10}((-1)-(-3)^5)=\frac{242}{10}=\frac{121}{5}$

2 Not possible, $s\neq 0$

3 Not possible, $x\neq\pm 1$, $1\in[0,2]$

4 $\int_0^1\frac{dx}{(2x+1)^3}=\int_0^1(2x+1)^{-3}dx$

$=-\frac{1}{4}\left[(2x+1)^{-2}\right]_0^1=-\frac{1}{4}\left[\frac{1}{(2x+1)^2}\right]_0^1$

$=-\frac{1}{4}\left(\frac{1}{9}-1\right)=\frac{2}{9}$

5 Not possible, $x\neq -1$

6 $\int_0^1\left(\frac{3}{3x+4}-\frac{2}{x+1}\right)dx=[\ln|3x+4|-2\ln|x+1|]_0^1$

$=(\ln 7-2\ln 2)-(\ln 4-2\ln 1)$

$=\ln 7-\ln 4-\ln 4$

$=\ln\frac{7}{16}$

7 $\int_{-1}^1\frac{e^x+4}{e^x}dx=\int_{-1}^1(1+4e^{-x})dx$

$=\left[x-4e^{-x}\right]_{-1}^1$

$=(1-4e^{-1})-(-1-4e)$

$=2-\frac{4}{e}+4e$

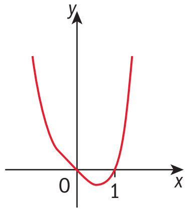
8 $\int_0^2 10^xdx=\frac{1}{\ln 10}\left[10^x\right]_0^2$

$=\frac{1}{\ln 10}(100-1)$

$=\frac{99}{\ln 10}$

Exercise 7H

1 $y = x^4 - x = x(x^3 - 1)$



$$\int_{-1}^{0}(x^4 - x)\mathrm{d}x = \left[\frac{x^5}{5} - \frac{x^2}{2}\right]_{-1}^{0}$$

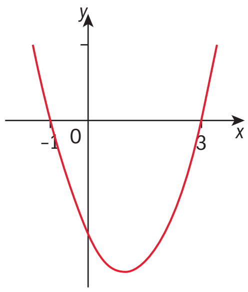
$$= 0 - \left(\frac{-1}{5} - \frac{1}{2}\right) = \frac{7}{10}$$

$$\int_{0}^{1}(x^4 - x)\,\mathrm{d}x = \left[\frac{x^5}{5} - \frac{x^2}{2}\right]_{0}^{1}$$

$$= \frac{1}{5} - \frac{1}{2} = -\frac{3}{10}$$

$\therefore$ area $= \frac{7}{10} + \frac{3}{10} = 1$ sq. unit

2 $y = x^2 - 2x - 3 = (x - 3)(x + 1)$



$$A = \int_{-3}^{-1}(x^2 - 2x - 3)\mathrm{d}x$$

$$= \left[\frac{x^3}{3} - x^2 - 3x\right]_{-3}^{-1}$$

$$= \left(\frac{-1}{3} - 1 + 3\right) - (-9 - 9 + 9)$$

$$= \frac{32}{3} \text{ sq. units}$$

3 $\displaystyle\int_{-1}^{1}(x^2 - 2x - 3)\,\mathrm{d}x = \left[\frac{x^3}{3} - x^2 - 3x\right]_{-1}^{1}$

$$= \left(\frac{1}{3} - 1 - 3\right) - \left(\frac{-1}{3} - 1 + 3\right) = \frac{-16}{3}$$

$\therefore$ required area $= \frac{32}{3} + \frac{16}{3} = 16$ sq. units

4 $\displaystyle\int_{\ln 3}^{3}(\mathrm{e}^x - 3)\,\mathrm{d}x = \left[\mathrm{e}^x - 3x\right]_{\ln 3}^{3}$

$$= \mathrm{e}^3 - 9 - 3 + 3\ln 3$$

$$= \mathrm{e}^3 + 3\ln 3 - 12$$

$$\int_{0}^{\ln 3}(\mathrm{e}^x - 3)\,\mathrm{d}x = \left[\mathrm{e}^x - 3x\right]_{0}^{\ln 3}$$

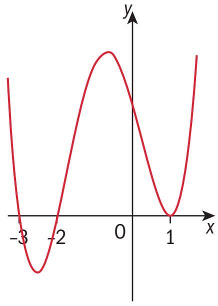
$$= 3 - 3\ln 3 - 1$$

$$= -(3\ln 3 - 2)$$

$\therefore$ area $= \mathrm{e}^3 + 3\ln 3 - 12 + 3\ln 3 - 2$

$= \mathrm{e}^3 + 6\ln 3 - 14$

5 $y = x^4 + 3x^3 - 3x^2 - 7x + 6$



$$\int_{-3}^{-2}(x^4 + 3x^3 - 3x^2 - 7x + 6)\mathrm{d}x$$

$$= \left[\frac{x^5}{5} + \frac{3x^4}{4} - x^3 - \frac{7x^2}{2} + 6x\right]_{-3}^{-2}$$

$$= \left(-\frac{32}{5} + 12 + 8 - 14 - 12\right)$$

$$-\left(-\frac{243}{5} + \frac{243}{4} + 27 - \frac{63}{2} - 18\right)$$

$$= -12.4 - (-10.35) = -2.05$$

$$\int_{-2}^{1}(x^4 + 3x^3 - 3x^2 - 7x + 6)\mathrm{d}x$$

$$= \left[\frac{x^5}{5} + \frac{3x^4}{4} - x^3 - \frac{7x^2}{2} + 6x\right]_{-2}^{1}$$

$$= \left(\frac{1}{5} + \frac{3}{4} - 1 - \frac{7}{2} + 6\right) - (-12.4) = 14.85$$

area $= 2.05 + 14.85 = 16.9$ sq. units

6 $y = \sqrt{4 - x} \quad x = 0, \quad x = 4$

$$A = \int_{0}^{4}(4 - x)^{\frac{1}{2}}\,\mathrm{d}x = \left[-\frac{2}{3}(4 - x)^{\frac{3}{2}}\right]_{0}^{4}$$

$$= -\frac{2}{3}\left(0 - 4^{\frac{3}{2}}\right) = \frac{16}{3} \text{ sq. units}$$

7 $y = \frac{1}{x^2} + 1 \quad x = \frac{1}{2}, x = 5$

$$A = \int_{\frac{1}{2}}^{5}(x^{-2} + 1)\,\mathrm{d}x = \left[\frac{-1}{x} + x\right]_{\frac{1}{2}}^{5}$$

$$= \left(\frac{-1}{5} + 5\right) - \left(-2 + \frac{1}{2}\right)$$

$= 6.3$ sq. units

8 $y = 2^x \quad x = 1, x = 2$

$$A = \int_{1}^{2} 2^x\,\mathrm{d}x = \frac{1}{\ln 2}\left[2^x\right]_1^2$$

$$= \frac{1}{\ln 2}(4 - 2) = \frac{2}{\ln 2} \text{ sq. units}$$

9 $y = 2\mathrm{e}^{-x+1} - 1 \quad x = 0, x = 3$

$$A = \int_{0}^{3}|2\mathrm{e}^{-x+1} - 1|\,\mathrm{d}x = 3.32 \text{ sq. units}$$

10 $y = \frac{1}{x + 2} \quad x = -1, x = 2$

$$A = \int_{-1}^{2}\frac{1}{x + 2}\,\mathrm{d}x = \left[\mathrm{In}\,|x + 2|\right]_{-1}^{2}$$

$= \ln 4 - \ln 1 = \ln 4 = 2\ln 2$ sq. units

11 $y = \frac{2}{3-4x}$ $x = 1, x = 3$

$\int_1^3 \frac{2}{3-4x}\,dx = \frac{2}{-4}\left[\text{In}\,|3-4x|\right]_1^3$

$= -\frac{1}{2}(\ln 9 - \ln 1) = -\frac{1}{2}\ln 9 = -\ln 3$

$\therefore$ area = ln3 sq. units

12 $y = -x^3 + 6x^2 + x - 30$ x-intercepts: $-2, 3, 5$

$\int_{-2}^{3} (-x^3 + 6x^2 + x - 30)dx$

$= \left[\frac{-x^4}{4} + 2x^3 + \frac{x^2}{2} - 30x\right]_{-2}^{3}$

$= \left(\frac{-81}{4} + 54 + \frac{9}{2} - 90\right) - (-4 - 16 + 2 + 60)$

$= -51.75 - 42$

$= -93.75$

$\int_3^5 (-x^3 + 6x^2 + x - 30)dx = \left[\frac{-x^4}{4} + 2x^3 + \frac{x^2}{2} - 30x\right]_3^5$

$= \left(\frac{-625}{4} + 250 + \frac{25}{2} - 150\right) - (-51.75)$

$= -43.75 + 51.75 = 8$

$\therefore$ area $= 93.75 + 8 = 101.75$ sq. units

13 $y = \begin{cases} x^2 & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$

Area $= \int_0^1 x^2 dx + \int_1^2 (2-x)dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$

$= \frac{1}{3} + (4-2) - \left(2 - \frac{1}{2}\right) = \frac{5}{6}$ sq. units

14 $y = \begin{cases} \sqrt{x} & 0 \le x \le 1 \\ x^2 & 1 \le x \le 2 \end{cases}$

Area $= \int_0^1 x^{\frac{1}{2}} dx + \int_1^2 x^2\, dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2$

$= \frac{2}{3} + \frac{8}{3} - \frac{1}{3}$

$= 3$ sq. units

Exercise 7I

1 $y = x^2 + 1$ $y = 1,$ $y = 10$

$x = \sqrt{y-1}$ $A = \int_1^{10} (y-1)^{\frac{1}{2}}\,dy = \left[\frac{2}{3}(y-1)^{\frac{3}{2}}\right]_1^{10} = \frac{2}{3}(9)^{\frac{3}{2}}$

$= 18$ sq. units

2 $y = \sqrt{x}$ $y = 0,$ $y = 4$

$x = y^2$

$A = \int_0^4 y^2\,dy = \left[\frac{y^3}{3}\right]_0^4 = \frac{64}{3}$ sq. units

3 $y = \sqrt{4-x}$ $y = 0,$ $y = 2$

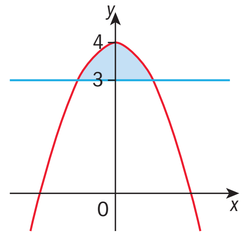
$y^2 = 4 - x$

$x = 4 - y^2$

$A = \int_0^2 (4 - y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_0^2$

$= 8 - \frac{8}{3} = \frac{16}{3}$ sq.units

4 $y = 4 - x^2$ $y = 3,$ $y = 4$



$x^2 = 4 - y$

$x = (4-y)^{\frac{1}{2}}$

$A = 2\int_3^4 (4-y)^{\frac{1}{2}}dy = 2\left[\frac{-2}{3}(4-y)^{\frac{3}{2}}\right]_3^4$

$= \frac{-4}{3}(0-1) = \frac{4}{3}$ sq.units

5 $y = \frac{1}{\sqrt{-x+4}}$ $y = \frac{1}{2}$ $y = 2$

$-x + 4 = \frac{1}{y^2}$ $x = 4 - \frac{1}{y^2}$

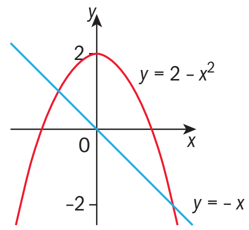
$A = \int_{\frac{1}{2}}^{2} \left(4 - \frac{1}{y^2}\right) dy = \left[4y + \frac{1}{y}\right]_{\frac{1}{2}}^{2}$

$= \left(8 + \frac{1}{2}\right) - (2+2)$

$= 4\frac{1}{2}$ sq. units

Exercise 7J

1



$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or -1

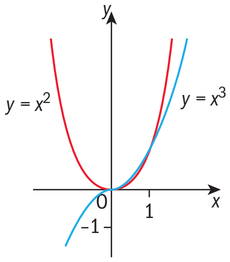
Area $= \int_{-1}^{2} (2 - x^2 + x)\,dx$

$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$

$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$

$= \frac{9}{2}$ sq. units

2



$x^3 = x^2$

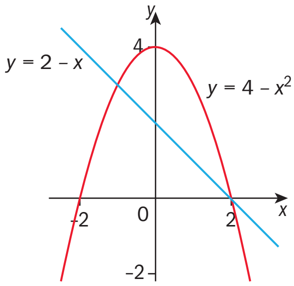
$x^2(x-1) = 0$

$x = 0$ or 1

$A = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ sq. units

3



$4 - x^2 = 2 - x$

$x^2 - x - 2 = 0$

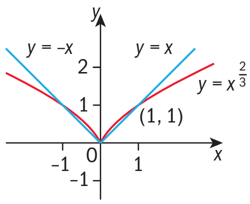
$(x-2)(x+1) = 0$

$x = -1$ or 2

Area $= \int_{-1}^{2} (4 - x^2 - (2 - x))\,dx = \int_{-1}^{2} (2 - x^2 + x)\,dx$

$= \frac{9}{2}$ sq. units (see qn. 1)

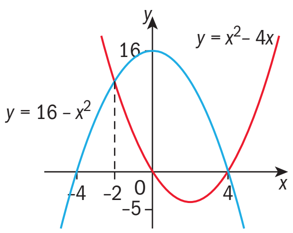
4



Area $= 2\int_0^1 \left(x^{\frac{2}{3}} - x\right)dx = 2\left[\frac{3}{5}x^{\frac{5}{3}} - \frac{x^2}{2}\right]_0^1$

$= 2\left(\frac{3}{5} - \frac{1}{2}\right) = \frac{1}{5}$

5



$16 - x^2 = x^2 - 4x$

$2x^2 - 4x - 16 = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4$ or -2

$A = \int_{-2}^{4} (16 - x^2 - (x^2 - 4x))\,dx$

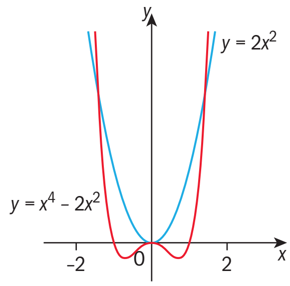
$= \int_{-2}^{4} (16 - 2x^2 + 4x)\,dx$

$= \left[16x - \frac{2}{3}x^3 + 2x^2\right]_{-2}^{4}$

$= \left(64 - \frac{128}{3} + 32\right) - \left(-32 + \frac{16}{3} + 8\right)$

$= 72$ sq. units

6



$x^4 - 2x^2 = 2x^2$

$x^4 - 4x^2 = 0$

$x^2(x^2 - 4) = 0$

$x^2(x-2)(x+2) = 0$

$x = 0, \pm 2$

$A = \int_{-2}^{2} (2x^2 - (x^4 - 2x^2))\,dx = \int_{-2}^{2} (4x^2 - x^4)\,dx$

$= \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left(\frac{-32}{3} + \frac{32}{5}\right)$

$= \frac{128}{15}$ sq. units

7 $2x^3 + 5x^2 + x - 2 = 8 - 4x - 20x^2 - 8x^3$

$10x^3 + 25x^2 + 5x - 10 = 0$

$2x^3 + 5x^2 + x - 2 = 0$

$x = -2, -1, \frac{1}{2}$

Area $= \int_{-2}^{-1} (10x^3 + 25x^2 + 5x - 10)\,dx$

$+ \int_{-1}^{\frac{1}{2}} (-10x^3 - 25x^2 - 5x + 10)\,dx$

$= \left[\frac{5x^4}{2} + \frac{25x^3}{3} + \frac{5x^2}{2} - 10x\right]_{-2}^{-1}$

$+ \left[\frac{-5x^4}{2} - \frac{25x^3}{3} - \frac{5x^2}{2} + 10x\right]_{-1}^{\frac{1}{2}}$

$= \left(\frac{5}{2} - \frac{25}{3} + \frac{5}{2} + 10\right) - \left(40 - \frac{200}{3} + 10 + 20\right)$

$+ \left(-\frac{5}{32} - \frac{25}{24} - \frac{5}{8} + 5\right) - \left(-\frac{5}{2} + \frac{25}{3} - \frac{5}{2} - 10\right)$

$= \frac{20}{3} - \frac{10}{3} + \frac{305}{96} + \frac{20}{3} = \frac{1265}{96}$

$= 13.2$ sq. units (3 sf)

8 $x^4 - 4 = \frac{1}{1+x}\ (x > 0)$

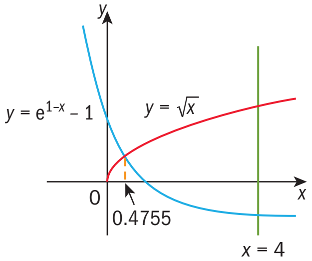
$(x^4 - 4)(1 + x) = 1$

$x^5 + x^4 - 4x - 5 = 0$

$x = 1.449$

$A = \int_0^{1.449} \left(\frac{1}{1+x} - x^4 + 4\right) dx = 5.41$ sq. units

9



$A = \int_{0.4755}^{4} (\sqrt{x} - e^{1-x} + 1)\, dx = 7.00$ sq. units

10 $y = x^2\ x = \sqrt{y}$

$\int_a^4 y^{\frac{1}{2}}\, dy = \int_0^a y^{\frac{1}{2}}\, dy$

$\left[\frac{2}{3} y^{\frac{3}{2}}\right]_a^4 = \left[\frac{2}{3} y^{\frac{3}{2}}\right]_0^a$

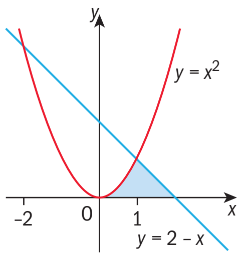
$4^{\frac{3}{2}} - a^{\frac{3}{2}} = a^{\frac{3}{2}}$

$2a^{\frac{3}{2}} = 8$

$a^{\frac{3}{2}} = 4$

$a = 4^{\frac{2}{3}}$

11



$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

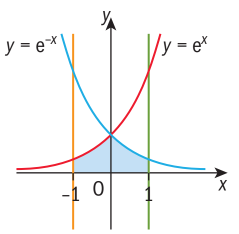
$x = -2$ or 1

$A = \int_0^1 x^2\, dx + \int_1^2 (2 - x)\, dx$

$A = \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{3} + (4 - 2) - \left(2 - \frac{1}{2}\right)$

$= \frac{5}{6}$ sq. units

12

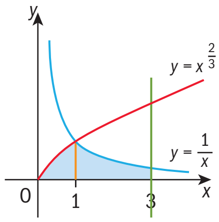


$\text{Area} = 2\int_0^1 e^{-x}\, dx$

$= 2\left[-e^{-x}\right]_0^1$

$= 2(-e^{-1} + 1) = 2\left(1 - \frac{1}{e}\right)$ or 1.26 sq. units

13



$A = \int_0^1 x^{\frac{2}{3}}\, dx + \int_1^3 \frac{1}{x}\, dx$

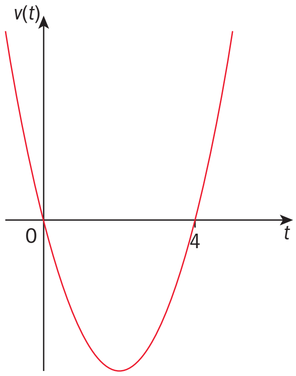
$= \left[\frac{3}{5} x^{\frac{5}{3}}\right]_0^1 + \left[\ln|x|\right]_1^3$

$= \frac{3}{5}(1 - 0) + \ln 3 - \ln 1$

$= \frac{3}{5} + \ln 3$ sq. units

Exercise 7K

1 $v(t) = t(t - 4)$

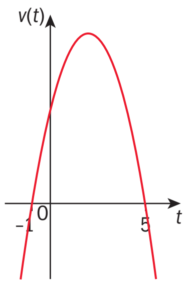


$\text{distance} = \left|\int_0^4 (t^2 - 4t)\, dt\right|$

$= \left|\left[\frac{t^3}{3} - 2t^2\right]_0^4\right|$

$= \left|\frac{64}{3} - 32\right| = \left|\frac{-32}{3}\right| = \frac{32}{3}$ m

2 $v(t) = 5 + 4t - t^2 = (1 + t)(5 - t)$



a $\text{distance} = \int_0^1 (5 + 4t - t^2) dt$

$= \left[5t + 2t^2 - \frac{t^3}{3}\right]_0^1$

$= 5 + 2 - \frac{1}{3} = \frac{20}{3}$ m

b distance $= \int_1^5 (5 + 4t - t^2)\,dt$

$+ \left| \int_5^6 (5 + 4t - t^2)\,dt \right|$

$= \left[5t + 2t^2 - \frac{t^3}{3} \right]_1^5 + \left| \left[5t + 2t^2 - \frac{t^3}{3} \right]_5^6 \right|$

