4.4 Exponential models

Exponential functions and their graphs

→ In an **exponential function**, the independent variable is the **exponent** (or **power**).

Here are some examples of **exponential functions:**

$$f(x) = 2^x, \quad f(x) = 5(3)^x + 2, \quad g(x) = 5^{-x} - 3, \quad h(x) = \left(\frac{1}{3}\right)^x + 1$$

Investigation – exponential graphs

1 The number of water lilies in a pond doubles every week. In week one there were 4 water lilies in the pond.
Draw a table and write down the number of water lilies in the pond each week up to week 12.
Plot the points from the table on a graph of number of lilies against time.
Draw a smooth curve through all the points.

Time is the dependent variable, so it goes on the horizontal axis.

The graph is an example of an increasing exponential graph.

2 A radioactive substance has a half-life of two hours. This means that every two hours its radioactivity halves.
A Geiger counter reading of the radioactive substance is taken at time $t = 0$. The reading is 6000 counts per second.
Two hours later ($t = 2$) the reading is 3000 counts per second.
What will the readings be at $t = 4$, $t = 6$, $t = 8$ and $t = 10$?
Plot the points on a graph of counts per second against time and join them with a smooth curve.

Could the number of water lilies in a pond keep doubling forever? Will the radioactivity of the substance ever reach zero?

This graph is an example of a decreasing exponential graph.

Does the shape of a ski slope form an exponential function? Investigate ski slopes on the internet to find out what the function is.

Graphs of exponential functions $f(x) = a^x$ where $a \in \mathbb{Q}^+$, $a \neq 1$

$\mathbb{Q}^+$ is the set of positive rational numbers.

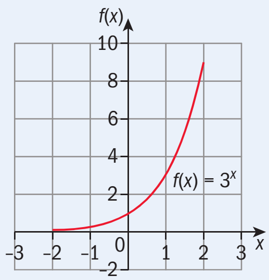
Why can $a \neq 1$? What kind of function would you get if $a = 1$?

Example 19

Draw a graph of the function $f(x) = 3^x$ for $-2 \leq x \leq 2$

Answers

Method 1: By hand

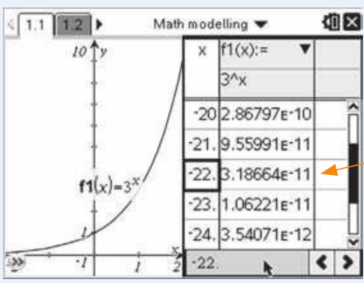


Draw a table of values.

x	−2	−1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Plot the points.
Draw a smooth curve through all the points.
This is an increasing exponential function.

Method 2: Using a GDC



For help with graphing exponential functions on your GDC see Chapter 12, Section 4.3.

3.1866E−11
= 0.000 000 000 031 866

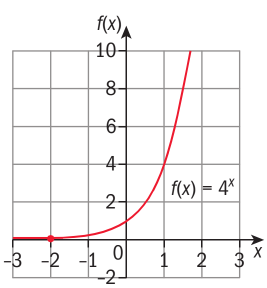
GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can check what happens when the values of *x* get very small or very large using the table of values on your GDC.

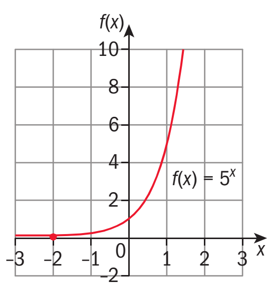
Look at the graph in Example 19. As the values of x get smaller, the curve gets closer and closer to the x-axis. The x-axis ($y = 0$) is a horizontal **asymptote** to the graph. At $x = 0$, $f(x) = 1$. As the values of x get very large, $f(x)$ gets larger even more quickly. We say that $f(x)$ tends to infinity. The function is an **increasing** exponential function.

An asymptote is a line that the curve approaches but never touches.

Here are some more graphs of **increasing** exponential functions.



▲ $f(x) = 4^x$



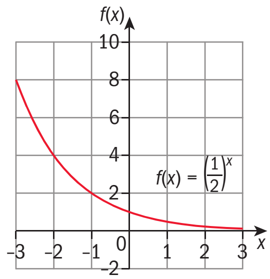
▲ $f(x) = 5^x$

All these graphs pass through the point (0, 1) and have $y = 0$ (the x-axis) as a horizontal asymptote.

Graphs of exponential functions $f(x) = a^x$ where $0 < a < 1$

What happens if a is a positive proper fraction?

Here is the graph of $y = \left(\frac{1}{2}\right)^x$.