$= \left(25 + 50 - \frac{125}{3} \right) - \frac{20}{3} + \left| (30 + 72 - 72) \right.$

$\left. - \left(25 + 50 - \frac{125}{3} \right) \right|$

$= \frac{80}{3} + \left| \frac{-10}{3} \right| = 30\,\text{m}$

3 $a(t) = 1 - e^{-2t} \quad 0 \le t \le 3$

$v(t) = t + \frac{1}{2}e^{-2t} + c$

$v(0) = 0 \quad \therefore 0 = \frac{1}{2} + c \quad \therefore c = -\frac{1}{2}$

$v(t) = t + \frac{1}{2}e^{-2t} - \frac{1}{2}$

distance $= \int_0^3 \left(t + \frac{1}{2}e^{-2t} - \frac{1}{2} \right) dt$

$= \left[\frac{t^2}{2} - \frac{1}{4}e^{-2t} - \frac{1}{2}t \right]_0^3 = 3.25\,\text{m}$

4 $v(t) = 10 + 5e^{-0.5t}$

a $a(t) = -2.5e^{-0.5t} < 0 \;\therefore$ always negative

b distance $= \int_0^2 (10 + 5e^{-0.5t})\,dt$

$= 26.3\,\text{m}$

Exercise 7L

1 $y = (x - 1)^2 - 1 = x^2 - 2x$

$v = \pi \int_0^1 (x^2 - 2x)^2\,dx = \pi \int_0^1 (x^4 - 4x^3 + 4x^2)\,dx$

$= \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_0^1$

$= \frac{8}{15}\pi$ cu. units or 1.68 cu. units

2 $y = 1 + \sqrt{x}$

$v = \pi \int_0^2 (1 + \sqrt{x})^2\,dx = \pi^2 (1 + 2\sqrt{x} + x)\,dx$

$= \pi \left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2} \right]_0^2$

$= \pi \left(2 + \frac{4}{3}(2\sqrt{2}) + 2 \right) = \pi \left(4 + \frac{4}{3}(2\sqrt{2}) \right)$

$= \frac{4\pi}{3}(3 + 2\sqrt{2})$ cu. units or 24.4 cu. units

3 $y = \frac{x^2}{2} \quad x^2 = 2y$

$v = \pi \int_0^2 2y\,dy = \pi \left[y^2 \right]_0^2 = 4\pi$ cu. units

4 $y = \sqrt{2x - x^2} \quad y^2 = 2x - x^2$

$v = \pi \int_1^2 (2x - x^2)\,dx = \pi \left[x^2 - \frac{x^3}{3} \right]_1^2$

$= \pi \left[\left(4 - \frac{8}{3} \right) - \left(1 - \frac{1}{3} \right) \right] = \frac{2\pi}{3}$ cu. units

5 $y = x^{\frac{3}{2}} \quad x = y^{\frac{2}{3}} \quad x^2 = y^{\frac{4}{3}}$

$v = \pi \int_1^3 y^{\frac{4}{3}}\,dy = \pi \left[\frac{3}{7} y^{\frac{7}{3}} \right]_1^3$

$= \frac{3\pi}{7}(3^{\frac{7}{3}} - 1) = 16.1$ cu. units

6 $y = \frac{x}{12}\sqrt{36 - x^2} \quad y^2 = \frac{x^2}{144}(36 - x^2) = \frac{x^2}{4} - \frac{x^4}{144}$

$v = \pi \int_0^6 \left(\frac{x^2}{4} - \frac{x^4}{144} \right) dx = \pi \left[\frac{x^3}{12} - \frac{x^5}{720} \right]_0^6$

$= \pi (18 - 10.8)$

$= 7.2\pi$ cu. units

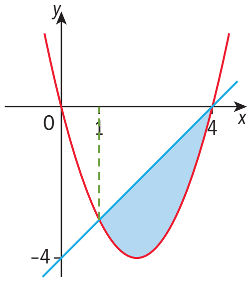
Exercise 7M

1 $y = x \quad y = \frac{x}{2}$

$v = \pi \int_2^5 \left(x^2 - \frac{x^2}{4} \right) dx = \pi \int_2^5 \frac{3x^2}{4}\,dx$

$= \pi \left[\frac{x^3}{4} \right]_2^5 = \pi \left(\frac{125}{4} - 2 \right) = \frac{117\pi}{4}$ cu. units

2



$x - 4 = x^2 - 4x$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ or 4

$v = \pi \int_1^4 ((x^2 - 4x)^2 - (x - 4)^2)\,dx$

$= \pi \int_1^4 (x^4 - 8x^3 + 16x^2 - x^2 + 8x - 16)\,dx$

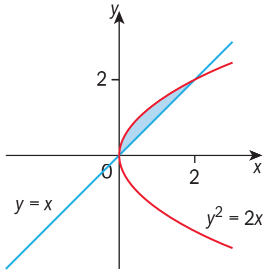
$= \pi \int_1^4 (x^4 - 8x^3 + 15x^2 + 8x - 16)\,dx$

$= \pi \left[\frac{x^5}{5} - 2x^4 + 5x^3 + 4x^2 - 16x \right]_1^4$

$= \pi \left[\left(\frac{1024}{5} - 512 + 320 + 64 - 64 \right) - \left(\frac{1}{5} - 2 + 5 + 4 - 16 \right) \right]$

$= \frac{108\pi}{5}$ cu. units

3



$x^2 = 2x \quad x(x-2) = 0$

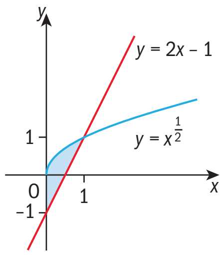
$x = 0$ or $2 \quad y = 0$ or 2

$$v = \pi\int_0^2\left(y^2 - \left(\frac{y^2}{2}\right)^2\right)dy$$

$$v = \pi\int_0^2\left(y^2 - \frac{y^4}{16}\right)dy$$

$$v = \pi\left[\frac{y^3}{3} - \frac{y^5}{20}\right]_0^2 = \pi\left(\frac{8}{3} - \frac{32}{20}\right) = \frac{16}{15}\pi \text{ cu. units}$$

4



$y = 2x - 1 \quad y = x^{\frac{1}{2}}$

$x = \frac{y+1}{2} \quad x = y^2$

$x^2 = \frac{(y+1)^2}{4} \quad x^2 = y^4$

$$v = \pi\int_{-1}^{1}\frac{(y+1)^2}{4}\,dy - \pi\int_0^1 y^4\,dy$$

$$= \pi\left[\frac{(y+1)^3}{12}\right]_{-1}^{1} - \pi\left[\frac{y^5}{5}\right]_0^1$$

$$= \pi\left(\frac{8}{12}\right) - \pi\left(\frac{1}{5}\right) = \frac{7\pi}{15} \text{ cu. units}$$

Review exercise

1 $\frac{dy}{dx} = ax + \frac{b}{x^2}$ $(-1, 2)$ $(-2, 0)$ = stationary point

when $x = -2$, $\frac{dy}{dx} = 0$ $\therefore -2a + \frac{b}{4} = 0$

$\therefore b = 8a$ (1)

$y = \frac{ax^2}{2} - \frac{b}{x} + c$

$(-2, 0)$ $0 = 2a + \frac{b}{2} + c$ (2)

$(-1, 2)$ $2 = \frac{a}{2} + b + c$ (3)

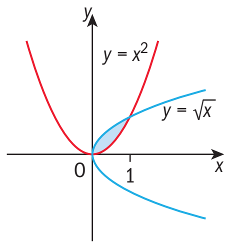
(2) − (3) $-2 = \frac{3a}{2} - \frac{b}{2}$ $b = 8a$

$\therefore -2 = \frac{3a}{2} - 4a \Rightarrow -4 = 3a - 8a \Rightarrow -4 = -5a$

$a = \frac{4}{5} \quad b = \frac{32}{5} \quad c = -2a - \frac{b}{2} = -\frac{8}{5} - \frac{16}{5} = \frac{-24}{5}$

$y = \frac{2}{5}x^2 - \frac{32}{5x} - \frac{24}{5}$

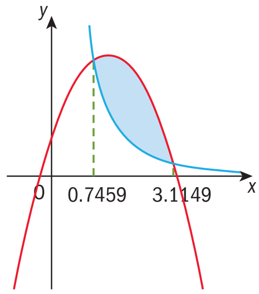
2



$$\text{Area} = \int_0^1\left(x^{\frac{1}{2}} - x^2\right)dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

3



$$v = \pi\int_{0.7459}^{3.1149}\left((1 + 3x - x^2)^2 - \left(\frac{2}{x}\right)^2\right)dx$$

$= 41.3$ cu. units

4 **a** $\int_1^2\left(x + \frac{1}{x^2} - \frac{1}{x^4}\right)dx = \int_1^2 (x + x^{-2} - x^{-4})\,dx$

$$= \left[\frac{x^2}{2} - x^{-1} + \frac{x^{-3}}{3}\right]_1^2 = \left[\frac{x^2}{2} - \frac{1}{x} + \frac{1}{3x^3}\right]_1^2$$

$$= \left(2 - \frac{1}{2} + \frac{1}{24}\right) - \left(\frac{1}{2} - 1 + \frac{1}{3}\right) = \frac{41}{24}$$

b $\int_1^4 \frac{5x^4 - 4}{\sqrt{x}}\,dx = \int_0^4\left(5x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}\right)dx$

$$= \left[2x^{\frac{5}{2}} - 8x^{\frac{1}{2}}\right]_1^4 = \left(2(4)^{\frac{5}{2}} - 8(4)^{\frac{1}{2}}\right) - (2 - 8)$$

$= (64 - 16) - (-6) = 54$

c $\int_1^2 \frac{1}{x-3}\,dx = [\ln|x-3|]_1^2 = \ln 1 - \ln 2 = -\ln 2$

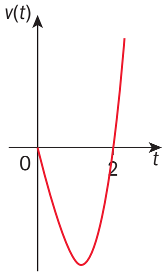
d $\int_1^e \frac{1}{1-4x}\,dx = -\frac{1}{4}[\ln|1-4x|]_1^e$

$= -\frac{1}{4}(\ln|1 - 4e| - \ln 3)$

$= -\frac{1}{4}(\ln(4e - 1) - \ln 3) = -\frac{1}{4}\ln\left(\frac{4e-1}{3}\right)$

Review exercise

1 $v(t) = t^3 - 4t = t(t^2 - 4) = t(t-2)(t+2)$



$$\int_0^2 (t^3 - 4t)\,dt = \left[\frac{t^4}{4} + 2t^2\right]_0^2$$

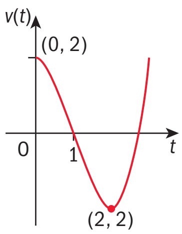
$$= 4 - 8 = -4$$

$$\int_2^3 (t^3 - 4t)\,dt = \left[\frac{t^4}{4} - 2t^2\right]_2^3$$

$$= \left(\frac{81}{4} - 18\right) - (-4) = \frac{25}{4}$$

$\therefore$ total distance $= 4 + \frac{25}{4} = \frac{41}{4}$ m

2 $v(t) = t^3 - 3t^2 + 2$



$$\int_0^1 (t^3 - 3t^2 + 2)\,dt = \left[\frac{t^4}{4} - t^3 + 2t\right]_0^1$$

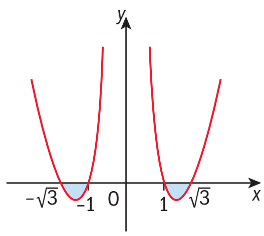
$$= \frac{1}{4} - 1 + 2 = \frac{5}{4}$$

$$\int_1^2 (t^3 - 3t^2 + 2)\,dt = \left[\frac{t^4}{4} - t^3 + 2t\right]_1^2$$

$$= (4 - 8 + 4) - \left(\frac{5}{4}\right) = \frac{-5}{4}$$

$\therefore$ total distance $= \frac{5}{4} + \frac{5}{4} = \frac{5}{2}$ m

3



$y = x^2 - 4 + \frac{3}{x^2}$

$x^2 - 4 + \frac{3}{x^2} = 0$

$x^4 - 4x^2 + 3 = 0$

$(x^2 - 1)(x^2 - 3) = 0 \quad x = \pm 1, \pm\sqrt{3}$

$$\int_1^{\sqrt{3}} \left(x^2 - 4 + \frac{3}{x^2}\right) dx = \left[\frac{x^3}{3} - 4x - \frac{3}{x}\right]_1^{\sqrt{3}}$$

$$= (\sqrt{3} - 4\sqrt{3} - \sqrt{3}) - \left(\frac{1}{3} - 4 - 3\right) = -4\sqrt{3} + \frac{20}{3}$$

$\therefore$ total area $= 2\left(4\sqrt{3} - \frac{20}{3}\right) = 8\sqrt{3} - \frac{40}{3}$ sq. units

4 **a** $\displaystyle\int \frac{3x^4 + 6}{x^2}\,dx = \int 3x^2 + \frac{6}{x^2}\,dx = x^3 - \frac{6}{x} + c$

b $\displaystyle\int \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) dx = \int \left(x^2 - \frac{1}{x^2}\right) dx = \frac{x^3}{3} + \frac{1}{x} + c$

c $\displaystyle\int \frac{1}{2 - 3x}\,dx = \frac{-1}{3} \ln|2 - 3x| + c$

d $\displaystyle\int \frac{2}{\sqrt{1-4x}}\,dx = \int 2(1-4x)^{-\frac{1}{2}}\,dx = \frac{2}{-4}\,\frac{(1-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -\sqrt{1-4x} + c$

e $\displaystyle\int \left(2e^{-3x} + \sqrt[3]{e^x}\right) dx = \int \left(2e^{-3x} + e^{\frac{x}{3}}\right) dx$

$= \frac{-2}{3} e^{-3x} + 3e^{\frac{x}{3}} + c = \frac{-2}{3} e^{-3x} + 3\sqrt[3]{e^x} + c$

5

```
        x + 2
2x − 1 ) 2x² + 3x
         2x² − x
              4x
              4x − 2
                   2
```

$\therefore \frac{2x^2 + 3x}{2x - 1} = x + 2 + \frac{2}{2x - 1}$

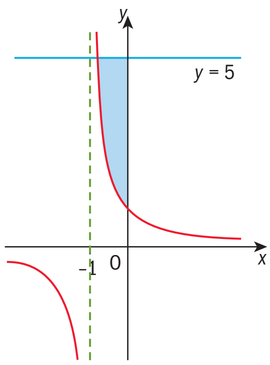
$$\int_1^2 \frac{2x^2 + 3x}{2x - 1}\,dx = \int_1^2 \left(x + 2 + \frac{2}{2x - 1}\right) dx$$

$$= \left[\frac{x^2}{2} + 2x + \ln|2x - 1|\right]_1^2$$

$= (2 + 4 + \ln 3) - \left(\frac{1}{2} + 2 + \ln 1\right)$

$= \frac{7}{2} + \ln 3$

6



$y = \frac{1}{x+1} \quad \therefore x + 1 = \frac{1}{y}$

$x = \frac{1}{y} - 1$

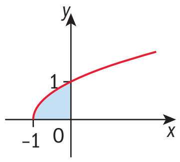
$$\int_1^5 \left(\frac{1}{y} - 1\right) dy = [\ln y - y]_1^5$$

$= (\ln 5 - 5) - (\ln 1 - 1)$

$= \ln 5 - 4$

$\therefore$ area $= 4 - \ln 5$ sq. units

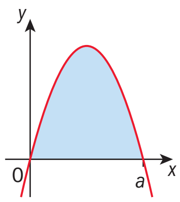
7



$$A = \int_{-1}^{0} (x+1)^{\frac{1}{2}}\,dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^{0}$$

$= \frac{2}{3}$ sq. units

8



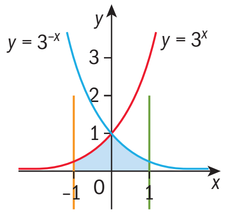
$$\int_0^a (3ax - 3x^2)\,dx = 4$$

$$\left[\frac{3ax^2}{2} - x^3\right]_0^a = 4$$

$$\frac{3a^3}{2} - a^3 = 4$$

$\frac{a^3}{2} = 4 \quad a^3 = 8 \quad \therefore a = 2$

9



$$v = 2\pi\int_0^1 (3^{-x})^2\,dx = 2\pi\int_0^1 3^{-2x}\,dx$$

$$= \frac{2\pi}{-2\ln 3}\left[3^{-2x}\right]_0^1$$

$$= -\frac{\pi}{\ln 3}(3^{-2} - 1)$$

$$= \frac{\pi}{\ln 3}\left(\frac{8}{9}\right)$$

$= \frac{8\pi}{9\ln 3}$ cu. units

9 The power of calculus

Answers

Skills check

1 a $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}} = \dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} = \cos^2\theta - \sin^2\theta = \cos^2\theta$

$\therefore \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

b $\tan 2\theta = \tan(\theta+\theta) = \dfrac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\therefore \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

2 a $f(x) = 3e^{2x} - 2x^2$, $f'(x) = 6e^{2x} - 4x$

b $g(x) = (x+1)\ln(x^2+2x+1)$

$= (x+1)\ln(x+1)^2 = 2(x+1)\ln(x+1)$

$g'(x) = \dfrac{2(x+1)}{(x+1)} + 2\ln(x+1) = 2 + 2\ln(x+1)$

c $h(x) = \dfrac{e^{x^2}}{x+1}$ $h'(x) = \dfrac{(x+1)2xe^{x^2} - e^{x^2}}{(x+1)^2}$

$= \dfrac{e^{x^2}(2x^2+2x-1)}{(x+1)^2}$

Exercise 9A

Proofs using differentiation from first principles

Exercise 9B

1 a $y = \cot x$ $\dfrac{dy}{dx} = -\csc^2 x$

b $y = \csc x$ $\dfrac{dy}{dx} = -\csc x\cot x$

c $y = \sin 3x$ $\dfrac{dy}{dx} = 3\cos 3x$

d $y = \tan(5x-3)$ $\dfrac{dy}{dx} = 5\sec^2(5x-3)$

e $y = \cos(8-3x)$ $\dfrac{dy}{dx} = 3\sin(8-3x)$

f $y = \csc\left(\dfrac{x-3}{4}\right)$ $\dfrac{dy}{dx} = -\dfrac{1}{4}\csc\left(\dfrac{x-3}{4}\right)\cot\left(\dfrac{x-3}{4}\right)$

g $y = \cot\left(\dfrac{7-2x}{13}\right)$ $\dfrac{dy}{dx} = \dfrac{2}{13}\csc^2\left(\dfrac{7-2x}{13}\right)$

2 a $y = \sin(x^5-3)$ $\dfrac{dy}{dx} = 5x^4\cos(x^5-3)$

b $y = \cos(e^x)$ $\dfrac{dy}{dx} = -e^x\sin(e^x)$

c $y = \csc(x^2+11)$ $\dfrac{dy}{dx} = -2x\csc(x^2+11)\cot(x^2+11)$

d $y = \cot(4x^3 - 2x^2 + 7x + 17)$

$\dfrac{dy}{dx} = -(12x^2 - 4x + 7)\csc^2(4x^3 - 2x^2 + 7x + 17)$

e $y = \tan(\ln(2x+1))$, $\dfrac{dy}{dx} = \dfrac{2}{2x+1}\sec^2(\ln(2x+1))$

f $y = \sec\left(\sqrt{e^x+1}\right)$

$\dfrac{dy}{dx} = \dfrac{1}{2}e^x(e^x+1)^{-\frac{1}{2}}\sec\left(\sqrt{e^x+1}\right)\tan\left(\sqrt{e^x+1}\right)$

$= \dfrac{e^x\sec\left(\sqrt{e^x+1}\right)\tan\left(\sqrt{e^x+1}\right)}{2\left(\sqrt{e^x+1}\right)}$