> A **proper fraction** is a fraction where the numerator is smaller than the denominator.

- For an increasing exponential function, the y-values increase as the x-values increase from left to right.
- For a decreasing exponential function, the y-values decrease as the x-values increase from left to right.

This graph also passes through the point (0, 1) and has $y = 0$ (the x-axis) as a horizontal asymptote. However, this is an example of a **decreasing** exponential function.

Exercise 4R

Draw the graphs of these functions using your GDC.
For each, write down the coordinates of the point where the curve intersects the y-axis and the equation of the horizontal asymptote.

1 $f(x) = 2^x$ **2** $f(x) = 6^x$ **3** $f(x) = 8^x$

4 $f(x) = \left(\frac{1}{3}\right)^x$ **5** $f(x) = \left(\frac{1}{5}\right)^x$

Investigation – graphs of $f(x) = ka^x$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

Use your GDC to draw the graphs of

1 $f(x) = 2(3)^x$ **2** $f(x) = 3\left(\frac{1}{2}\right)^x$ **3** $f(x) = -3(2)^x$

For each graph, write down

a the value of k in the equation $f(x) = ka^x$
b the point where the graph crosses the y-axis
c the equation of the horizontal asymptote.

What do you notice?

Investigation – graphs of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

Use your GDC to draw the graphs of

1 $f(x) = 2^x + 3$ **2** $f(x) = 3\left(\frac{1}{2}\right)^x - 4$ **3** $f(x) = -2(3)^x + 5$

for $-3 \le x \le 3$.

For each graph, write down

a the values of k and c in the equation $f(x) = ka^x + c$

b the point where the graph crosses the y-axis

c the equation of the horizontal asymptote.

Work out $k + c$ for each graph. What do you notice?

→ In general, for the graph of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

- the line $y = c$ is a **horizontal asymptote**
- the curve passes through the point $(0, k + c)$.

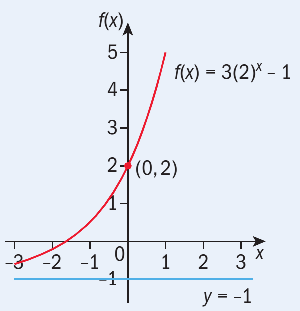
Sketching an exponential graph

- Draw and label the axes.
- Label the point where the graph crosses the y-axis.
- Draw in the asymptotes.

Example 20

Sketch the graph of the function $f(x) = 3(2)^x - 1$

Answer



Comparing $f(x) = 3(2)^x - 1$ *to*
$f(x) = ka^x + c$:
$k = 3$
$a = 2$
$c = -1$
$y = c$ *is a horizontal asymptote* $\Rightarrow$
$y = -1$
The curve passes through the point $(0, k + c) \Rightarrow (0, 3 - 1)$ *or* $(0, 2)$.

Exercise 4S

For each function, write down

a the coordinates of the point where the curve cuts the y-axis

b the equation of the horizontal asymptote.

Hence, sketch the graph of the function.

1 $f(x) = 2^x$ **2** $f(x) = 6^x$

3 $f(x) = \left(\frac{1}{3}\right)^x$ **4** $f(x) = \left(\frac{1}{5}\right)^x$

5 $f(x) = 3(2)^x + 4$

6 $f(x) = -2(4)^x - 1$

7 $f(x) = -1(2)^x + 3$

8 $f(x) = 4(3)^x - 2$

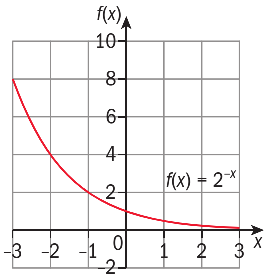
9 $f(x) = 0.5(2)^x + 3$

10 $f(x) = 2(0.5)^x + 1$

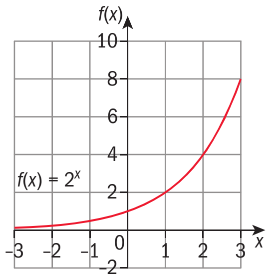
11 $f(x) = 0.4^x + 1$

12 $f(x) = 2(0.1)^x - 1$

Graphs of $f(x) = a^{-x} + c$ where $a \in \mathbb{Q}^+$ and $a \neq 1$

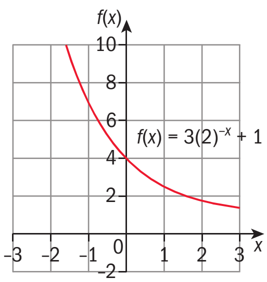


▲ Graph of $f(x) = 2^{-x}$.

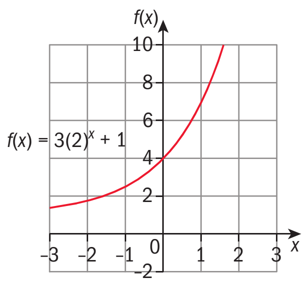


▲ Graph of $f(x) = 2^x$.

The graph of $f(x) = 2^{-x}$ is a reflection in the y-axis of the graph of $f(x) = 2^x$.



▲ Graph of $f(x) = 3(2)^{-x} + 1$.



▲ Graph of $f(x) = 3(2)^x + 1$.

$k = 3$ and $c = 1$.
Notice that $3 + 1 = 4$.

The curves pass through the point (0, 4) and the horizontal asymptote is $y = 1$.

→ In general, for the graph of $f(x) = ka^{-x} + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

- the line $y = c$ is a horizontal asymptote
- the curve passes through the point $(0, k + c)$
- the graph is a reflection in the y-axis of $g(x) = ka^x + c$.

Exercise 4T

For each function, write down

a the coordinates of the point where the curve cuts the y-axis

b the equation of the horizontal asymptote.

Hence, sketch the graph of the function.

1 $f(x) = 4(2)^{-x} + 2$

2 $f(x) = -4^{-x} + 1$

3 $f(x) = -2(2)^{-x} + 3$

4 $f(x) = 3(2)^{-x} - 2$

5 $f(x) = 0.5(3)^{-x} + 2$

6 $f(x) = 0.5^{-x} + 1$

7 $f(x) = 2(0.1)^{-x} - 1$

8 $f(x) = 0.4^{-x} + 2$

9 $f(x) = 3(0.2)^{-x} + 4$

10 $f(x) = 5(3)^{-x} - 2$

Applications of exponential functions

Many real-life situations involving growth and decay can be modeled by exponential functions.

Example 21

The length, l cm, of a pumpkin plant increases according to the equation

$$l = 4(1.2)^t$$

where t is the time in days.

a Copy and complete the table. Give your answers correct to 3 sf.

t	0	2	4	6	8	10	12	14	16
l									

b Draw a graph of l against t for $0 \le t \le 20$ and $0 \le l \le 100$.

c How long is the pumpkin plant when $t = 0$?

d How long will the pumpkin plant be after 3 weeks?

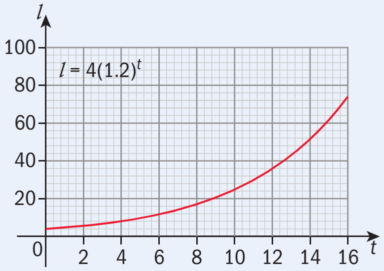
Answers

a

t	0	2	4	6	8	10	12	14	16
l	4	5.8	8.3	11.9	17.2	24.8	35.7	51.4	74

Substitute each value of t into the equation to find the corresponding value of l.

b



Draw and label the axes.
Put t on the horizontal axis.
Put l on the vertical axis.
Plot the points from the table and join with a smooth curve.

c When $t = 0$, $l = 4$ cm.

Read the value of l that corresponds to t = 0 from the table.

d 3 weeks = 21 days

So, $l = 4(1.2)^{21} = 184$ cm (to 3 sf).

For the equation, time is given in days, so convert from weeks.
Substitute t = 21 into the equation.

Example 22

Hubert invests 3000 euros in a bank at a rate of 5% per annum compounded yearly.
Let y be the amount he has in the bank after x years.

a Draw a graph to represent how much Hubert has in the bank after x years. Use a scale of 0–10 years on the x-axis and 2500–5000 euros on the y-axis.

b How much does he have after 4 years?

c How many years is it before Hubert has 4000 euros in the bank?

Answers

a The compound interest formula is:

$$y = 3000\left(1 + \frac{5}{100}\right)^x$$

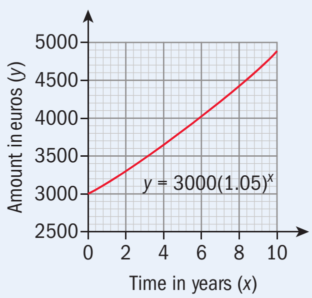
$y = 3000(1.05)^x$

where x = number of years.

This problem can be represented by a compound interest function.

Draw a table of values.

Time (x years)	Amount (y euros)
0	3000
2	3307.50
4	3646.52
6	4020.29
8	4432.37
10	4886.68



Draw and label the axes.