$= \dfrac{e^x\sin\left(\sqrt{e^x+1}\right)\sec^2\left(\sqrt{e^x+1}\right)}{2\left(\sqrt{e^x+1}\right)}$

g $y = \sin(\cos(\tan x))$

$\dfrac{dy}{dx} = -\sec^2 x\sin(\tan x)\cos(\cos(\tan x))$

Exercise 9C

1 a $y = (2x-1)\cos x$

$\dfrac{dy}{dx} = 2\cos x - (2x-1)\sin x$

b $y = (3x - x^2)\sin 2x$

$\dfrac{dy}{dx} = (3-2x)\sin 2x + 2(3x - x^2)\cos 2x$

c $y = e^{1-x}\tan x$

$\dfrac{dy}{dx} = e^{1-x}\sec^2 x - e^{1-x}\tan x = e^{1-x}(\sec^2 x - \tan x)$

d $y = \dfrac{\sin x}{x}$, $\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x}{x^2}$

e $y = \dfrac{2x+3}{\sin 2x}$, $\dfrac{dy}{dx} = \dfrac{2\sin 2x - 2(2x+3)\cos 2x}{\sin^2 2x}$

f $y = \dfrac{\tan x}{\sqrt{2-x}}$

$\dfrac{dy}{dx} = \left(\sqrt{2-x}\sec^2 x + \dfrac{1}{2}(2-x)^{-\frac{1}{2}}\tan x \times \dfrac{1}{(2-x)}\right)$

$\dfrac{dy}{dx} = \dfrac{2(2-x)\sec^2 x + \tan x}{2(2-x)^{\frac{3}{2}}}$

2 a $y = \sin 2x \qquad x = \frac{\pi}{6}$

$\frac{dy}{dx} = 2\cos 2x = 2\cos\frac{\pi}{3} = 1$

b $y = \cos 3x \qquad x = \frac{7\pi}{12}$

$\frac{dy}{dx} = -3\sin 3x = -3\sin\frac{7\pi}{4} = \frac{3}{\sqrt{2}}$

c $y = \tan(-x) = -\tan x \qquad x = \frac{5\pi}{4}$

$\frac{dy}{dx} = -\sec^2 x = -\sec^2\frac{5\pi}{4} = -2$

d $y = (x-2)\sin x \qquad x = 0$

$\frac{dy}{dx} = \sin x + (x-2)\cos x$

$= \sin 0 + (-2)\cos 0 = -2$

e $y = -3x\cos x \qquad x = \frac{\pi}{2}$

$\frac{dy}{dx} = -3\cos x + 3x\sin x$

$= -3\cos\frac{\pi}{2} + 3\frac{\pi}{2}\sin\frac{\pi}{2} = \frac{3\pi}{2}$

f $y = x^2\tan x \quad x = \frac{3x}{4}$

$\frac{dy}{dx} = 2x\tan x + x^2\sec^2 x$

$= \frac{3\pi}{2}\tan\left(\frac{3\pi}{4}\right) + \frac{9\pi^2}{16}\sec^2\frac{3\pi}{4}$

$= -\frac{3\pi}{2} + \frac{9\pi^2}{8}$

g $y = e^x\sec x \quad x = 0 \quad \frac{dy}{dx} = e^x\sec x\tan x + e^x\sec x$

$= \sec 0\tan 0 + \sec 0 = 1$

3 a $y = \sin^2\alpha + \cos^2\alpha = 1 \qquad \frac{dy}{d\alpha} = 0$

b $y = \frac{\tan\beta}{\sin\beta} = \sec\beta \qquad \frac{dy}{d\beta} = \sec\beta\tan\beta$

c $y = \frac{2\tan 2\theta}{1-\tan^2\theta} = \tan 4\theta \qquad \frac{dy}{d\theta} = 4\sec^2 4\theta$

d $y = \frac{\sin\rho + \sin 2\rho}{\cos\rho + \cos 2\rho} = \frac{2\sin\frac{3\rho}{2}\cos\frac{\rho}{2}}{2\cos\frac{3\rho}{2}\cos\frac{\rho}{2}}$

$y = \tan\frac{3\rho}{2} \qquad \frac{dy}{dx} = \frac{3}{2}\sec^2\frac{3\rho}{2}$

e $y = \frac{(\sin\varphi\sin 2\varphi - \cos\varphi)\sec\varphi}{\sin\varphi - \cos\varphi}$

$= \frac{2\sin^2\varphi - 1}{\sin\varphi - \cos\varphi} = \frac{\sin^2\varphi - \cos^2\varphi}{\sin\varphi - \cos\varphi}$

$= \frac{(\sin\varphi - \cos\varphi)(\sin\varphi + \cos\varphi)}{\sin\varphi - \cos\varphi}$

$y = \sin\varphi + \cos\varphi \quad \frac{dy}{d\varphi} = \cos\varphi - \sin\varphi$

Exercise 9D

1 a $y = \arccos x \;\therefore\; x = \cos y$

$\frac{dx}{dy} = -\sin y \quad \therefore \quad \frac{dx}{dy} = -\frac{1}{\sin y}$

$= -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \quad f'(x) = -\frac{1}{\sqrt{1-x^2}}$

b $f(x) = \arcsin 3x, \; f'(x) = \frac{3}{\sqrt{1-9x^2}}$

c $f(x) = \arctan(2x+1), \; f'(x) = \frac{2}{1+(2x+1)^2}$

$= \frac{2}{1+4x^2+4x+1}$

$f'(x) = \frac{1}{2x^2+2x+1}$

2 a $y = 2x\arcsin x$

$\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^2}} + 2\arcsin x$

b $y = \frac{\arccos x}{x}$

$\frac{dy}{dx} = \frac{\frac{-x}{\sqrt{1-x^2}} - \arccos x}{x^2}$

$\frac{dy}{dx} = \frac{-x - \sqrt{1-x^2}\arccos x}{x^2\sqrt{1-x^2}}$

c $y = (2x+1)\arctan x$

$\frac{dy}{dx} = 2\arctan x + \frac{2x+1}{1+x^2}$

d $y = \sqrt{1-x^2}\arcsin x$

$\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} - 2x\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\arcsin x$

$\frac{dy}{dx} = 1 - \frac{x\arcsin x}{\sqrt{1-x^2}}$

e $y = (4x^2+1)\arctan 2x$

$\frac{dy}{dx} = 8x\arctan 2x + \frac{2(4x^2+1)}{1+4x^2}$

$\frac{dy}{dx} = 8x\arctan 2x + 2$

3 a $\frac{d}{dx}(\arcsin x + \arccos x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

b $\frac{d}{dx}(\arctan x + \arctan(-x)) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

c $\frac{d}{dx}\left(2\arctan x - \arcsin\frac{2x}{x^2+1}\right)$

$= \frac{2}{1+x^2} - \left(\frac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2}\right)\frac{1}{\sqrt{1-\frac{4x^2}{(x^2+1)^2}}}$

$= \frac{2}{1+x^2} - \frac{(2-2x^2)}{(x^2+1)^2}\frac{(x^2+1)}{\sqrt{(x^2+1)^2-4x^2}}$

$= \frac{2}{1+x^2} - \frac{2(1-x^2)}{(x^2+1)\sqrt{(x^2-1)^2}} = 0$

4 **a** $x = \sin y$

$1 = \cos y\frac{dy}{dx} \quad \therefore \quad \frac{dy}{dx} = \frac{1}{\cos y}$

b $x + y = \tan y$

$1 + \frac{dy}{dx} = \sec^2 y\frac{dy}{dx}$

$\frac{dy}{dx}(\sec^2 y - 1) = 1 \Rightarrow \frac{dy}{dx}\tan^2 y = 1 \Rightarrow \frac{dy}{dx} = \cot^2 y$

c $x + \sin x = y + \cos y$

$1 + \cos x = \frac{dy}{dx}(1 - \sin y) \quad \therefore \quad \frac{dy}{dx} = \frac{1+\cos x}{1-\sin y}$

d $e^{\sin y} = x^2$

$e^{\sin y}\cos y\frac{dy}{dx} = 2x \quad \therefore \quad \frac{dy}{dx} = \frac{2x}{e^{\sin y}\cos y}$

e $\cos y = \frac{x}{y} \Rightarrow y\cos y = x$

$\frac{dy}{dx}(\cos y - y\sin y) = 1 \quad \therefore \quad \frac{dy}{dx} = \frac{1}{\cos y - y\sin y}$

f $\ln(xy) = \tan 2y \Rightarrow \ln x + \ln y = \tan 2y$

$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 2\sec^2 2y\frac{dy}{dx}$

$\frac{dy}{dx}\left(2\sec^2 2y - \frac{1}{y}\right) = \frac{1}{x}$

$\frac{dy}{dx} = \frac{\frac{1}{x}}{2\sec^2 2y - \frac{1}{y}} = \frac{y}{2xy\sec^2 2y - x}$

Exercise 9E

1 **a** $f(x) = \tan 3x \quad P(0, 0)$

$f'(x) = 3\sec^2 3x \quad f'(0) = 3$

$y = 3x$

b $f(x) = \sin(2x) - 1 \qquad P\left(\frac{\pi}{3}, y\right)$

$y = \sin\left(\frac{2\pi}{3}\right) - 1 = \frac{\sqrt{3}}{2} - 1 \quad P\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2} - 1\right)$

$f'(x) = 2\cos(2x) \quad f'\left(\frac{\pi}{3}\right) = -1$

$y - \frac{\sqrt{3}}{2} + 1 = -1\left(x - \frac{\pi}{3}\right)$

$y = -x + \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1$

c $f(x) = 2\cos\left(\frac{x}{2}\right) - e^{2x} \qquad P(0, 1)$

$f'(x) = -\sin\left(\frac{x}{2}\right) - 2e^{2x} \qquad \therefore f'(0) = -2$

$y = -2x + 1$

d $f(x) = \ln\left(\tan\left(\frac{x}{3}\right)\right) + 2 \qquad P\left(\frac{3\pi}{4}, y\right)$

$y = \ln\left(\tan\frac{\pi}{4}\right) + 2 = 2 \qquad P\left(\frac{3\pi}{4}, 2\right)$

$f'(x) = \frac{\frac{1}{3}\sec^2\left(\frac{x}{3}\right)}{\tan\left(\frac{x}{3}\right)} \qquad f'\left(\frac{3\pi}{4}\right) = \frac{1}{3}\sec^2\left(\frac{\pi}{4}\right) = \frac{2}{3}$

$y - 2 = \frac{2}{3}\left(x - \frac{3\pi}{4}\right) \qquad y = \frac{2}{3}x - \frac{\pi}{2} + 2$

2 **a** $f(x) = \cos(2x) \qquad P(0, 1)$

$f'(x) = -2\sin(2x), \qquad f'(0) = 0$

equation of normal is $x = 0$

b $f(x) = \tan(4x) \qquad P\left(\frac{\pi}{16}, y\right)$

$y = \tan\frac{\pi}{4} = 1 \qquad P\left(\frac{\pi}{16}, 1\right)$

$f'(x) = 4\sec^2(4x), \quad f'\left(\frac{\pi}{16}\right) = 4\sec^2\left(\frac{\pi}{4}\right) = 8$

$y - 1 = \frac{-1}{8}\left(x - \frac{\pi}{16}\right)$

$y = \frac{-1}{8}x + \frac{\pi}{128} + 1$

c $f(x) = 2e^x\sin\left(\frac{x}{2}\right) \quad P(0, y) \quad y = 0 \quad P(0, 0)$

$f'(x) = e^x\cos\left(\frac{x}{2}\right) + 2e^x\sin\left(\frac{x}{2}\right)$

$f'(0) = 1 \qquad y = -x$

d $f(x) = x\cos(2x) - 3 \quad P\left(\frac{\pi}{2}, y\right)$

$y = \frac{\pi}{2}\cos\pi - 3 = -\frac{\pi}{2} - 3 \qquad P\left(\frac{\pi}{2}, -\frac{\pi}{2} - 3\right)$

$f'(x) = -2x\sin(2x) + \cos(2x)$

$f'\left(\frac{\pi}{2}\right) = -1 \qquad y + \frac{\pi}{2} + 3 = x - \frac{\pi}{2}$

$y = x - \pi - 3$

3 $\ln(x) = \tan y \quad P(1, 0)$

$\frac{1}{x} = \sec^2 y\frac{dy}{dx} \qquad \therefore \quad \frac{dy}{dx} = \frac{\cos^2 y}{x}$

At P, $\frac{dy}{dx} = 1 \quad y = x - 1$

4 $y + y^2 = \sin 2x \quad P(0, -1)$

$\frac{dy}{dx}(1 + 2y) = 2\cos 2x$

$\frac{dy}{dx} = \frac{2\cos 2x}{1+2y} = \frac{2}{-1} = -2$

$y + 1 = \frac{1}{2}x \qquad y = \frac{1}{2}x - 1$

5 $y = \cos(x^2)$

a $(0.974, 0.583)$

b $y = \cos(x^2)$

$\frac{dy}{dx} = -2x\sin(x^2)$

$x = 0.97407123, \ \frac{dy}{dx} = -1.5833\ldots$

$y - 0.58264678 = -1.5833(x - 0.97407123)$

$y = -1.58x + 2.12$

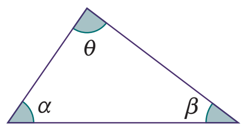
$y = e^{x^2} - 2$

$\frac{dy}{dx} = 2xe^{x^2} = 5.03136\ldots$

$y - 0.58264678 = 5.03136(x - 0.97407123)$

$y = 5.03x - 4.32$

c



$\tan\alpha = 5.03136\ldots$

$\alpha = 1.3746$

$\tan\beta = 1.5833\ldots$

$\beta = 1.00747$

$\theta = \pi - \alpha - \beta = 0.760$ rads

6 $e^y = \sin x + 1 \qquad P(-\pi, 0)$

$e^y\frac{dy}{dx} = -\cos x \quad \frac{dy}{dx} = \frac{-\cos x}{e^y}$

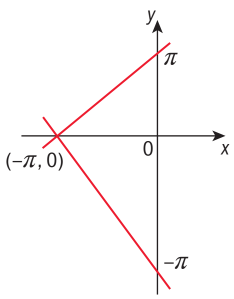
At P, $\frac{dy}{dx} = \frac{1}{e^0} = 1$

Tangent: $y = x + \pi$

Normal: $y = x - \pi$

Area $= \frac{1}{2} \times 2\pi \times \pi$

$= \pi^2$



Exercise 9F

1 a $f(x) = \tan x \quad f'(x) = \sec^2 x$

$f''(x) = 2\sec x \sec x \tan x = 2\sec^2 x \tan x$

$f''\left(\frac{\pi}{3}\right) = 2(2)^2\sqrt{3} = 8\sqrt{3}$

b $f(x) = x\sin x, \quad f'(x) = \sin x + x\cos x$

$f''(x) = \cos x + \cos x - x\sin x$

$= 2\cos x - x\sin x$

$f''(0) = 2$

c $f(x) = (x^2 + 1)\cos x$

$f'(x) = 2x\cos x - (x^2 + 1)\sin x$

$f''(x) = 2\cos - 2x\sin x - [+(x^2 + 1)\cos x + 2x\sin x]$

$= 2\cos x - 4x\sin x - (x^2 + 1)\cos x$

$f''(0) = 2 - 1 = 1$

d $f(x) = \sqrt{x}\cos\frac{x}{2}$

$f'(x) = -\frac{1}{2}\sqrt{x}\sin\frac{x}{2} + \frac{1}{2}x^{-\frac{1}{2}}\cos\frac{x}{2}$

$= \frac{-x\sin\frac{x}{2} + \cos\frac{x}{2}}{2\sqrt{x}}$

$$f''(x) = \frac{2\sqrt{x}\left(-\sin\frac{x}{2} - \frac{x}{2}\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right) - \left(-x\sin\frac{x}{2} + \cos\frac{x}{2}\right)x^{\frac{1}{2}}}{4x}$$

$$f''(1) = \frac{2\left(-\frac{3}{2}\sin\frac{1}{2} - \frac{-1}{2}\cos\frac{1}{2}\right) - \left(-\sin\frac{1}{2} + \cos\frac{1}{2}\right)}{4}$$

$$= \frac{-3\sin\frac{1}{2} - \cos\frac{1}{2} + \sin\frac{1}{2} - \cos\frac{1}{2}}{4}$$

$$= \frac{1}{4}\left(-2\sin\frac{1}{2} - 2\cos\frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(\sin\frac{1}{2} + \cos\frac{1}{2}\right)$$

e $f(x) = e^x\sin 2x \quad f'(x) = e^x(2\cos 2x + \sin 2x)$

$f''(x) = e^x(-4\sin 2x + 2\cos 2x + 2\cos 2x + \sin 2x)$

$= e^x(4\cos 2x - 3\sin 2x)$

$f''\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}(0 - 3) = -3e^{\frac{\pi}{4}}$

f $f(x) = 2x\sec x, \quad f'(x) = 2\sec x + 2x\sec x\tan x$

$= 2\sec x(1 + x\tan x)$

$f''(x) = 2\sec x(\tan x + x\sec^2 x)$

$+ 2\sec x\tan x(1 + x\tan x)$

$f''(\pi) = -2(0 + \pi) + 2(-1)(0) = -2\pi$

2 a $f(x) = \cos x \quad f'(x) = -\sin x$

$f''(x) = -\cos x \quad f^{(3)}(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 4k - 3 \\ -\cos x, & n = 4k - 2 \\ \sin x, & n = 4k - 1 \\ \cos x, & n = 4k \end{cases} \qquad k \in \mathbb{Z}^+$$

b $g(x) = \sin 3x \quad g'(x) = 3\cos 3x$
$g''(x) = -9\sin 3x \quad g^{(3)}(x) = -27\cos 3x$

$$g^{(n)}(x) = \begin{cases} 3^n \cos 3x & n = 4k-3 \\ -3^n \sin 3x & n = 4k-2 \\ -3^n \cos 3x & n = 4k-1 \\ 3^n \sin 3x & n = 4k \end{cases} \quad k \in \mathbb{Z}^+$$

c $h(x) = \cos(ax + b) \quad h'(x) = -a\sin(ax + b)$
$h''(x) = -a^2\cos(ax + b)$
$h^{(3)}(x) = a^3\sin(ax + b)$

$$h^{(n)}(x) = \begin{cases} -a^n \sin(ax+b) & n = 4k-3 \\ -a^n \cos(ax+b) & n = 4k-2 \\ a^n \sin(ax+b) & n = 4k-1 \\ a^n \cos(ax+b) & n = 4k \end{cases} \quad k \in \mathbb{Z}^+$$

3 $f(x) = \sin 2x \quad a_n = f^{(n-1)}\left(\frac{\pi}{8}\right) \quad n = 1, 2, 3, \ldots$

a $a_1 = f\left(\frac{\pi}{8}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = 2\cos 2x \quad a_2 = f'\left(\frac{\pi}{8}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f''(x) = -4\sin 2x \quad a_3 = f''\left(\frac{\pi}{8}\right) = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$

$f^{(3)}(x) = -8\cos 2x \quad a_4 = f^{(3)}\left(\frac{\pi}{8}\right) = -\frac{8}{\sqrt{2}} = -4\sqrt{2}$

$\frac{1}{\sqrt{2}}, \sqrt{2}, -2\sqrt{2}, -4\sqrt{2}$

b $\frac{1}{\sqrt{2}}(1 + 2 - 4 - 8 + 16 + 32 - 64$
$- 128 + 256 + 512)$
$= \frac{615}{\sqrt{2}}$ or $\frac{615\sqrt{2}}{\sqrt{2}}$

4 **a** P(n): $f(x) = \sin x \Rightarrow f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$,
$n = 0, 1, 2, \ldots$
P(0): $f(x) = \sin x$

Assume P(k): $f^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$

Prove P(k + 1) $f^{(k+1)}(x) = \cos\left(x + \frac{k\pi}{2}\right)$
$= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$
$= \sin\left(x + (k+1)\frac{\pi}{2}\right)$

$\therefore$ P(k) $\Rightarrow$ P(k + 1) and P(0) is true
$\therefore$ by induction,

$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), n = 0, 1, 2, \ldots.$

b P(n): $g(x) = \cos x \Rightarrow g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right)$,
$n = 0, 1, 2, \ldots$

P(0): $g(x) = \sin\left(x + \frac{\pi}{2}\right) = \cos x$

Assume P(k): $g^{(k)}(x) = \sin\left(x + \frac{(k+1)\pi}{2}\right)$

Prove P(k + 1) $g^{(k+1)}(x) = \cos\left(x + \frac{(k+1)\pi}{2}\right)$

$= \sin\left(x + \frac{(k+1)\pi}{2} + \frac{\pi}{2}\right)$

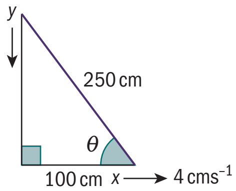
$= \sin\left(x + \frac{(k+2)\pi}{2}\right)$

$\therefore$ P(k) $\Rightarrow$ P(k + 1) and P(0) is true
$\therefore$ by induction,

$g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right), n = 0, 1, 2, \ldots.$

Exercise 9G

1



$\frac{dx}{dt} = 4 \quad \cos\theta = \frac{x}{250}$

$-\sin\theta\frac{d\theta}{dt} = \frac{1}{250}\frac{dx}{dt}$

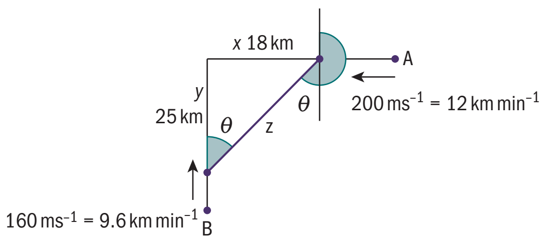
$x = 100 \Rightarrow \cos\theta = \frac{100}{250} = \frac{2}{5}$

$\sin\theta = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$

$-\frac{\sqrt{21}}{5}\frac{d\theta}{dt} = \frac{4}{250} \quad \therefore \quad \frac{d\theta}{dt} = -0.0175\,\text{cs}^{-1}$

the angle is decreasing at a rate of 0.0175 cs^{-1}

2



a $x = 18 - 12t \Rightarrow \frac{dx}{dt} = -12$

$y = 25 - 9.6t \Rightarrow \frac{dy}{dt} = -9.6$

$z^2 = x^2 + y^2 \quad \therefore 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$
$t = 0.5 \Rightarrow x = 12, y = 20.2,$
$z = \sqrt{552.04} = 23.4955\ldots$

$23.4955\frac{dz}{dt} = 12(-12) + 20.2(-9.6)$

$\therefore \frac{dz}{dt} = -14.4\text{ km min}^{-1} \approx -240\text{ ms}^{-1}$
Approaching each other at 240 ms^{-1}

b bearing $= \pi + \theta$

$\therefore$ rate of change of bearing $= \frac{d\theta}{dt}$

$\tan\theta = \frac{x}{y} \quad \Rightarrow \quad \sec^2\theta \frac{d\theta}{dt} = \left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)y^2$

$t = 1 \Rightarrow x = 6,\ y = 15.4,\ \tan\theta = \frac{6}{15.4}$

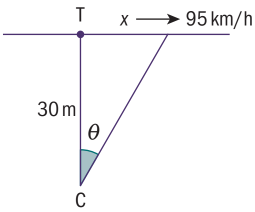
$\sec^2\theta = 1 + \left(\frac{6}{15.4}\right)^2 = \frac{6829}{5929}$

$\frac{6829}{5929}\frac{d\theta}{dt} = \frac{15.4(-12) - 6(-9.6)}{15.4^2}$

$\frac{d\theta}{dt} = -0.466\ \text{c min}^{-1}$

bearing is decreasing at a rate of 0.466 c min^{-1}

3 a



$\frac{dx}{dt} = 95000\ \text{mh}^{-1}$

$\tan\theta = \frac{x}{30}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$

$x = 0,\ \theta = 0 \Rightarrow \sec^2\theta = 1 \Rightarrow \frac{d\theta}{dt} = \frac{95000}{30}$

$= 31.667\ \text{ch}^{-1}$

$= 52.8\ \text{c min}^{-1}$ or $0.880\ \text{cs}^{-1}$

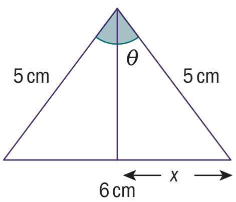
b After 1 sec, $x = \frac{95000}{3600} = 26.338 \ldots$ m

$\tan\theta = \frac{26.388\ldots}{30} = 0.8796\ldots \quad \sec^2\theta = 1.7737\ldots$

$1.7737\ldots\frac{d\theta}{dt} = \frac{95000}{30}$

$\therefore \frac{d\theta}{dt} = 1785.3$ c/h $= 29.8\ \text{c min}^{-1}$ or $0.496\ \text{cs}^{-1}$

4 a



$\frac{dx}{dt} = -0.05\ \text{cms}^{-1}$

$\frac{d}{dt}(2\theta) = 2\frac{d\theta}{dt}$

$\sin\theta = \frac{x}{5}$

$\cos\theta\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

At $t = 0$, $\cos\theta = \frac{4}{5}$

$\therefore \frac{d\theta}{dt} = \frac{1}{5}\times -0.05$

$\therefore \frac{d\theta}{dt} = -0.0125$

$\therefore$ angle is decreasing at a rate of $0.0125\ \text{cs}^{-1}$

b $\theta = 30°$ when equilateral

$\therefore \cos 30° \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt} = -0.01$

$\therefore \frac{d\theta}{dt} = -0.115$

$\therefore$ angle 2θ is decreasing at $0.0231\ \text{cs}^{-1}$

5 a $\frac{dv}{dt} = -2\ \text{cm}^3\ \text{min}^{-1}$

$v = \frac{4}{3}\pi r^3 \quad \frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$

$r = 12, \quad -2 = 4\pi(12)^2\frac{dr}{dt} \quad \frac{dr}{dt} = -\frac{1}{288\pi}\ \text{cm min}^{-1}$

$= -0.00111\ \text{cm min}^{-1}$

radius is decreasing at $0.00111\ \text{cm min}^{-1}$

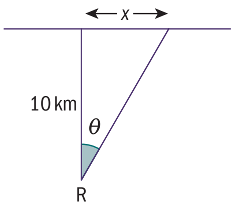
b $A = 4\pi r^2 \quad \frac{dA}{dt} = 8\pi r\frac{dr}{dt}$

$r = 4, \quad -2 = 4\pi(4)^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-1}{32\pi}\ \text{cm min}^{-1}$

$\frac{dA}{dt} = 8\pi(4)\left(\frac{-1}{32\pi}\right) = -1\ \text{cm}^2\ \text{min}^{-1}$,

decreasing at $1\ \text{cm}^2\ \text{min}^{-1}$,

6 a



$\frac{dx}{dt} = 1025\ \text{km h}^{-1}$

$\sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt}$

$x = 8,\ \tan\theta = \frac{4}{5}\ \sec^2\theta = 1 + \frac{16}{25} = \frac{41}{25}$

$\frac{41}{25}\frac{d\theta}{dt} = 102.5 \Rightarrow \frac{d\theta}{dt} = 62.5\ \text{ch}^{-1}$

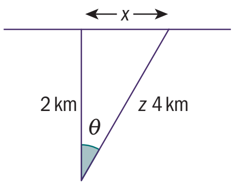
$\frac{d\theta}{dt} = 0.01761\ \text{cs}^{-1}$

$= 0.995\ \text{deg s}^{-1}$

b $x = 0,\ \theta = 0,\ \sec^2\theta = 1$

$\frac{d\theta}{dt} = 102.5\ \text{ch}^{-1} = 0.028472 \ldots\ \text{cs}^{-1} = 1.63\ \text{deg s}^{-1}$

7 a



$\frac{dx}{dt} = 75\ \text{km h}^{-1}$

$z^2 = 4 + x^2$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$

when $z = 4$, $16 = 4 + x^2 \quad \therefore \quad x = \sqrt{12}$

$4\frac{dz}{dt} = \sqrt{12}(75)$

$\frac{dz}{dt} = \frac{75\sqrt{3}}{2} = 65.0\ \text{km h}^{-1}$

b $\tan\theta = \frac{x}{2} \quad \sec^2\theta\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$

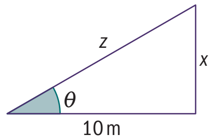
$x = \sqrt{12}, \ \tan\theta = \sqrt{3}, \quad \sec^2\theta = 1 + \left(\sqrt{3}\right)^2 = 4$

$4\frac{d\theta}{dt} = \frac{75}{2}$

$\frac{d\theta}{dt} = \frac{75}{8} = 9.375 \text{ c/h} = 0.00260 \text{ c sec}^{-1}$

$= 0.1 \text{ deg s}^{-1}$ (nearest tenth)

8 a



$\frac{dz}{dt} = 5 \text{ ms}^{-1}$

$\cos\theta = \frac{10}{z}$

$-\sin\theta\frac{d\theta}{dt} = \frac{-10}{z^2}\frac{dz}{dt}$

$z = 20, \ \cos\theta = \frac{10}{20} = \frac{1}{2}, \ \sin^2\theta = 1 - \frac{1}{4} = \frac{3}{4},$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}\frac{d\theta}{dt} = \frac{10}{20^2}(5) \Rightarrow \frac{d\theta}{dt} = \frac{1}{4\sqrt{3}} = 0.144 \text{ cs}^{-1}$

or 8.27 deg s^{-1}

b $z^2 = 100 + x^2$

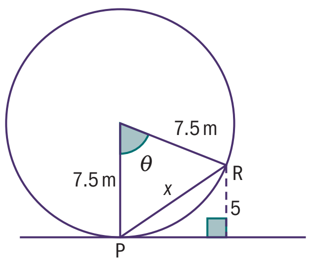
$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$

$z = 20, \ x^2 = 300 \Rightarrow x = 10\sqrt{3}$

$20(5) = 10\sqrt{3}\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{10}{\sqrt{3}} = 5.77 \text{ ms}^{-1}$

9



$\frac{d\theta}{dt} = \frac{4\pi}{60} = \frac{\pi}{15}\text{cs}^{-1}$

$x^2 = 7.5^2 + 7.5^2 - 2 \times 7.5^2 \times \cos\theta$

$x^2 = 112.5 - 112.5\cos\theta$

$2x\frac{dx}{dt} = 112.5\sin\theta\frac{d\theta}{dt}$

When height is 5 m, $\cos\theta = \frac{2.5}{7.5} = \frac{1}{3},$

$\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore \ \sin\theta = \frac{2\sqrt{2}}{3}$

$x^2 = 112.5 - 112.5\left(\frac{1}{3}\right) = 75 \qquad \therefore x = 5\sqrt{3}$

$10\sqrt{3}\frac{dx}{dt} = 112.5\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{\pi}{15}\right)$

$\therefore \ \frac{dx}{dt} = \frac{\pi\sqrt{2}}{2\sqrt{3}} = 1.28 \text{ ms}^{-1}$

Exercise 9H

1 a $\int \sin 3x \, dx = -\frac{1}{3}\cos 3x + c$

b $\int \cos(2x+1) \, dx = \frac{1}{2}\sin(2x+1) + c$

c $\int \sec^2 3x \, dx = \frac{1}{3}\tan 3x + c$

d $\int \sec^2(1-x) \, dx = -\tan(1-x) + c$

e $\int \sin\left(\frac{5x-1}{3}\right)dx = -\frac{3}{5}\cos\left(\frac{5x-1}{3}\right) + c$

f $\int \cos\left(\frac{3x+2}{7}\right)dx = \frac{7}{3}\sin\left(\frac{3x+2}{7}\right) + c$

2 a $\int (1 - 2\cos^2 x) \, dx = \int -\cos 2x \, dx = -\frac{1}{2}\sin 2x + c$

b $\int (1 + \tan^2 x) \, dx = \int \sec^2 x \, dx = \tan x + c$

c $\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x) \, dx$

$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

d $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\therefore \int \cos^2 x \, dx = \frac{1}{2}\int (1 + \cos 2x) \, dx$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

e $\int (1 - 2\sin^2(2x))dx = \int \cos 4x \, dx = \frac{1}{4}\sin 4x + c$

f $\int (2 + 2\tan^2(5x)) \, dx = \int 2\sec^2(5x) \, dx$

$= \frac{2}{5}\tan 5x + c$

g $\int (1 + \tan^2 x)(1 - \sin^2 x) \, dx = \int \sec^2 x (\cos^2 x) \, dx$

$= \int 1 \, dx = x + c$

h $\int 4\sin^2 x \cos^2 x \, dx = \int \sin^2(2x) \, dx$

$= \frac{1}{2}\int (1 - \cos 4x) \, dx$

$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c = \frac{1}{2}x - \frac{1}{8}\sin 4x + c$

Exercise 9I

1 a $\int (2\sin x - 3\cos x) \, dx = -2\cos x - 3\sin x + c$

b $\int (x^2 - 7\sin x) \, dx = \frac{1}{3}x^3 + 7\cos x + c$

c $\int\left(4e^x - \frac{1}{3}\sec^2 x\right) dx = 4e^x - \frac{1}{3}\tan x + c$

d $\int(1-\sqrt{2x}+7\sin 3x)dx = x - \sqrt{2}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{7}{3}\cos 3x + c$

$= x - \frac{2\sqrt{2}x^{\frac{3}{2}}}{3} - \frac{7}{3}\cos 3x + c$

e $\int\left(\frac{5}{2x}+\sec^2\left(\frac{x}{3}\right)\right)dx = \frac{5}{2}\ln|x| + 3\tan\frac{x}{3} + c$

f $\int\left(\frac{x}{x+1}-\sin\left(\frac{3x}{4}\right)\right)dx = \int\left(1-\frac{1}{x+1}-\sin\left(\frac{3x}{4}\right)\right)dx$

$= x - \ln|x+1| + \frac{4}{3}\cos\frac{3x}{4} + c$

g $\int\left(2^x + 5\sin\frac{x}{2} - \cos\frac{2x}{3}\right)dx$

$= \frac{2^x}{\ln 2} - 10\cos\frac{x}{2} - \frac{3}{2}\sin\frac{2x}{3} + c$

h $\int(3^{-2x} - 11\sec^2(11x))\, dx$

$= \frac{-3^{-2x}}{2\ln 3} - \tan(11x) + c$

Exercise 9J

1 a $f'(x) = 5 - 2\cos x \quad f(0) = 0$

$f(x) = 5x - 2\sin x + c$

$0 = c \quad \therefore f(x) = 5x - 2\sin x$

b $f'(x) = 4x - 6\sin 2x \quad f(0) = 1$

$f(x) = 2x^2 + 3\cos 2x + c$

$1 = 3 + c \quad \therefore c = -2 \quad f(x) = 2x^2 + 3\cos 2x - 2$

c $f'(x) = 3\cos x - 2\sec^2 x \quad f\left(\frac{\pi}{6}\right) = \frac{-2\sqrt{3}}{3}$

$f(x) = 3\sin x - 2\tan x + c$

$\frac{-2\sqrt{3}}{3} = \frac{3}{2} - \frac{2\sqrt{3}}{3} + c \quad \therefore c = \frac{-3}{2}$

$f(x) = 3\sin x - 2\tan x - \frac{3}{2}$

d $f'(x) = 3x^2 - 2e^x + \cos 4x \quad f(0) = -5$

$f(x) = x^3 - 2e^x + \frac{1}{4}\sin 4x + c$

$-5 = -2 + c \quad \therefore c = -3$

$f(x) = x^3 - 2e^x + \frac{1}{4}\sin 4x - 3$

e $f'(x) = \frac{3}{x} + \cos(3x) - 4 \quad f(1) = \frac{\sin 3}{3}$

$f(x) = 3\ln|x| + \frac{1}{3}\sin(3x) - 4x + c$

$\frac{\sin 3}{3} = \left(\frac{1}{3}\right)\sin 3 - 4 + c \quad \therefore c = 4$

$f(x) = 3\ln|x| + \frac{1}{3}\sin(3x) - 4x + 4$

f $f'(x) = \frac{7}{3-4x} - 8x + 4e^{2x-1} \quad f\left(\frac{1}{2}\right) = -1$

$f(x) = \frac{-7}{4}\ln|3-4x| - 4x^2 + 2e^{2x-1} + c$

$-1 = -1 + 2 + c \quad \therefore c = -2$

$f(x) = \frac{-7}{4}\ln|3-4x| - 4x^2 + 2e^{2x-1} - 2$

2 a $f''(x) = 4\sin x \quad f'\left(\frac{\pi}{3}\right) = 0, \; f(0) = 1$

$f'(x) = -4\cos x + c_1$

$0 = -2 + c_1 \quad \therefore c_1 = 2$

$f'(x) = -4\cos x + 2$

$f(x) = -4\sin x + 2x + c_2$

$1 = c_2 \quad \therefore f(x) = -4\sin x + 2x + 1$

b $f''(x) = 1 + \cos x \quad f'(0) = 3, \quad f(1) = -\cos(1)$

$f'(x) = x + \sin x + c_1$

$3 = c_1 \quad f'(x) = x + \sin x + 3$

$f(x) = \frac{x^2}{2} - \cos x + 3x + c_2$

$-\cos(1) = \frac{1}{2} - \cos(1) + 3 + c_2, \; c = -\frac{7}{2}$

$f(x) = \frac{1}{2}x^2 - \cos x + 3x - \frac{7}{2}$

c $f''(x) = e^{1-x} + \sin(1-x), \quad f'(1) = 2, \quad f(1) = 2$

$f'(x) = -e^{1-x} + \cos(1-x) + c_1$

$2 = -1 + 1 + c_1 \quad \therefore c_1 = 2$

$f'(x) = -e^{1-x} + \cos(1-x) + 2$

$f(x) = e^{1-x} - \sin(1-x) + 2x + c_2$

$2 = 1 + 2 + c_2 \quad \therefore c_2 = -1$

$f(x) = e^{1-x} - \sin(1-x) + 2x - 1$

d $f''(x) = e^{2x} + \sin(2x) + x^3 - 2x + 1, \quad f'(0) = 2, f(0) = 2$

$f'(x) = \frac{1}{2}e^{2x} - \frac{1}{2}\cos(2x) + \frac{1}{4}x^4 - x^2 + x + c_1$

$2 = \frac{1}{2} - \frac{1}{2} + c_1 \quad \therefore c_1 = 2$

$f'(x) = \frac{1}{2}e^{2x} - \frac{1}{2}\cos(2x) + \frac{1}{4}x^4 - x^2 + x + 2$

$f(x) = \frac{1}{4}e^{2x} - \frac{1}{4}\sin 2x + \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + c_2$

$2 = \frac{1}{4} + c_2 \therefore c_2 = \frac{7}{4}$

$f(x) = \frac{1}{4}e^{2x} - \frac{1}{4}\sin 2x + \frac{1}{20}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + \frac{7}{4}$

Exercise 9K

1 a $\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}}(2x - \sin x)\, dx = \left[x^2 + \cos x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \left(\frac{\pi^2}{4} + 0\right) - \left(\frac{\pi^2}{9} + \frac{1}{2}\right) = \frac{5\pi^2}{36} - \frac{1}{2}$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(5 + \cos x)\, dx = \left[5x + \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \left(\frac{5\pi}{2} + 1\right) - \left(\frac{5\pi}{6} + \frac{1}{2}\right) = \frac{5\pi}{3} + \frac{1}{2}$

c $\int_0^{\frac{\pi}{4}}(2\sec^2 x + 1)\,dx = \left[2\tan x + x\right]_0^{\frac{\pi}{4}}$

$= \left(2 + \frac{\pi}{4}\right) - (0) = 2 + \frac{\pi}{4}$

d $\int_0^{\frac{\pi}{3}}(e^x + 2\sin x)\,dx = \left[e^x - 2\cos x\right]_0^{\frac{\pi}{3}}$

$= (e^{\frac{\pi}{3}} - 1) - (1-2) = e^{\frac{\pi}{3}}$

e $\int_{-2\pi}^{2\pi}\left(3^{-x} + \frac{1}{4}\cos\frac{x}{4}\right)dx = \left[\frac{-3^{-x}}{\ln 3} + \sin\frac{x}{4}\right]_{-2\pi}^{2\pi}$