Plot the points and join them with a smooth curve.

b After 4 years Hubert has $3000(1.05)^4 = 3646.52$ euros.

Substitute $x = 4$ into the formula.

c Hubert has 4000 euros in the bank after 6 years.

You need to find the value of x for $y = 4000$ euros.
*From the table of values in part **a** you can see that after 6 years the amount is 4020.29.*
Check the amount after 5 years:
$y = 3000(1.05)^5 = 3828.84$
This is less than 4000 euros.

The **compound interest** formula is an exponential (growth) function.

You will learn more about compound interest in Chapter 7.

Exercise 4U

EXAM-STYLE QUESTIONS

1 Sketch the graphs of $f(x) = 2^x + 0.5$ and $g(x) = 2^{-x} + 0.5$ for $-3 \le x \le 3$.

a Write down the coordinates of the point of intersection of the two curves.

b Write down the equation of the horizontal asymptote to both graphs.

2 The value of a car decreases every year according to the function

$V(t) = 26\,000x^t$

where V is the value of the car in euros, t is the number of years after it was first bought and x is a constant.

a Write down the value of the car when it was first bought.

b After one year the value of the car is 22 100 euros. Find the value of x.

c Calculate the number of years that it will take for the car's value to fall to less than 6000 euros.

3 The equation $M(t) = 150(0.9)^t$ gives the amount, in grams, of a radioactive material kept in a laboratory for t years.

a Sketch the graph of the function $M(t)$ for $0 \le t \le 100$.

b Write down the equation of the horizontal asymptote to the graph of $M(t)$.

c Find the mass of the radioactive material after 20 years.

d Calculate the number of years that it will take for the radioactive material to have a mass of 75 grams.

4 The area, A m^2, covered by a certain weed is measured at 06:00 each day.

On the 1st June the area was 50 m^2.

Each day the area of the weeds grew by the formula

$A(t) = 50(1.06)^t$

where t is the number of days after 1st June.

a Sketch the graph of $A(t)$ for $-4 \le t \le 20$.

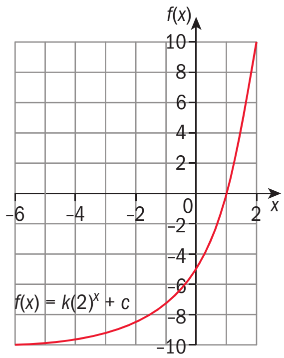
b Explain what the negative values of t represent.

c Calculate the area covered by the weeds at 06:00 on 15th June.

d Find the value of t when the area is 80 m^2.

EXAM-STYLE QUESTIONS

5 The graph shows the function $f(x) = k(2)^x + c$.
Find the values of c and k.



6 The temperature, T, of a cup of coffee is given by the function

$$T(t) = 18 + 60(2)^{-t}$$

where T is measured in °C and t is in minutes.

- **a** Sketch the graph of $T(t)$ for $0 \le t \le 10$.
- **b** Write down the temperature of the coffee when it is first served.
- **c** Find the temperature of the coffee 5 minutes after serving.
- **d** Calculate the number of minutes that it takes the coffee to reach a temperature of 40 °C.
- **e** Write down the temperature of the room where the coffee is served. Give a reason for your answer.

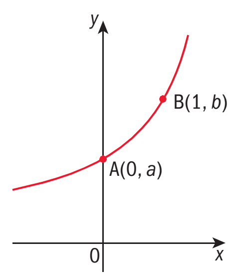
7 The value, in USD, of a piece of farm machinery depreciates according to the formula

$$D(t) = 18\,000(0.9)^t \quad \text{where } t \text{ is the time in years.}$$

- **a** Write down the initial cost of the machine.
- **b** Find the value of the machine after 5 years.
- **c** Calculate the number of years that it takes for the value of the machine to fall below 9000 USD.

8 The graph of the function $f(x) = \frac{2^x}{a}$ passes through the points $(0, b)$ and $(2, 0.8)$. Calculate the values of a and b.

9 The diagram shows the graph of $y = 2^x + 3$. The curve passes through the points A(0, a) and B(1, b).



- **a** Find the value of a and the value of b.
- **b** Write down the equation of the asymptote to the curve.

10 A function is represented by the equation $f(x) = 2(3)^x + 1$.
Here is a table of values of $f(x)$ for $-2 \le x \le 2$.

x	−2	−1	0	1	2
f(x)	1.222	*a*	3	7	*b*

- **a** Calculate the value a and the value of b.
- **b** Draw the graph of $f(x)$ for $-2 \le x \le 2$.
- **c** The domain of $f(x)$ is the real numbers. What is the range?