$= \left(\frac{-3^{-2\pi}}{\ln 3} + 1\right) - \left(\frac{-3^{2\pi}}{\ln 3} - 1\right) = \frac{3^{2\pi} - 3^{-2\pi}}{\ln 3} + 2$

f $\int_0^{\frac{\pi}{2}}\left(\frac{e^{3x}}{3} - \frac{2\sin 2x}{5}\right)dx = \left[\frac{e^{3x}}{9} + \frac{1}{5}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{1}{9}e^{\frac{3\pi}{2}} - \frac{1}{5}\right) - \left(\frac{1}{9} + \frac{1}{5}\right) = \frac{1}{9}(e^{\frac{3\pi}{2}} - 1) - \frac{2}{5}$

g $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(1 - \frac{x}{2} + 2\sin 2x\right)dx = \left[x - \frac{x^2}{4} - \cos 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{4} - \frac{\pi^2}{64} - 0\right) - \left(-\frac{\pi}{4} - \frac{\pi^2}{64} - 0\right) = \frac{\pi}{2}$

h $\int_0^{\frac{\pi}{2}}(2^x + 3\cos 6x)\,dx = \left[\frac{2^x}{\ln 2} + \frac{1}{2}\sin 6x\right]_0^{\frac{\pi}{12}}$

$= \left(\frac{2^{\frac{\pi}{12}}}{\ln 2} + \frac{1}{2}\right) - \left(\frac{1}{\ln 2} + 0\right) = \frac{2^{\frac{\pi}{12}} - 1}{\ln 2} + \frac{1}{2}$

i $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}}(x^2 + 2\sec^2 2x)\,dx = \left[\frac{x^3}{3} + \tan 2x\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}}$

$= \left(\frac{\pi^3}{1536} + 1\right) - \left(-\frac{\pi^3}{1536} - 1\right) = \frac{\pi^3}{768} + 2$

j $\int_0^{\pi}(16e^{8x} + 9\sin 3x)\,dx = \left[2e^{8x} - 3\cos 3x\right]_0^{\pi}$

$= (2e^{8\pi} + 3) - (2 - 3) = 2e^{8\pi} + 4$

Exercise 9L

1 a $\int 2x\sin x^2\,dx = -\cos x^2 + c$

b $\int 3x^2\sqrt{x^3+3}\,dx = \frac{2}{3}(x^3+3)^{\frac{3}{2}} + c$

c $\int (3-4x)e^{1+3x-2x^2}\,dx = e^{1+3x-2x^2} + c$

d $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$

$= -\ln|\cos x| + c$ or $\ln|\sec x| + c$

e $\int 2\cos 2x\, e^{\sin 2x}\,dx = e^{\sin 2x} + c$

f $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}dx = e^{\sqrt{x}} + c$

g $\int 2^x \ln 2\sin(2^x)\,dx = -\cos(2^x) + c$

h $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx = \frac{1}{2}(\arcsin x)^2 + c$

i $\int \frac{2\arctan 2x}{1-4x^2}dx = \frac{1}{2}(\arctan 2x)^2 + c$

2 a $\int x\cos x^2\,dx = \frac{1}{2}\sin x^2 + c$

b $\int x^5\sqrt[3]{x^6-1}\,dx = \frac{1}{6}(x^6-1)^{\frac{4}{3}}\frac{3}{4} + c$

$= \frac{1}{8}(x^6-1)^{\frac{4}{3}} + c$

c $\int (x+2)e^{3x^2+12x-7} = \frac{1}{6}e^{3x^2+12x-7} + c$

d $\int \frac{\tan(5x+4)}{5}dx = \int \frac{\sin(5x+4)}{5\cos(5x+4)}dx$

$= \frac{-1}{25}\ln|\cos(5x+4)| + c$ or $\frac{1}{25}\ln|\sec(5x+4)| + c$

e $\int \sin 3x \cdot 3^{\cos 3x}\,dx = \frac{-3^{\cos 3x}}{3\ln 3} + c$

f $\int \frac{\sin\sqrt[4]{x}}{\sqrt[4]{x^3}}dx = -4\cos\sqrt[4]{x} + c$

g $\int 5^x\cos(5^x)\,dx = \frac{\sin(5^x)}{\ln 5} + c$

h $\int \frac{e^{2x} + e^{-2x}}{e^{-2x} - e^{2x}}dx = -\frac{1}{2}\ln\left|e^{-2x} - e^{2x}\right| + c$

i $\int \frac{\sqrt{\arctan\frac{x}{3}}}{9+x^2}dx = \frac{2}{9}\left(\arctan\frac{x}{3}\right)^{\frac{3}{2}} + c$

j $\int (x^2+x)\cos\left(x^3 + \frac{3}{2}x^2\right)dx = \frac{1}{3}\sin\left(x^3 + \frac{3}{2}x^2\right) + c$

k $\int \frac{\arcsin^2(2x+1)}{\sqrt{-x-x^2}}dx = \frac{1}{3}\arcsin^3(2x+1) + c$

Exercise 9M

1 $\int_0^1 3x^2(x^3-1)^4\,dx = \left[\frac{1}{5}(x^3-1)^5\right]_0^1 = 0 - \left(\frac{-1}{5}\right) = \frac{1}{5}$

2 $\int_0^3 \frac{2x}{x^2+1}dx = \left[\ln|x^2+1|\right]_0^3 = \ln 10 - \ln 1 = \ln 10$

3 $\int_0^{\frac{\pi}{6}} \cos x\sqrt{\sin x}\,dx = \left[\frac{2}{3}(\sin x)^{\frac{3}{2}}\right]_0^{\frac{\pi}{6}}$

$= \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$

4 $\int_1^{e^3} \frac{\ln x}{x}dx = \left[\frac{1}{2}(\ln x)^2\right]_1^{e^3} = \frac{1}{2}(\ln e^3)^2 - 0 = \frac{9}{2}$

5 $\int_0^{\ln 2} \frac{e^x}{e^x+1}dx = \left[\ln(e^x+1)\right]_0^{\ln 2} = \ln(e^{\ln 2}+1) - \ln 2 = \ln\frac{3}{2}$

6 $\int_0^{\frac{\pi}{6}} 2\tan 2x\,dx = \left[-\ln(\cos 2x)\right]_0^{\frac{\pi}{6}} = -\ln\left(\frac{1}{2}\right) = \ln 2$

7 $\int_0^1 (x^2+x)\cos\left(x^3+\frac{3}{2}x^2\right)dx = \frac{1}{3}\left[\sin\left(x^3+\frac{3}{2}x^2\right)\right]_0^1$

$= \frac{1}{3}\sin\frac{5}{2}$

8 $\int_0^3 2^x\sqrt{2^x+1}\,dx = \frac{1}{\ln 2}\left[\frac{2}{3}(2^x+1)^{\frac{3}{2}}\right]_0^3$

$= \frac{2}{3\ln 2}\left(9^{\frac{3}{2}}-2^{\frac{3}{2}}\right) = \frac{2}{3\ln 2}(27-2\sqrt{2})$

Exercise 9N

1 $\int xe^x\,dx = xe^x - \int e^x\,dx$ $\quad u = x \quad \frac{dv}{dx} = e^x$

$= xe^x - e^x + c$ $\quad \frac{du}{dx} = 1 \quad v = e^x$

$= e^x(x-1)+c$

2 $\int (2x+9)\cos x\,dx$ $\quad u = 2x+9$

$= (2x+9)\sin x - \int 2\sin x\,dx$ $\quad \frac{dv}{dx} = \cos x$

$= (2x+9)\sin x + 2\cos x + c$ $\quad \frac{du}{dx} = 2$

$v = \sin x$

3 $\int (2-5x)\sin x\,dx$ $\quad u = 2-5x$

$\frac{dv}{dx} = \sin x$

$= -(2-5x)\cos x - \int 5\cos x\,dx$ $\quad \frac{du}{dx} = -5$

$v = -\cos x$

$=(5x-2)\cos x - 5\sin x + c$

4 $\int (3x-1)e^{3x}\,dx$ $\quad u = 3x-1$

$\frac{dv}{dx} = e^{3x}$

$= (3x-1)\frac{1}{3}e^{3x} - \int e^{3x}\,dx$ $\quad \frac{du}{dx} = 3$

$v = \frac{1}{3}e^{3x}$

$= \frac{1}{3}(3x-1)e^{3x} - \frac{1}{3}e^{3x} + c$

$= \frac{1}{3}e^{3x}(3x-2)+c$

5 $\int (4x-7)e^{4x-1}\,dx$ $\quad u = 4x-7$

$= \frac{1}{4}e^{4x-1}(4x-7) - \int e^{4x-1}\,dx$ $\quad \frac{dv}{dx} = e^{4x-1}$

$= \frac{1}{4}e^{4x-1}(4x-7) - \frac{1}{4}e^{4x-1} + c$ $\quad \frac{du}{dx} = 4$

$= \frac{1}{4}e^{4x-1}(4x-8) + c$ $\quad v = \frac{1}{4}e^{4x-1}$

$= e^{4x-1}(x-2)+c$

6 $\int \frac{x+3}{2}\sin(2x+3)dx$ $\quad u = \frac{x+3}{2}$

$\frac{dv}{dx} = \sin(2x+3)$

$= -\frac{1}{4}(x+3)\cos(2x+3)$ $\quad \frac{du}{dx} = \frac{1}{2}$

$+ \int \frac{1}{4}\cos(2x+3)\,dx$ $\quad v = -\frac{1}{2}\cos(2x+3)$

$= -\frac{1}{4}(x+3)\cos(2x+3) + \frac{1}{8}\sin(2x+3)+c$

7 $\int \frac{3-x}{4}\cos\left(\frac{x}{4}\right)dx$ $\quad u = \frac{3-x}{4}$

$\frac{dv}{dx} = \cos\left(\frac{x}{4}\right)$

$= (3-x)\sin\left(\frac{x}{4}\right) + \int \sin\left(\frac{x}{4}\right)dx$ $\quad \frac{du}{dx} = \frac{-1}{4}$

$v = 4\sin\left(\frac{x}{4}\right)$

$= (3-x)\sin\left(\frac{x}{4}\right) - 4\cos\left(\frac{x}{4}\right)+c$

8 $\int x2^x\,dx$ $\quad u = x \quad \frac{dv}{dx} = 2^x$

$\frac{2^x x}{\ln 2} - \int \frac{2^x}{\ln 2}dx$ $\quad \frac{du}{dx} = 1 \quad v = \frac{2^x}{\ln 2}$

$= \frac{2^x x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + c$

$= \frac{2^x(x\ln 2 - 1)}{(\ln 2)^2} + c$

9 $\int (1-x)5^x\,dx$ $\quad u = 1-x$

$= \frac{(1-x)5^x}{\ln 5} + \int \frac{5^x}{\ln 5}dx$ $\quad \frac{dv}{dx} = 5^x$

$= \frac{(1-x)5^x}{\ln 5} + \frac{5^x}{(\ln 5)^2} + c$ $\quad \frac{du}{dx} = -1$

$= \frac{5^x((1-x)\ln 5 + 1)}{(\ln 5)^2} + c$ $\quad v = \frac{5^x}{\ln 5}$

10 $\int \frac{(2-x)}{7.3^x}dx = \frac{1}{7}\int (2-x)3^{-x}\,dx$ $\quad u = 2-x$

$= \frac{1}{7}\left[\frac{-3^{-x}(2-x)}{\ln 3} - \int \frac{3^{-x}}{\ln 3}dx\right]$ $\quad \frac{dv}{dx} = 3^{-x}$

$= \frac{1}{7}\left[\frac{3^{-x}(x-2)}{\ln 3} + \frac{3^{-x}}{(\ln 3)^2}\right] + c$ $\quad \frac{du}{dx} = -1$

$= \frac{3^{-x}((x-2)\ln 3 + 1)}{7(\ln 3)^2} + c$ $\quad v = \frac{-3^{-x}}{\ln 3}$

11 $\int \frac{4x\cdot 3^x}{5^x}dx = \int 4x\left(\frac{3}{5}\right)^x dx$ $\quad u = 4x \quad \frac{dv}{dx} = \left(\frac{3}{5}\right)^x$

$= 4x\left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)} - \int 4\left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)}dx$ $\quad \frac{du}{dx} = 4$

$v = \left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)}$

$$= \frac{4x \cdot 3^x}{5^x \ln\left(\frac{3}{5}\right)} - 4\left(\frac{3}{5}\right)^x \frac{1}{\left(\ln\left(\frac{3}{5}\right)\right)^2} + c$$

$$= \frac{4 \cdot 3^x \left(x \ln\left(\frac{3}{5}\right) - 1\right)}{5^x \left(\ln\left(\frac{3}{5}\right)\right)^2} + c$$

Exercise 9O

1 $\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$

$u = \ln x$, $\frac{dv}{dx} = x$, $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{x^2}{2}$

$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$

2 $\int (3x + 2) \ln x \, dx$

$u = \ln x$, $\frac{dv}{dx} = 3x + 2$, $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{3x^2}{2} + 2x$

$= \left(\frac{3x^2}{2} + 2x\right) \ln x - \int \left(\frac{3x}{2} + 2\right) dx$

$= \left(\frac{3x^2}{2} + 2x\right) \ln x - \frac{3x^2}{4} - 2x + c$

3 $\int (1 - x) \ln x \, dx$

$u = \ln x$, $\frac{dv}{dx} = 1 - x$, $\frac{du}{dx} = \frac{1}{x}$, $v = x - \frac{x^2}{2}$

$= \left(x - \frac{x^2}{2}\right) \ln x - \int \left(1 - \frac{x}{2}\right) dx$

$= \left(x - \frac{x^2}{2}\right) \ln x - x + \frac{x^2}{4} + c$

4 $\int x \ln (4x) \, dx$

$u = \ln (4x)$, $\frac{dv}{dx} = x$, $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{x^2}{2}$

$= \frac{x^2}{2} \ln (4x) - \int \frac{x}{2} dx$

$= \frac{x^2}{2} \ln (4x) - \frac{x^2}{4} + c$

$= \frac{x^2}{4} (2 \ln(4x) - 1) + c$

5 $\int (3x - 2) \ln\left(\frac{x}{5}\right) dx$

$u = \ln\left(\frac{x}{5}\right)$, $\frac{dv}{dx} = 3x - 2$, $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{3x^2}{2} - 2x$

$= \left(\frac{3x^2}{2} - 2x\right) \ln\left(\frac{x}{5}\right) - \int \left(\frac{3x}{2} - 2\right) dx$

$= \left(\frac{3x^2}{2} - 2x\right) \ln\left(\frac{x}{5}\right) - \frac{3x^2}{4} + 2x + c$

6 $\int (3 + 4x) \ln(3 + 4x) dx$

Let $u = 3 + 4x$, $\frac{du}{dx} = 4$

$\int (3 + 4x) \ln(3 + 4x) \, dx = \int u \ln u \, \frac{1}{4} du$

$= \frac{1}{4}\left[\frac{u^2}{2} \ln u - \frac{u^2}{4}\right] + c$ (using result from qn1)

$= \frac{u^2}{16}(2 \ln u - 1) + c$

$= \frac{(3 + 4x)^2}{16}(2 \ln(3 + 4x) - 1) + c$

7 Let $t = 4 - 11x$, $\frac{dt}{dx} = -11$

$x = \frac{1}{11}(4 - t)$

$\therefore\ 5 + 7x = 5 + \frac{7}{11}(4 - t) = \frac{83}{11} - \frac{7t}{11}$

$\int (5 + 7x) \ln (4 - 11) dx = \frac{-1}{11}$

$\int \frac{(83 - 7t)}{11} \ln t \, dt$

$= \frac{1}{121} \int (7t - 83) \ln t \cdot dt$

$u = \ln t$, $\frac{dv}{dt} = 7t - 83$, $\frac{du}{dt} = \frac{1}{t}$, $v = \frac{7t^2}{2} - 83t$

$= \frac{1}{121}\left[\left(\frac{7t^2}{2} - 83t\right) \ln t - \int \left(\frac{7t}{2} - 83\right) dt\right]$

$= \frac{1}{121}\left[\left(\frac{7t^2}{2} - 83t\right) \ln t - \frac{7t^2}{4} + 83t\right] + c$

$= \frac{1}{121}\left[\left(\frac{7}{2}(4 - 11x)^2 - 83(4 - 11x)\right) \ln(4 - 11x) - \frac{7}{4}(4 - 11x)^2 + 83(4 - 11x)\right] + c$

$= \frac{1}{121}\left[\left(\frac{7}{2}(16 - 88x + 121x^2) - 332 + 913x\right) \ln(4 - 11x) - \frac{7}{4}(16 - 88x + 121x^2) + 332 - 913x\right] + c$

$= \frac{1}{121}\left[\left(\frac{847}{2}x^2 + 605x - 276) \ln(4 - 11x)\right) - \frac{847}{4}x^2 - 759x + 304\right] + c$

8 $\int x^2 \ln x dx$

$u = \ln x$, $\frac{dv}{dx} = x^2$, $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{x^3}{3}$

$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + c$

9 $\int (2 - x + x^2)\ln(3x)\,dx$ $\quad u = \ln(3x)$, $\frac{dv}{dx} = 2 - x + x^2$

$= \left(2x - \frac{x^2}{2} + \frac{x^3}{3}\right)\ln(3x) - \int\left(2 - \frac{x}{2} + \frac{x^2}{3}\right)dx$ $\quad \frac{du}{dx} = \frac{1}{x}$, $v = 2x - \frac{x^2}{2} + \frac{x^3}{3}$

$= \left(2x - \frac{x^2}{2} + \frac{x^3}{3}\right)\ln(3x) - 2x + \frac{x^2}{4} - \frac{x^3}{9} + c$

Exercise 9P

1 $\int \log x\,dx$ $\quad u = \log x$ $\quad \frac{dv}{dx} = 1$

$= x\log x - \int \frac{1}{\ln 10}dx$ $\quad \frac{du}{dx} = \frac{1}{x\ln 10}$ $\quad v = x$

$= x\log x - \frac{x}{\ln 10} + c$

2 $\int \log_a x\,dx$ $\quad u = \log_a x$ $\quad \frac{dv}{dx} = 1$

$= x\log_a x - \int \frac{1}{\ln a}dx$ $\quad \frac{du}{dx} = \frac{1}{x\ln a}$ $\quad v = x$

$= x\log_a x - \frac{x}{\ln a} + c$

3 $\int \arctan x\,dx$ $\quad u = \arctan x$, $\frac{dv}{dx} = 1$

$= x\arctan x - \int \frac{x}{1+x^2}dx$ $\quad \frac{du}{dx} = \frac{1}{1+x^2}$ $\quad v = x$

$= x\arctan x - \frac{1}{2}\ln\left|1 + x^2\right| + c$

4 $\int \arccos x\,ds$ $\quad u = \arccos$, $\frac{dv}{dx} = 1$

$= x\arccos x + \int \frac{x}{\sqrt{1-x^2}}dx$ $\quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\quad v = x$

$= x\arccos x - \frac{1}{2}\cdot 2(1-x^2)^{\frac{1}{2}} + c$

$= x\arccos x - \sqrt{1-x^2} + c$

5 $\int 2x\arctan x\,dx$ $\quad u = \arctan x$, $\frac{dv}{dx} = 2x$

$= x^2\arctan x - \int \frac{x^2}{1+x^2}dx$ $\quad \frac{du}{dx} = \frac{1}{1+x^2}$ $\quad v = x^2$

$= x^2\arctan x - \int 1 - \frac{1}{1+x^2}dx$

$= x^2\arctan x - x + \arctan x + c$

$= (x^2 + 1)\arctan x - x + c$

6 $\int x^2\arcsin x\,dx$ $\quad u = \arcsin x$, $\frac{dv}{dx} = x^2$

$= \frac{x^3}{3}\arcsin x - \int \frac{x^3}{3\sqrt{1-x^2}}dx$ $\quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\quad v = \frac{x^3}{3}$

Let $t = 1 - x^2$ $\quad \frac{dt}{dx} = -2x$

$\therefore\ x\,dx = \frac{-1}{2}dt$

$x^2 = 1 - t$

$\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx = \frac{1}{3}\int \frac{1-t}{t^{\frac{1}{2}}}\left(\frac{-1}{2}\right)dt = \frac{-1}{6}\int (t^{\frac{-1}{2}} - t^{\frac{1}{2}})dt$

$= \frac{-1}{6}\left(2t^{\frac{1}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right) + c = \frac{1}{3}\left(\frac{1}{3}t^{\frac{3}{2}} - t^{\frac{1}{2}}\right) + c$

$= \frac{1}{9}t^{\frac{1}{2}}(t-3) + c = \frac{1}{9}\sqrt{1-x^2}(1-x^2-3) + c$

$= \frac{-1}{9}(x^2+2)\sqrt{1-x^2} + c$

$\therefore\ \int x^2\arcsin x\,dx = \frac{x^3}{3}\arcsin x + \frac{1}{9}(x^2+2)\sqrt{1-x^2} - c$

Exercise 9Q

1 $\int x^2e^x dx$ $\quad u = x^2$ $\quad \frac{dv}{dx} = e^x$

$= x^2e^x - \int 2xe^x dx$ $\quad \frac{dv}{dx} = 2x$ $\quad v = e^x$

$= x^2e^x - [2xe^x - \int 2e^x dx]$ $\quad v = 2x$ $\quad \frac{dv}{dx} = e^x$

$= x^2e^x - 2xe^x + 2e^x + c$ $\quad \frac{du}{dx} = 2$ $\quad v = e^x$

$= e^x(x^2 - 2x + 2) + c$

2 $\int (x^2 + 1)\sin x\,dx$ $\quad u = x^2 + 1$, $\frac{dv}{dx} = \sin x$

$= -(x^2 + 1)\cos x + \int 2x\cos x\,dx$ $\quad \frac{du}{dx} = 2x$, $v = -\cos x$

$= -(x^2+1)\cos x + 2x\sin x - \int 2\sin x\,dx$ $\quad u = 2x$, $\frac{dv}{dx} = \cos x$

$= -(x^2+1)\cos x + 2x\sin x + 2\cos x + c$ $\quad \frac{du}{dx} = 2$, $v = \sin x$

$= (1 - x^2)\cos x + 2x\sin x + c$

3 $\int (2x - x^2)\cos x\,dx$ $\quad u = 2x - x^2$, $\frac{dv}{dx} = \cos x$

$(2x - x^2)\sin x + \int (2x - 2)\sin x\,dx$ $\quad \frac{du}{dx} = 2 - 2x$, $v = \sin x$

$= (2x - x^2)\sin x - (2x - 2)\cos x + \int 2\cos x\,dx$ $\quad u = 2x - 2$

$= (2x - x^2)\sin x - (2x - 2)\cos x + 2\sin x + c$ $\quad \frac{du}{dx} = 2$

$= (2x - x^2 + 2)\sin x + (2 - 2x)\cos x + c$ $\quad v = -\cos x$

4 $\int (1 + x - x^2)e^{2x}dx$

$u = 1 + x - x^2$, $\frac{dv}{dx} = e^{2x}$, $\frac{du}{dx} = 1 - 2x$, $v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}(1 + x - x^2)e^{2x} + \int (2x - 1)\frac{1}{2}e^{2x}\,dx$

$u = 2x - 1$, $\frac{dv}{dx} = \frac{1}{2}e^{2x}$, $\frac{du}{dx} = 2$, $v = \frac{1}{4}e^{2x}$

$= \frac{1}{2}(1 + x - x^2)e^{2x} + \frac{1}{4}(2x - 1)e^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$= \frac{1}{2}(1 + x - x^2)e^{2x} + \frac{1}{4}(2x - 1)e^{2x} - \frac{1}{4}e^{2x} + c$

$= \frac{1}{2}e^{x}(1 + x - x^2 + x - \frac{1}{2} - \frac{1}{2}) + c$

$= \frac{1}{2}e^{x}(2x - x^2) + c$

5 $\int (2x^2 + x + 3)\cos(2x)\,dx$

$u = 2x^2 + x + 3$, $\frac{dv}{dx} = \cos(2x)$, $\frac{du}{dx} = 4x + 1$, $v = \frac{1}{2}\sin(2x)$

$= \frac{1}{2}(2x^2 + x + 3)\sin 2x - \frac{1}{2}\int (4x + 1)\sin 2x\,dx$

$u = 4x + 1$, $\frac{dv}{dx} = \sin 2x$, $\frac{du}{dx} = 4$, $v = -\frac{1}{2}\cos 2x$

$= \frac{1}{2}(2x^2 + x + 3)\sin 2x - \frac{1}{2}\left[\frac{-1}{2}(4x+1)\cos 2x + \int 2\cos 2x\,dx\right]$

$= \frac{1}{2}(2x^2 + x + 3)\sin 2x + \frac{1}{4}(4x + 1)\cos 2x - \frac{1}{2}\sin 2x + c$

$= \frac{1}{2}(2x^2 + x + 2)\sin 2x + \frac{1}{4}(4x+1)\cos 2x + c$

6 $\int x^2\sin(1 - 2x)dx$

$u = x^2$, $\frac{dv}{dx} = \sin(1 - 2x)$, $\frac{du}{dx} = 2x$, $v = -\frac{1}{2}\cos(1 - 2x)$

$= \frac{1}{2}x^2\cos(1 - 2x) - \int x\cos(1 - 2x)\,dx$

$u = x$, $\frac{dv}{dx} = \cos(1 - 2x)$, $\frac{du}{dx} = 1$, $v = -\frac{1}{2}\sin(1 - 2x)$

$= \frac{1}{2}x^2\cos(1 - 2x) - \left[-\frac{1}{2}x\sin(1-2x) + \int \frac{1}{2}\sin(1-2x)\,dx\right]$

$= \frac{1}{2}x^2\cos(1-2x) + \frac{1}{2}x\sin(1 - 2x) - \frac{1}{4}\cos(1 - 2x) + c$

$= \frac{1}{4}(2x^2 - 1)\cos(1-2x) + \frac{1}{2}x\sin(1-2x) + c$

7 $\int x^2 3^x dx$

$u = x^2$, $\frac{dv}{dx} = 3^x$, $\frac{du}{dx} = 2x$, $v = \frac{3^x}{\ln 3}$

$= \frac{x^2 3^x}{\ln 3} - \int 2x \cdot \frac{3^x}{\ln 3}dx$

$u = 2x$, $\frac{dv}{dx} = \frac{3^x}{\ln 3}$, $\frac{du}{dx} = 2$, $v = \frac{3^x}{(\ln 3)^2}$

$= \frac{x^2 3^x}{\ln 3} - \left[2x \cdot \frac{3^x}{(\ln 3)^2} - \int 2 \cdot \frac{3^x}{(\ln 3)^2}dx\right]$

$= \frac{x^2 3^x}{\ln 3} - 2x \cdot \frac{3^x}{(\ln 3)^2} + \frac{2.3^x}{(\ln 3)^3} + c$

$= \frac{3^x}{(\ln 3)^3}[x^2(\ln 3)^2 - 2x\ln 3 + 2] + c$

8 $\int (1 + x^3)e^{\frac{x}{2}}\,dx$

$u = 1 + x^3$, $\frac{dv}{dx} = e^{\frac{x}{2}}$, $\frac{du}{dx} = 3x^2$, $v = 2e^{\frac{x}{2}}$

$= 2(1 + x^3)e^{\frac{x}{2}} - \int 6x^2 e^{\frac{x}{2}}\,dx$

$u = 6x^2$, $\frac{dv}{dx} = e^{\frac{x}{2}}$, $\frac{du}{dx} = 12x$, $v = 2e^{\frac{x}{2}}$

$= 2(1 + x^3)e^{\frac{x}{2}} - 12x^2e^{\frac{x}{2}} + \int 24xe^{\frac{x}{2}}\,dx$

$u = 24x$, $\frac{dv}{dx} = e^{\frac{x}{2}}$, $\frac{du}{dx} = 24$, $v = 2e^{\frac{x}{2}}$

$= 2(1 + x^3)e^{\frac{x}{2}} - 12x^2e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - \int 48e^{\frac{x}{2}}\,dx$

$= 2(1 + x^3)e^{\frac{x}{2}} - 12x^2e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c$

$= e^{\frac{x}{2}}[2x^3 - 12x^2 + 48x - 94] + c$

9 $\int (x^3 + x^2)\sin 5x\,dx$

$u = (x^3 + x^2)$, $\frac{dv}{dx} = \sin 5x$, $\frac{du}{dx} = 3x^2 + 2x$, $v = \frac{-1}{5}\cos 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \int (3x^2 + 2x)\frac{1}{5}\cos 5x\, dx$ $\quad u = 3x^2 + 2x \quad \frac{dv}{dx} = \frac{1}{5}\cos 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \frac{1}{25}(3x^2 + 2x)\sin 5x - \int (6x + 2)\frac{1}{25}\sin 5x\, dx$ $\quad \frac{du}{dx} = 6x + 2 \quad v = \frac{1}{25}\sin 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \frac{1}{25}(3x^2 + 2x)\sin 5x + \frac{1}{125}(6x + 2)\cos 5x - \int \frac{6}{125}\cos 5x\, dx$ $\quad u = 6x + 2 \quad \frac{dv}{dx} = \frac{1}{25}\sin 5x \quad du = 6 \quad v = \frac{1}{25}\cos 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \frac{1}{25}(3x^2 + 2x)\sin 5x + \frac{1}{125}(6x + 2)\cos 5x - \frac{6}{125}\sin 5x + c$

$= \frac{1}{125}(-25x^3 - 25x^2 + 6x + 2)\cos 5x + \frac{1}{625}(75x^2 + 50x - 6)\sin 5x + c$

10 $\int x^4\cos x\, dx$ $\quad u = x^4 \quad \frac{dv}{dx} = \cos x$

$= x^4\sin x - \int 4x^3\sin x\, dx$ $\quad \frac{du}{dx} = 4x^3 \quad v = \sin x$

$= x^4\sin x + 4x^3\cos x - \int 12x^2\cos x\, dx$

$= x^4\sin x + 4x^3\cos x - 12x^2\sin x + \int 24x\sin x\, dx$

$= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + \int 24\cos x\, dx$

$= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x + c$

$= (x^4 - 12x^2 + 24)\sin x + (4x^3 - 24x)\cos x + c$

11 $\int x^5 e^{2x}x\, dx$ $\quad u = x^5 \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 5x^4 \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^5e^{2x} - \int \frac{5}{2}x^4e^{2x}dx$ $\quad u = \frac{5}{2}x^4 \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 10x^3 \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^5e^{2x} - \frac{5}{4}x^4e^{2x} + \int 5x^3e^{2x}dx$ $\quad u = 5x^3 \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 15x^2 \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^5e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{5}{2}x^3e^{2x} - \int \frac{15}{2}x^2e^{2x}dx$ $\quad u = \frac{15}{2}x^2 \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 15x \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^5e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{5}{2}x^3e^{2x} - \frac{15}{4}x^2e^{2x} + \int \frac{15}{2}xe^{2x}dx$ $\quad u = \frac{15}{2}x \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = \frac{15}{2} \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^5e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{5}{2}x^3e^{2x} - \frac{15}{4}x^2e^{2x} + \frac{15}{4}xe^{2x} - \int \frac{15}{4}e^{2x}$

$= \frac{e^{2x}}{8}(4x^5 - 10x^4 + 20x^3 - 30x^2 + 30x - 15) + c$

Exercise 9R

1 $\int \sin x e^x\, dx$ $\quad u = \sin x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = \cos x \quad v = e^x$

$= e^x\sin x - \int e^x\cos x\, dx$ $\quad u = \cos x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = -\sin x \quad v = e^x$

$= e^x\sin x - e^x\cos x - \int \sin x e^x\, dx$

$2\int \sin x e^x\, dx = e^x(\sin x - \cos x) + c$

$\int \sin x e^x dx = \frac{1}{2}e^x(\sin x - \cos x) + c$

2 $\int e^{2x}\cos x\, dx$ $\quad u = e^{2x} \quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2e^{2x} \quad v = \sin x$

$= e^{2x}\sin x - \int 2e^{2x}\sin x\, dx$ $\quad u = 2e^{2x} \quad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 4e^{2x} \quad v = -\cos x$

$= e^{2x}\sin x + 2e^{2x}\cos x - \int 4e^{2x}\cos x\, dx$

$5\int e^{2x}\cos x\, dx = e^{2x}(\sin x + 2\cos x)$

$\therefore \int e^{2x}\cos x\, dx = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$

3 $\int \cos 3x e^{4x}\,dx$ $\quad u = \cos 3x \quad \frac{dv}{dx} = e^{4x}$

$\frac{du}{dx} = -3\sin 3x \quad v = \frac{1}{4}e^{4x}$

$\int \cos 3x e^{4x}\,dx = \frac{1}{4}e^{4x}\cos 3x \quad u = \frac{3}{4}\sin 3x \quad \frac{dv}{dx} = e^{4x}$

$+ \int \frac{3}{4}\sin 3x e^{4x}\,dx \quad \frac{du}{dx} = \frac{9}{4}\cos 3x \quad v = \frac{1}{4}e^{4x}$

$= \frac{1}{4}e^{4x}\cos 3x + \frac{3}{16}e^{4x}\sin 3x - \int \frac{9}{16}\cos 3x e^{4x}dx$

$\frac{25}{16}\int \cos 3x e^{4x}dx = \frac{e^{4x}}{16}(4\cos 3x + 3\sin 3x)$

$\int \cos 3x e^{4x}dx = \frac{1}{25}e^{4x}(4\cos 3x + 3\sin 3x) + c$

4 $\int \frac{\sin(2x)}{e^x}dx = \int \sin 2x \cdot e^{-x}\,dx \quad u = \sin 2x \quad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = 2\cos 2x \quad v = -e^{-x}$

$= -e^{-x}\sin 2x + \int 2\cos 2x e^{-x}\,dx \quad u = 2\cos 2x \quad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = -4\sin 2x \quad v = -e^{-x}$

$= -e^{-x}\sin 2x - 2e^{-x}\cos 2x - \int 4\sin 2x e^{-x}\,dx$

$5\int \frac{\sin 2x}{e^x}dx = -e^{-x}(\sin 2x + 2\cos 2x)$

$\therefore \int \frac{\sin 2x}{e^x}dx = \frac{-1}{5}e^{-x}(\sin 2x + 2\cos 2x) + c$

Exercise 9S

1 $\int x\sqrt{x+2}\,dx \quad u = x + 2 \quad dx = du$

$= \int (u-2)u^{\frac{1}{2}}\,du$

$= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{15}u^{\frac{3}{2}}(3u-10) + c$

$= \frac{2}{15}(x+2)^{\frac{3}{2}}(3x-4) + c$

2 $\int 3x\sqrt{1-2x}\,dx \quad u = 1 - 2x$

$x = \frac{1}{2}(1-u) \quad dx = -\frac{1}{2}du$

$= \int \frac{3}{2}(1-u)u^{\frac{1}{2}}\left(\frac{-1}{2}\right)du$

$= \frac{-3}{4}\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$

$= \frac{-3}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + c$

$= \frac{-3}{60}\left(10u^{\frac{3}{2}} - 6u^{\frac{5}{2}}\right) + c$

$= \frac{-u^{\frac{3}{2}}}{10}(5-3u) + c$

$= \frac{-(1-2x)^{\frac{3}{2}}}{10}(2+6x) + c = -\frac{1}{5}(1-2x)^{\frac{3}{2}}(1+3x) + c$

3 $\int 5x^2\sqrt{3+4x}\,dx \quad u = 3 + 4x$

$x = \frac{1}{4}(u-3) \quad dx = \frac{1}{4}du$

$= \int \frac{5}{16}(u^2 - 6u + 9)\frac{u^{\frac{1}{2}}}{4}du$

$= \frac{5}{64}\int \left(u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}}\right)du$

$= \frac{5}{64}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{12}{5}u^{\frac{5}{2}} + 6u^{\frac{3}{2}}\right) + c$

$= \frac{1}{448}\left(10u^{\frac{7}{2}} - 84u^{\frac{5}{2}} + 210u^{\frac{3}{2}}\right) + c$

$= \frac{u^{\frac{3}{2}}}{224}(5u^2 - 42u + 105) + c$

$= \frac{(3+4x)^{\frac{3}{2}}}{224}(5(9 + 24x + 16x^2) - 42(3 + 4x) + 105) + c$

$= \frac{(3+4x)^{\frac{3}{2}}}{224}(24 - 48x + 80x^2) + c$

4 $x\sqrt[3]{x+3}\,dx$

$= \int u = x + 3 \; dx = du$

$= \int (u-3)u^{\frac{1}{3}}\,du \; (u^{\frac{4}{3}} - 3u^{\frac{1}{3}})du$

$= \frac{3}{7}u^{\frac{7}{3}} - \frac{9}{4}u^{\frac{4}{3}} + c$

$= \frac{3u^{\frac{4}{3}}}{28}(4u-21) + c$

$= \frac{3}{28}(x+3)^{\frac{4}{3}}(4x-9) + c$

5 $\int x^2\sqrt[4]{x+1}\,dx \quad u = x + 1 \quad dx = du$

$x = u - 1$

$= \int (u^2 - 2u + 1)u^{\frac{1}{4}}\,du = \int \left(u^{\frac{9}{4}} - 2u^{\frac{5}{4}} + u^{\frac{1}{4}}\right)du$

$= \frac{4}{13}u^{\frac{13}{4}} - \frac{8}{9}u^{\frac{9}{4}} + \frac{4}{5}u^{\frac{5}{4}} + c$

$= \frac{4u^{\frac{5}{4}}}{585}(45u^2 - 130u + 117) + c$

$= \frac{4}{585}(x+1)^{\frac{5}{4}}(45u^2 - 130u + 117) + c$

$= \frac{4}{585}(x+1)^{\frac{5}{4}}(45x^2 - 40x + 32) + c$

6 $\int x^3 \sqrt[5]{1-x}\, dx$ $u = 1 - x$ $dx = -du$
$x = 1 - u$

$= -\int (1 - 3u + 3u^2 - u^3)u^{\frac{1}{5}}\, du$

$= -\int \left(u^{\frac{1}{5}} - 3u^{\frac{6}{5}} + 3u^{\frac{11}{5}} - u^{\frac{16}{5}}\right) du$

$= -\frac{5}{6}u^{\frac{6}{5}} + \frac{15}{11}u^{\frac{11}{5}} - \frac{15}{16}u^{\frac{16}{5}} + \frac{5}{21}u^{\frac{21}{5}} + c$

$= -\frac{5}{3696}u^{\frac{6}{5}}(616 - 1008u + 693u^2 - 176u^3) + c$

$= -\frac{5}{3696}(1-x)^{\frac{6}{5}}(616 - 1008(1-x) + 693(1 - 2x + x^2)$
$\quad - 176(1 - 3x + 3x^2 - x^3)) + c$

$= -\frac{5(1-x)^{\frac{6}{5}}}{3696}(125 + 150x + 165x^2 + 176x^3) + c$

Exercise 9T

1 $\int \cos^3 x\, dx = \int (1 - \sin^2 x)\cos x\, dx$

$= \int (\cos x - \sin^2 x \cos x)\, dx$

$= \sin x - \frac{1}{3}\sin^3 x + c$

2 $\int \cos^4 x\, dx = \int (\cos^2 x)^2\, dx$

$= \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx$

$= \int \frac{1 + 2\cos 2x + \cos^2 2x}{4}\, dx$

$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cdot\frac{1 + \cos 4x}{2}\right) dx$

$= \int \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$

$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

3 $\int \sin^5\left(\frac{x}{5}\right) dx = \int \left(1 - \cos^2\left(\frac{x}{5}\right)\right)^2 \sin\left(\frac{x}{5}\right) dx$

$= \int \left(\sin\frac{x}{5} - 2\cos^2\frac{x}{5}\sin\frac{x}{5} + \cos^4\frac{x}{5}\sin\frac{x}{5}\right) dx$

$= -5\cos\frac{x}{5} + \frac{10}{3}\cos^3\frac{x}{5} - \cos^5\frac{x}{5} + c$

4 $\int 48\cos^6(2x)\, dx = \int 48(\cos^2(2x))^3\, dx$

$= \int 48\left(\frac{1 + \cos 4x}{2}\right)^3 dx$

$= \int 6(1 + 3\cos 4x + 3\cos^2 4x + \cos^3 4x)\, dx$

$= \int (6 + 18\cos 4x + 9(1 + \cos 8x)$
$\quad + 6\cos 4x(1 - \sin^2 4x))\, dx$

$= \int (15 + 24\cos 4x + 9\cos 8x - 6\cos 4x \sin^2 4x)\, dx$

$= 15x + 6\sin 4x + \frac{9}{8}\sin 8x - \frac{1}{2}\sin^3 4x + c$

Exercise 9U

1 $\int \sqrt{4 - x^2}\, dx$ $x = 2\sin\theta$ $dx = 2\cos\theta\, d\theta$

$4 - x^2 = 4 - 4\sin^2\theta = 4(1 - \sin^2\theta) = 4\cos^2\theta$

$\sqrt{4 - x^2} = 2\cos\theta$

$\int \sqrt{4 - x^2}\, dx = \int 2\cos\theta.\ 2\cos\theta\, d\theta = \int 4\cos^2\theta\, d\theta$

$= \int 2(1 + \cos 2\theta)\, d\theta$

$= 2\theta + \sin 2\theta + c$

$= 2\theta + 2\sin\theta\cos\theta + c$

$\theta = \arcsin\frac{x}{2}$ $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{x^2}{4}} = \frac{\sqrt{4 - x^2}}{2}$

$\int \sqrt{4 - x^2}\, dx = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4 - x^2} + c$

2 $\int \frac{1}{\sqrt{x^2 - 1}}\, dx$ $x = \sec\theta$ $dx = \sec\theta\tan\theta\, d\theta$

$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

$\sqrt{x^2 - 1} = \tan\theta$

$\int \frac{1}{\sqrt{x^2 - 1}}\, dx = \int \frac{1}{\tan\theta}\sec\theta\tan\theta\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln \sec\theta + \tan\theta\,| + c$

$= \ln\left|x + \sqrt{x^2 - 1}\right| + c$

3 $\int \sqrt{x^2 + 9}\, dx$ $x = 3\tan\theta$ $dx = 3\sec^2\theta\, d\theta$

$x^2 + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta$

$\sqrt{x^2 + 9} = 3\sec\theta$

$\int \sqrt{x^2 + 9}\, dx = \int 3\sec\theta\ 3\sec^2\theta\, d\theta$

$= \int 9\sec^3\theta = 9\int \sec\theta\sec^2\theta\, d\theta$

Using integration by parts:

$u = \sec\theta$ $\frac{dv}{d\theta} = \sec^2\theta$

$\frac{du}{d\theta} = \sec\theta\tan\theta$ $v = \tan\theta$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^2\theta\, d\theta + \int \sec\theta\, d\theta$$

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$$

$$\therefore \int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln \sec\theta + \tan\theta| + c$$

$$\therefore \int \sqrt{x^2+9}\, dx = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta$$

$$+ \tan\theta\,|\ln + c$$

$$\int \sqrt{x^2+9}\, dx = \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| + c$$

$$= \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|\sqrt{x^2+9}+x\right| + k$$

4 $\int \frac{3}{\sqrt{36-x^2}}\, dx \quad x = 6\sin\theta \qquad dx = 6\cos\theta\, d\theta$

$$36 - x^2 = 36(1 - \sin^2\theta) = 36\cos^2\theta$$

$$\sqrt{36-x^2} = 6\cos\theta$$

$$\int \frac{3}{\sqrt{36-x^2}}\, dx = \int \frac{3}{6\cos\theta}\,.\,6\cos\theta\, d\theta = \int 3\, d\theta$$

$$= 3\theta + c$$

$$= 3\arcsin\frac{x}{6} + c$$

5 $\int 3\sqrt{x^2-16}\, dx \quad x = 4\sec\theta \quad dx = 4\sec\theta\tan\theta d\theta$

$$x^2 - 16 = 16(\sec^2\theta - 1) = 16\tan^2\theta$$

$$\sqrt{x^2-16} = 4\tan\theta$$

$$\int 3\sqrt{x^2-16}\, dx = \int 12\tan\theta.4\sec\theta\tan\theta\, d\theta$$

$$= \int 48\sec\theta\tan^2\theta\, d\theta$$

$$= \int 48\sec\theta(\sec^2\theta - 1)d\theta$$

$$= \int (48\sec^3\theta - 48\sec\theta)\, d\theta$$

(see qn. 3 for $\int \sec^3\theta$)

$$= 48\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)$$

$$- 48\ln|\sec\theta + \tan\theta| + c$$

$$= 24\sec\theta\tan\theta - 24\ln|\sec\theta + \tan\theta| + c$$

$$= 6x\frac{1}{4}\sqrt{x^2-16} - 24\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + c$$

$$= \frac{3}{2}x\sqrt{x^2-16} - 24\ln\left|x + \sqrt{x^2-16}\right| + k$$

6 $\int \frac{5}{\sqrt{x^2+121}}\, dx \quad x = 11\tan\theta \quad dx = 11\sec^2\theta\, d\theta$

$$x^2 + 121 = 121(\tan^2\theta + 1) = 121\sec^2\theta$$

$$\sqrt{x^2+121} = 11\sec\theta$$

$$\int \frac{5}{\sqrt{x^2+121}}\, dx = \int \frac{5}{11\sec\theta}.\,11\sec^2\theta\, d\theta$$

$$= \int 5\sec\theta\, d\theta$$

$$= 5\ln|\sec\theta + \tan\theta| + c$$

$$= 5\ln\left|\frac{\sqrt{x^2+121}+x}{11}\right| + c$$

$$= 5\ln\left|\sqrt{x^2+121}+x\right| + k$$

7 $\int \frac{2}{\sqrt{81-4x^2}}\, dx \quad x = \frac{9}{2}\sin\theta \quad dx = \frac{9}{2}\cos\theta\, d\theta$

$$81 - 4x^2 = 81 - 81\sin^2\theta = 81\cos^2\theta$$

$$\sqrt{81-4x^2} = 9\cos\theta$$

$$\int \frac{2}{\sqrt{81-4x^2}}\, dx = \int \frac{2}{9\cos\theta}\,\frac{9}{2}\cos\theta\, d\theta = \int 1 d\theta$$

$$= \theta + c$$

$$= \arcsin\frac{2x}{9} + c$$

8 $\int \sqrt{3x^2-75}\, dx = \sqrt{3}\int \sqrt{x^2-25}\, dx$

$x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\, d\theta$

$x^2 - 25 = 25(\sec^2\theta - 1) = 25\tan^2\theta$

$$\int \sqrt{3x^2-75}\, dx = \sqrt{3}\int 5\tan\theta\cdot 5\sec\theta\tan\theta\, d\theta$$

$$= \sqrt{3}\int 25\sec\theta\tan^2\theta\, d\theta$$

$$= \sqrt{3}\int 25\sec\theta(\sec^2\theta - 1)\, d\theta$$

(see qn 3 for $\int \sec^3 d\theta$)

$$= 25\sqrt{3}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta|\right] + c$$

$$= \frac{25\sqrt{3}}{2}(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|) + c$$

$$= \frac{25\sqrt{3}}{2}\left(\frac{x}{s}\,\frac{\sqrt{x^2-25}}{5} - \ln\left|\frac{x+\sqrt{x^2-25}}{5}\right|\right) + c$$

$$= \frac{\sqrt{3}}{2}x\sqrt{x^2-25} - \frac{25\sqrt{3}}{2}\ln\left|x + \sqrt{x^2-25}\right| + k$$

9 $\int \frac{7}{\sqrt{7x^2+28}}\, dx = \sqrt{7}\int \frac{1}{\sqrt{x^2+4}}\, dx$

$x = 2\tan\theta \qquad dx = 2\sec^2\theta\, d\theta$

$x^2 + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$

$\sqrt{x^2+4} = 2\sec\theta$

$$\int \frac{7}{\sqrt{7x^2+28}}\,dx = \sqrt{7}\int \frac{1}{2\sec\theta}2\sec^2\theta\,d\theta$$

$$= \sqrt{7}\int \sec\theta\,d\theta$$

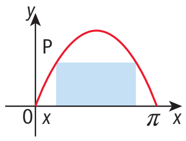
$$= \sqrt{7}\ln|\sec\theta + \tan\theta| + c$$

$$= \sqrt{7}\ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + c$$

$$= \sqrt{7}\ln\left|\sqrt{x^2+4}+x\right| + k$$

Exercise 9V

1



P(x, sin$x$)

A = $(\pi - 2x)\sin x$

Max. area = 1.12 when $x = 0.71046$

length = $\pi - 2x = 1.72$

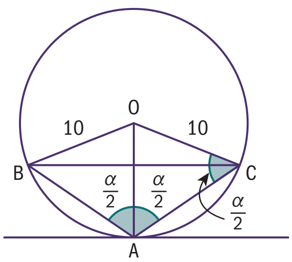
height = $\sin x = 0.652$

2 A (1, –1)   P (x, cos$x$)

$AP = \sqrt{(x-1)^2 + (\cos x + 1)^2}$

Min. distance = 1.11 (at the point (1.78, –0.025))

3 **a**



In Δ AOC, angle AOC = $\pi - \alpha$

area Δ AOC = $\frac{1}{2}$ .10.10 sin $(\pi - \alpha)$

= 50 sin α

Similarly, area AOB = 50sin α

In Δ BOC, angle BOC = $2\pi - 2\alpha$

area BOC = $\frac{1}{2}\cdot 10.10\sin(2\pi - 2\alpha)$

= –50 sin2α

∴ area ABC = 50 sinα + 50 sinα – (–50sin2α)

A (α) = 100 sinα + 50sin2α

= 100 sinα + 100 sinαcosα

A (α) = 100(1 + cosα) sinα

b For maximum area, $\alpha = 1.05$

4 **a** $t = 2\sqrt{\frac{15}{g\sin 2\theta}}$ we require t to be a minimum

$\theta = 0.785 = \frac{\pi}{4}$

b $\theta = 0.7854$ $1 = \frac{15}{\cos\theta}$

= 21.213m (nearest mm)

5 **a** $d(t) = \sin\left(\frac{\pi t}{6}\right) + \cos\left(\frac{\pi t}{6}\right)$

$v(t) = \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right) - \frac{\pi}{6}\sin\left(\frac{\pi t}{6}\right)$

$a(t) = \frac{-\pi^2}{36}\sin\left(\frac{\pi t}{6}\right)\frac{-\pi^2}{36}\cos\left(\frac{\pi t}{6}\right)$

$\therefore a(t) = \frac{-\pi^2}{36}d(t)$

∴ acceleration is proportional to displacement.

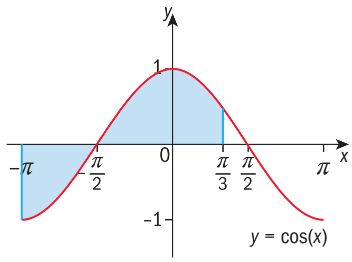
b Max. speed = 0.740 ms^{-1} when t = 4.50 s (velocity in negative and a minimum at this time).

6 Min. height = 1.82 m when x = 1.31 m

∴ the first pole is nearer to the point of minimum height.

Exercise 9W

1 **a**



$$\int_{-\pi}^{-\frac{\pi}{2}} \cos x\,dx = [\sin x]_{-\pi}^{\frac{-\pi}{2}}$$

$$= \sin\left(\frac{-\pi}{2}\right) - \sin(-\pi)$$

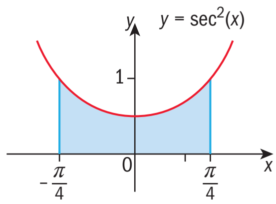
$$= -1 - 0 = -1$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{3}} \cos x\,dx = [\sin x]_{\frac{-\pi}{2}}^{\frac{\pi}{3}} = \sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= \frac{\sqrt{3}}{2} + 1$$

$\therefore$ total area = $2 + \frac{\sqrt{3}}{2}$

b

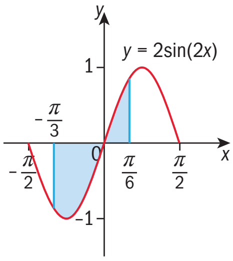


$$\text{Area} = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= 2[\tan x]_0^{\frac{\pi}{4}}$$

$$= 2\tan\frac{\pi}{4} = 2$$

c



$$\int_{-\frac{\pi}{2}}^{0} \sin 2x\ dx = [-\cos 2x]_{-\frac{\pi}{3}}^{0}$$

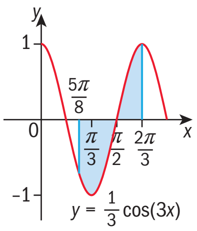
$$= -\cos 0 + \cos\left(-\frac{2\pi}{3}\right) = -1 + \left(-\frac{1}{2}\right) = -\frac{3}{2}$$

$$\int_{0}^{\frac{\pi}{6}} 2\sin 2x\ dx = [-\cos 2x]_{0}^{\frac{\pi}{6}}$$

$$= -\cos\frac{\pi}{3} + \cos 0 = -\frac{1}{2} + 1 = \frac{1}{2}$$

area = 2

d



$$\int_{\frac{5\pi}{18}}^{\frac{\pi}{2}} \frac{1}{3}\cos 3x\,dx = \frac{1}{9}[\sin 3x]_{\frac{5\pi}{18}}^{\frac{\pi}{2}}$$

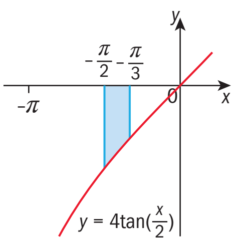
$$= \frac{1}{9}\left(\sin\frac{3\pi}{2} - \sin\frac{5\pi}{6}\right) = \frac{1}{9}\left(-1 - \frac{1}{2}\right) = -\frac{1}{6}$$

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{3}\cos 3x\,dx = \frac{1}{9}[\sin\ 3x]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = \frac{1}{9}\left(\sin 2\pi - \sin\frac{3\pi}{2}\right)$$

$$= \frac{1}{9}(0 - (-1)) = \frac{1}{9}$$

$$\text{area} = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$$

e



$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} 4\tan\left(\frac{x}{2}\right) dx = 8\left[\ln\left|\sec\frac{x}{2}\right|\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{3}}$$

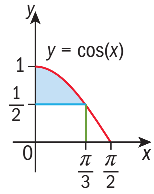
$$= 8\left(\ln\left|\sec\left(-\frac{\pi}{6}\right)\right| - \ln\left|\sec\left(-\frac{\pi}{4}\right)\right|\right)$$

$$= 8\left(\ln\frac{2}{\sqrt{3}} - \ln(\sqrt{2})\right)$$

$$= 8\ln\sqrt{\frac{2}{3}} = -8\ln\sqrt{\frac{3}{2}} = -4\ln\left(\frac{3}{2}\right)$$

$$\text{area} = 4\ \ln\left(\frac{3}{2}\right)$$

2 a



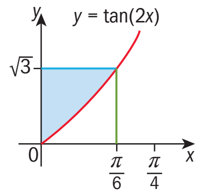
$$\text{Area} = \int_{0}^{\frac{\pi}{3}} \cos x\,dx - \frac{\pi}{6}$$

$$= [\sin x]_{0}^{\frac{\pi}{3}} - \frac{\pi}{6}$$

$$= \sin\frac{\pi}{3} - \sin 0 - \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

b



$$\text{Area} = \frac{\pi\sqrt{3}}{6} - \int_{0}^{\frac{\pi}{2}} \tan\ 2x\ dx$$

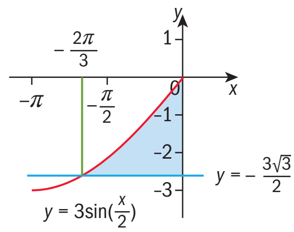
$$= \frac{\pi\sqrt{3}}{6} - \left[\frac{1}{2}\ln|\sec(2x)|\right]_{0}^{\frac{\pi}{6}}$$

$$= \frac{\pi\sqrt{3}}{5} - \frac{1}{2}\left[\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right]$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}\ln 2$$

c



$$3\sin\frac{x}{2} = -3\frac{\sqrt{3}}{2}$$

$$\sin\frac{x}{2} = -\frac{\sqrt{3}}{2}$$

$$\frac{x}{2} = -\frac{\pi}{3}$$

$$x = -\frac{2\pi}{3}$$

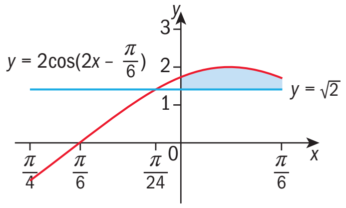
$$\int_{\frac{-2\pi}{3}}^{0} 3\sin\frac{x}{2}\ dx = -6\left[\cos\left(\frac{x}{2}\right)\right]_{\frac{-2\pi}{3}}^{0}$$

$$= -6\left[\cos\ 0 - \cos\left(-\frac{\pi}{3}\right)\right]$$

$$= -6\left(1 - \frac{1}{2}\right) = -3$$

$$\text{Area} = \frac{2\pi}{3}\left(\frac{3\sqrt{3}}{2}\right) - 3 = \sqrt{3}\pi - 3$$

d



$$2\cos\left(2x-\frac{\pi}{6}\right)=\sqrt{2}$$

$$2x-\frac{\pi}{6}=\frac{\pi}{4}$$

$$2x=\frac{5\pi}{12}$$

$$x=\frac{5\pi}{24}$$

$$\text{Area} = \int_0^{\frac{5\pi}{24}} 2\cos\left(2x-\frac{\pi}{6}\right)\,dx-\frac{5\pi}{24}(\sqrt{2})$$

$$=\left[\sin\left(2x-\frac{\pi}{6}\right)\right]_0^{\frac{5\pi}{24}}-\frac{5\pi}{24}(\sqrt{2})$$

$$=\sin\left(\frac{\pi}{4}\right)-\sin\left(-\frac{\pi}{6}\right)-\frac{5\pi\sqrt{2}}{24}$$

$$=\frac{\sqrt{2}+1}{2}-\frac{5\pi\sqrt{2}}{24}$$

e $\tan\frac{x}{3}=\frac{1}{\sqrt{3}}$

$$\frac{x}{3}=\frac{\pi}{6}$$

$$x=\frac{\pi}{2}$$

$$\text{Area} = \frac{\pi}{2\sqrt{3}} - \int_0^{\frac{\pi}{2}} \tan\frac{x}{3}\,dx$$

$$=\frac{\pi}{2\sqrt{3}}-\left[3\ln\left|\sec\frac{x}{6}\right|\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2\sqrt{3}}-3\left(\ln\left|\sec\frac{\pi}{6}\right|-\ln|\sec\ 0|\right)$$

$$=\frac{\pi}{2\sqrt{3}}-3\ln\frac{2}{\sqrt{3}}$$

3 $\text{Area} = \int_0^{2.51327}\left(\cos\frac{x}{2}-\cos 2x\right)dx = 2.38$

4 $\frac{dy}{dx}=\sec^2 x$

$\left(\frac{\pi}{4},1\right)$ $\frac{dy}{dx}=2$

Tangent: $y-1=2\left(x-\frac{\pi}{4}\right)$

$$y=2x-\frac{\pi}{2}+1$$

if $y=0,\ 2x=\frac{\pi}{2}-1$

$$x=\frac{\pi}{4}-\frac{1}{2}$$

$$\text{Area} = \int_0^{\frac{\pi}{4}}\tan x\,dx-\frac{1}{2}\left(\frac{1}{2}\right)$$

$$=\left[\ln|\sec x|\right]_0^{\frac{\pi}{4}}-\frac{1}{4}=\ln\left|\sec\frac{\pi}{4}\right|-\ln|\sec\ 0|-\frac{1}{4}$$

$$=\ln\sqrt{2}-\frac{1}{4}=\frac{1}{2}\ln 2-\frac{1}{4}$$

5 $\text{Area} = \int_0^{2.57915}\left|2\sin x - e^{\frac{x}{2}-4}-1\right|dx = 1.55$

6 $y=\frac{8}{4+x^2}$ $\quad y=\frac{x^2}{4}$

a $\frac{8}{4+x^2}=\frac{x^2}{4}\Rightarrow 32=4x^2+x^4$

$$x^4+4x^2-32=0$$

$$(x^2+8)(x^2-4)=0$$

$x^2=4,\ x=\pm 2$ $\quad$ (2, 1), (−2, 1)

b, c $\text{Area} = \int_{-2}^{2}\left(\frac{8}{4+x^2}-\frac{x^2}{4}\right)dx$

$$=\left[\frac{8}{2}\arctan\left(\frac{x}{2}\right)-\frac{x^3}{12}\right]_{-2}^{2}$$

$$=\left(4\arctan 1-\frac{2}{3}\right)-\left(4\arctan(-1)+\frac{2}{3}\right)$$

$$=\left(4\frac{\pi}{4}-\frac{2}{3}\right)-\left(-4\frac{\pi}{4}+\frac{2}{3}\right)$$

$$=2\pi-\frac{4}{3}$$

Exercise 9X

1 a $v=\pi\int_0^{\frac{\pi}{2}}\cos x\,dx=\pi[\sin x]_0^{\frac{\pi}{2}}=\pi\left(\sin\frac{\pi}{2}-\sin 0\right)$

$v=\pi$

b $v=\pi\int_0^{\frac{\pi}{4}}\sec^2 x\,dx=\pi[\tan x]_0^{\frac{\pi}{4}}=\pi\left(\tan\frac{\pi}{4}-\tan 0\right)$

$v=\pi$

c $v=\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\cos^2 x\ 2dx=\frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1+\cos 2x)\,dx$

$$=\frac{\pi}{2}\left[x+\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$=\frac{\pi}{2}\left[\left(\frac{5\pi}{6}+\frac{1}{2}\sin\frac{5\pi}{3}\right)-\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$=\frac{\pi}{2}\left[\frac{5\pi}{6}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right]=\frac{\pi}{2}\left[\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right]$$

$$=\pi\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$$

d $v=\pi\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\sin^2 x\,dx=\frac{\pi}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}(1-\cos 2x)\,dx$

$$=\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$=\frac{\pi}{2}\left[\left(\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}\right)-\left(\frac{\pi}{3}-\frac{1}{2}\sin\frac{2\pi}{3}\right)\right]$$

$$=\frac{\pi}{2}\left[\frac{2\pi}{3}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]$$

$$=\frac{\pi}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

2 a $v = \pi\int_0^1 \sin^2 y\, dy = \frac{\pi}{2}\int_0^1 (1 - \cos 2y)\, dy$

$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^1$

$= \frac{\pi}{2}\left[\left(1 - \frac{1}{2}\sin 2\right) - 0\right]$

$= \frac{\pi}{2}\left(1 - \frac{1}{2}\sin 2\right)$

b $y = \arcsin x, \quad 0 \le x \le 1$

$\Rightarrow x = \sin y, \quad 0 \le y \le \frac{\pi}{2}$

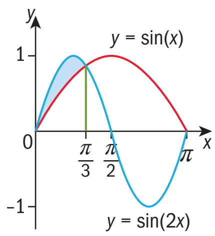
$v = \pi\int_0^{\frac{\pi}{2}} \sin^2 y\, dy = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2y)\, dy$

$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin \pi - (0)\right]$

$= \frac{\pi^2}{4}$

3 b



$v = \pi\int_0^{\frac{\pi}{3}} (\sin^2 2x - \sin^2 x)\, dx$

$\int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 4x - \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$

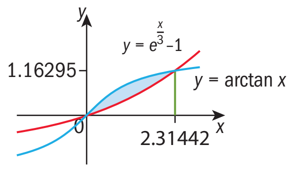
$= \frac{\pi}{2}\int_0^{\frac{\pi}{3}} (\cos 2x - \cos 4x)\, dx$

$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{2}\left[\frac{1}{2}\sin\frac{2\pi}{3} - \frac{1}{4}\sin\frac{4\pi}{3}\right]$

$= \frac{\pi}{2}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right) = \frac{3\pi\sqrt{3}}{16}$

c



$v = \pi\int_0^{2.31442} \left((\arctan x)^2 - (e^{\frac{x}{3}} - 1)^2\right) dx = 2.35$

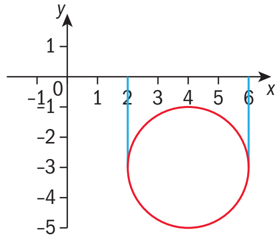
d $y = \arctan x \Rightarrow x2 + 1\ x = \tan y$

$y = e^{\frac{x}{3}} - 1\ \ e^{\frac{x}{3}} = y + 1\ \ x = 3\ln(y + 1)$

$v = \pi\int_0^{1.16295} ((3\ln(y + 1))^2 - \tan^2 y)\, dy = 4.18$

Exercise 9Y

1



$(x - 4)^2 + (y + 3)^2 = 4$

$(y + 3)^2 = 4 - (x - 4)^2$

$y + 3 = \pm\sqrt{4 - (x - 4)^2}$

$y = -3 \pm \sqrt{4 - (x - 4)^2}$

$V = \pi\int_2^6 \left(\left(-3 - \sqrt{4 - (x - 4)^2}\right)^2 - \left(-3 + \sqrt{4 - (x - 4)^2}\right)^2\right) dx$

$= \pi\int_2^6 \left(12\sqrt{4 - (x - 4)^2}\right) dx$

$= 12\pi\int_2^6 \sqrt{4 - (x - 4)^2}\, dx$

Let $x - 4 = 2\sin\theta\ \ dx = 2\cos\theta\, d\theta$

$x = 2 \Rightarrow \sin\theta = -1\ \ \theta = -\frac{\pi}{2}$

$x = 6 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\sqrt{4 - (x - 4)^2} = 2\cos\theta$

$v = 12\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta$

$= 24\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$

$= 24\pi\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$v = 24\pi^2$

2 $(x - 4)^2 + (y + 3)^2 = 4$

$(x - 4)^2 = 4 - (y + 3)^2$

$x - 4 = \pm\sqrt{4 - (y + 3)^2}$

$x = 4 \pm \sqrt{4 - (y + 3)^2}$

$v = \pi\int_{-5}^{-1} \left(\left(4 + \sqrt{4 - (y + 3)^2}\right)^2 - \left(4 - \sqrt{4 - (y + 3)^2}\right)^2\right) dy$

$= \pi\int_{-5}^{-1} 16\sqrt{4 - (y + 3)^2}\, dy$

Let $y + 3 = 2\sin\theta \quad dy = 2\cos\theta\, d\theta$

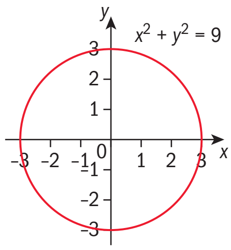
$y = -5 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$

$y = -1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\sqrt{4 - (y + 3)^2} = 2\cos\theta$

$v = 16\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta = 32\pi^2$(see qn.1)

3



$y^2 = 9 - x^2$

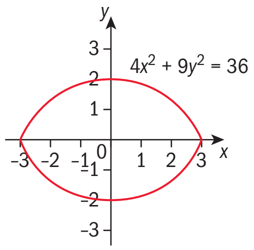
$v = \pi\int_{-3}^{3}(9 - x^2)\,dx$

$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$

$v = \pi\left[\left(27 - \frac{27}{3}\right) - \left(-27 + \frac{27}{3}\right)\right]$

$v = 36\pi$

4



$4x^2 + 9y = 36$

$y^2 = 4 - \frac{4}{9}x^2$

$v = \pi\int_{-3}^{-3}\left(4 - \frac{4}{9}x^2\right)dx$

$v = \pi\left[4x - \frac{4x^3}{27}\right]_{-3}^{3}$

$= \pi\,[(12 - 4) - (-12 + 4)] = 16\pi$

5 $x^2 = 9 - \frac{9}{4}y^2,\ v = \pi\int_{-2}^{2}\left(9 - \frac{9}{4}y^2\right)dy$

$v = \pi\left[9y - \frac{3}{4}y^3\right]_{-2}^{2}$

$\pi[(18 - 6) - (-18 + 6)] = 24\pi$

Review exercise

1 **a** $f(x) = (2x+3)\sin x$

$\Rightarrow f'(x) = 2\sin x + (2x+3)\cos x$

b $g(x) = e^x\cos 3x$

$\Rightarrow g'(x) = e^x\cos 3x + e^x\cdot(-\sin 3x)\cdot 3$

$= e^x(\cos 3x - 3\sin 3x)$

c $h(x) = \frac{\tan x}{2x^2} = \frac{1}{2}\tan x \cdot x^{-2}$

$\Rightarrow h'(x) = \frac{1}{2}\sec^2 x \cdot x^{-2} + \frac{1}{2}\tan x \cdot (-2x^{-3})$

$= \frac{x - 2\sin x\cos x}{2x^3\cos^2 x}$

$= \frac{x - \sin 2x}{2x^3\cos^2 x}$

2 $\sin y + e^{2x} = 1 \Rightarrow \cos y \cdot y' + e^{2x}\cdot 2 = 0.$

$\Rightarrow y' = \frac{-2e^{2x}}{\cos y}$

$\Rightarrow m = y'(0) = \frac{-2e^0}{\cos 0} = -2$

$T: y - 0 = -2(x - 0) \Rightarrow y = -2x$

3 $\int_{\frac{\pi}{4}}^{m}\sec^2 x\,dx = [\tan x]_{\frac{\pi}{4}}^{m} = \tan m - \tan\frac{\pi}{4}$

$= \tan m - 1 = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

$\Rightarrow \tan m - 1 = \sqrt{3} - 1 \Rightarrow \tan m = \sqrt{3} \Rightarrow m = \frac{\pi}{3}$

4 **a** $\int(2x-5)e^{2x}dx =$ Let $2x - 5 = u$, $e^{2x}dx = dv$ $\Rightarrow$ $2dx = du$, $\frac{1}{2}e^{2x} = v$,

$\frac{2x-5}{2}e^{2x} - \int e^{2x}dx = \left(x - \frac{5}{2}\right)e^{2x} - \frac{1}{2}e^{2x} + c$

$= (x - 3)e^{2x} + c$

b

	dv = cosx dx	**sign**
***u* = *x*² − 5**	v = sinx	+
2x	−cosx	−
2	−sinx	+

$(x^2 - 5x)\cos x\,dx = (x^2 - 5x)\sin x$

$+ 2x\cos x - 2\sin x + c;$

$= (x^2 - 5x - 2)\sin x + 2x\cos x + c$

c

	dv = eˣdx	**sign**
***u* = cosx**	v = eˣ	+
−3sin3x	eˣ	−
−9cos3x	eˣ	+

$\int e^x\cos 3x\,dx = e^x\cos 3x + e^x 3\sin 3x$

$-\,9\int e^x\cos 3x\,dx.$

$10\int e^x\cos 3x\,dx = e^x\cos 3x + 3e^x\sin 3x$

$\Rightarrow \int e^x\cos 3x\,dx = \frac{e^x}{10}(\cos 3x + 3\sin 3x)$

5 $A = \frac{1}{2}d^2 \Rightarrow \frac{dA}{dt} = d\cdot\frac{dd}{dt}$

$\Rightarrow \frac{dA}{dt} = \sqrt{5}\cdot 0.2 = \frac{\sqrt{5}}{5}$ cm²/s.

6 The curve $y = e^{2x-1}$ is given.

a $y = e^{2x-1} \Rightarrow y' = e^{2x-1}\cdot 2$

$m = y'(x_0) = e^{2x_0-1}\cdot 2 \Rightarrow T: y - y_0 = m(x - x_0)$

$y = 2e^{2x_0-1}(x - x_0) + y_0 \Rightarrow$

$y = 2e^{2x_0-1}x \underbrace{-2e^{2x_0-1}x_0 + e^{2x_0-1}}_{0}$

$e^{2x_0-1}(-2x_0 + 1) = 0 \Rightarrow x_0 = \frac{1}{2}$

$T: y = 2x$

b $\int_0^{\frac{1}{2}} \left(e^{2x-1} - 2x\right)dx = \left[\frac{1}{2}e^{2x-1} - x^2\right]_0^{\frac{1}{2}}$

$= \frac{1}{2} - \frac{1}{4} - \frac{1}{2e} = \frac{e-2}{4e}$

c $\int_0^{\frac{1}{2}} \left(\left(e^{2x-1}\right)^2 - (2x)^2\right)dx = \pi \int_0^{\frac{1}{2}} \left(e^{4x-2} - 4x^2\right)dx$

$= \pi\left[\frac{1}{4}e^{4x-2} - \frac{4}{3}x^3\right]_0^{\frac{1}{2}}$

$= \pi\left(\frac{1}{4} - \frac{1}{6} - \frac{1}{4e^2}\right) = \frac{\left(e^2 - 3\right)\pi}{12e^2}$

7 Let $x = 3\cos\theta \Rightarrow dx = -3\sin\theta d\theta, \quad \theta = \arccos\left(\frac{x}{3}\right)$

$\sqrt{9 - x^2} = \sqrt{9 - 9\cos^2\theta} = 3\sin\theta$

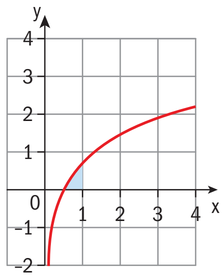
$\int \sqrt{9 - x^2}\, dx = -9\int \sin^2\theta d\theta - 9\int \frac{1 - \cos 2\theta}{2} d\theta$

$= -9\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]$

$= \frac{9}{2}\sin\left(\arccos\left(\frac{x}{3}\right)\right)\cos\left(\arccos\left(\frac{x}{3}\right)\right) - \frac{9}{2}\arccos\left(\frac{x}{3}\right) + c$

$= \frac{x}{2}\sqrt{9 - x^2} - \frac{9}{2}\arccos\left(\frac{x}{3}\right) + c$

8 **a**



b $y = \ln(2x) \Rightarrow x = \frac{1}{2}e^y, \quad x = 1 \Rightarrow y = \ln 2$

$\pi\int_0^{\ln 2} \left(1 - \left(\frac{1}{2}e^y\right)^2\right)dy = \pi\int_0^{\ln 2} \left(1 - \frac{e^{2y}}{4}\right)dy$

$= \pi\left[y - \frac{1}{8}e^{2y}\right]_0^{\ln 2}$

$= \pi\left(\ln 2 - \frac{4-1}{8}\right) = \frac{(8\ln 2 - 3)\pi}{8}$

9 **a** $s = \int_0^k v\, dt = \int_0^k 5e^{-\frac{2t}{3}}dt = 5\left[-\frac{3}{2}e^{-\frac{2t}{3}}\right]_0^k$

$= \frac{15}{2}\left(1 - e^{-\frac{2k}{3}}\right)$

b $\lim_{k\to\infty} s = \lim_{k\to\infty}\left(\frac{15}{2} - \frac{15}{2}e^{-\frac{2k}{3}}\right) = \frac{15}{2} - 0 = 7.5\text{ m}$

10 $x^2y^3 = \cos(\pi x) \Rightarrow 2xy^3 + x^2 3y^2 y' = -\sin(\pi x)\cdot\pi$

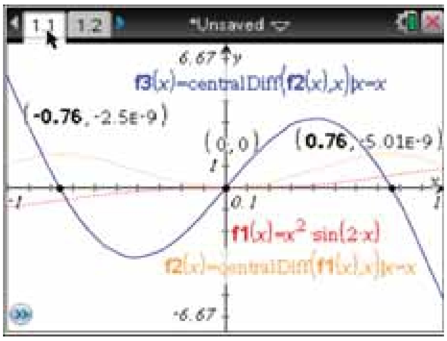
$2\cdot 1(-1)^3 + 1^2\cdot 3(-1)^2\cdot m_T = -\sin(\pi)\cdot\pi$

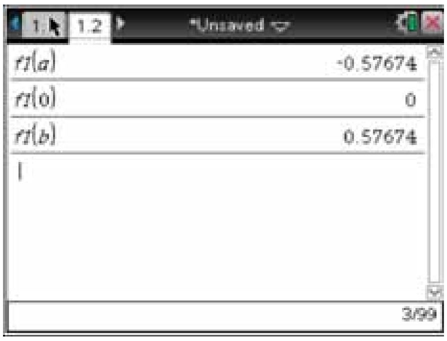
$\Rightarrow m_T = \frac{2}{3} \Rightarrow m_N = -\frac{3}{2}$

$N: y + 1 = -\frac{3}{2}(x - 1) \Rightarrow y = -\frac{3}{2}x + \frac{1}{2}$

Review exercise

1 We need to find the zeros of the second derivative of the function $y = x^2\sin 2x, -1 \le x \le 1$. We store the variables a and b and then to find the y-coordinates of the points of inflexion we input those values of x in the original function.





So the points of inflexion are $(-0.760, -0.577)$, $(0, 0)$ and $(0.760, 0.577)$.

2 $y^3 = \cos x \Rightarrow 3y^2y' = -\sin x \Rightarrow y' = -\frac{\sin x}{3y^2}$

When $x = 1 \Rightarrow y^3 = \cos 1 \Rightarrow y = \sqrt[3]{\cos 1} = 0.814$

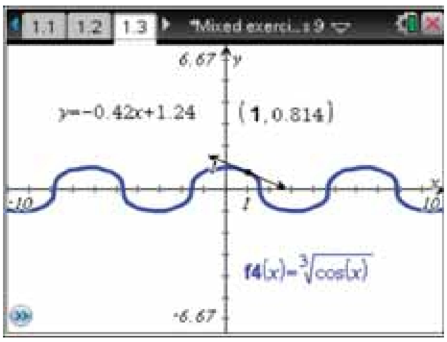
$m = -\frac{\sin 1}{3\sqrt[3]{\cos^2 1}} = -0.423$

$T: y - 0.814 = -0.423(x - 1) \Rightarrow$

$y = -0.423x + 1.24$

Checking:

First we find the explicit form of the curve and graph it on a GDC. $y^3 = \cos x \Rightarrow y = \sqrt[3]{\cos x}$



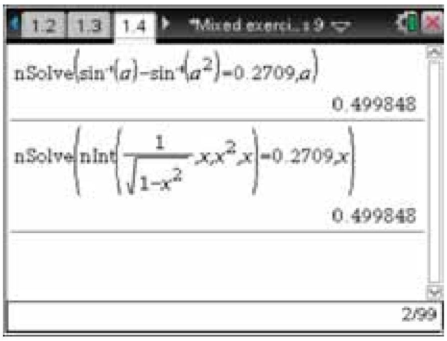
3 Find the value of a, $0 < a < 1$, such that

$$\int_{a^2}^{a} \frac{1}{\sqrt{1-x^2}}\,dx = 0.2709$$

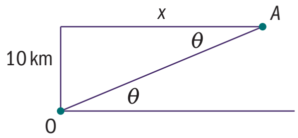
$$\int_{a^2}^{a} \frac{1}{\sqrt{1-x^2}}\,dx = [\arcsin x]_{a^2}^{a} = \arcsin a - \arcsin a^2 = 0.2709$$

We use a GDC to solve this equation, $a = 0.500$.

The result can be obtained directly on a calculator by using numerical integration.



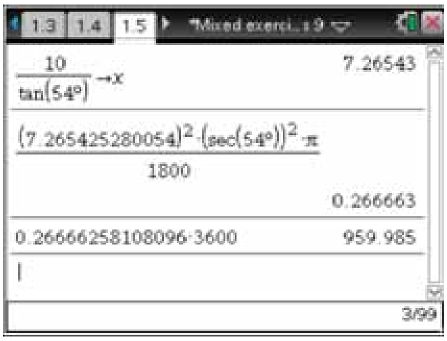
4



$$\tan\theta = \frac{10}{x} \Rightarrow x = \frac{10}{\tan\theta} = \frac{10}{\tan 54°} = 7.27$$

$$\tan\theta = \frac{10}{x} \Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{10}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{x^2\sec^2\theta}{10}\cdot\frac{d\theta}{dt} = -\frac{7.27^2\cdot\sec^2 54°}{10}\cdot\frac{\pi}{180} = -0.267\ \text{km}^{-1}$$



So the speed of the plane is 960 km/h.

5 First we graph the functions and identify the region. Then we find the point of intersection between the curves and store the x-coordinate to a variable called d.

$$V = \pi\int_0^{0.601} \left(\cos^2 x - (e^x - 1)^2\right)dx = 1.31$$