1. (6 points) Consider the function $f(x) = x^3 - 3x^2 - 9x + 10$, $x \in \mathbb{R}$.
   (a) (4 points) Find the equation of the straight line passing through the maximum and minimum points of the graph of f.
   (b) (2 points) Show that the inflection point of the graph of f also lies on this line.

2. (6 points) Find the equations of the tangents to the curve $x^2y - xy^2 = 12$ at the points where $y = 3$.

3. (6 points) Find the values of p and q which satisfy the following simultaneous equations:

$$\int_1^2 (px^2 + q)dx = 9$$

$$\int_1^2 (qx + p)dx = 6$$

4. (6 points) Find the area enclosed by the curve $2y = 3 + \ln x$ $(x > 0)$, and the lines $y = 1$, $y = 2$ and $x = 0$.

5. (7 points) Let $f(x) = |x^2 - 2x - 8|$.

   (a) (2 points) Find the equation of the tangent to the graph of f when $x = 2$.

   (b) (5 points) The tangent line found in part (a) meets the graph of f at two more points. Find the exact values of x-coordinates of these points.

6. (7 points) P is the point on the curve $y = x^3 - x$ with coordinates $(1, 0)$. The tangent to the curve at P intersects the curve again at Q. Find the area of the region bounded by the curve and the line segment PQ.

7. (7 points) Let $C = \int_0^{\pi} e^{-3x} \cos x dx$ and $S = \int_0^{\pi} e^{-3x} \sin x dx$. Using integration by parts show that $C = 3S$ and, using a similar method, express S in terms of C. Hence find the values of C and S.

8. (8 points)

(a) Show that $\dfrac{1}{x-1}+\dfrac{1}{x+1}=\dfrac{2}{x^2-1}$.

(b) Write $\dfrac{1}{x-2}+\dfrac{1}{x+2}$ and $\dfrac{1}{x-3}+\dfrac{1}{x+3}$ as a single fraction.

(c) Write $\dfrac{1}{x-n}+\dfrac{1}{x+n}$ as a single fraction.

(d) Find $\displaystyle\int_{11}^{12}\frac{20}{x^2-100}dx$ giving your answer in the form $\ln\left(\frac{p}{q}\right)$.

9. (7 points) The point P, with coordinates (p, q), lies on the graph of $\sqrt{x} + \sqrt{y} = \sqrt{a}$, where a is a positive constant. The tangent to the curve at P cuts the axes at $(m, 0)$ and $(0, n)$. Show that $m + n = a$.

10. (18 points) Let $\tan(\frac{x}{2}) = t$.

(a) i. Find an expression for $\cos^2(\frac{x}{2})$ in terms of t.

ii. Show that $\cos x = \dfrac{1-t^2}{1+t^2}$.

iii. Find an expression for $\sin x$ in terms of t.

iv. Find an expression for $\tan x$ in terms of t.

(b) For this part assume that $0 < x < \pi$

i. Show that $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$

ii. Use the substitution $\tan(\frac{x}{2}) = t$ to find $\displaystyle\int \frac{1}{1+\cos x}dx$.

iii. Find $\displaystyle\int \frac{1}{1+\sin x}dx$.

iv. Show that $\displaystyle\int \frac{1}{1+\sin x+\cos x}dx = \ln(1+\tan(\frac{x}{2})) + c$.

11. (20 points) Let S_k and C_k be defined by $S_k(x) = \sin kx$ and $C_k(x) = \cos kx$.

(a) Find $\int_{-\pi}^{\pi} (S_k(x))^2 dx$ and $\int_{-\pi}^{\pi} (C_k(x))^2 dx$.

(b) i. Find $\int_{-\pi}^{\pi} S_1(x)C_1(x)dx$ and $\int_{-\pi}^{\pi} S_2(x)C_2(x)dx$.

ii. Write down the value of $\int_{-\pi}^{\pi} S_k(x)C_k(x)dx$.

iii. Show that for any positive integers n, m, with $n \neq m$, $f(x) = S_n(x)C_m(x)$ is an odd function and hence find $\int_{-\pi}^{\pi} S_n(x)C_m(x)dx$.

(c) i. Show that $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$.

ii. For positive integers n, m, with $n \neq m$, find $\int_{-\pi}^{\pi} C_n(x)C_m(x)dx$.

(d) For positive integers n, m, with $n \neq m$, find $\int_{-\pi}^{\pi} S_n(x)S_m(x)dx$.

(e) Calculate:

i. $\int_{2\pi}^{6\pi} \cos(3x)\cos(5x)dx$.

ii. $\int_{4\pi}^{12\pi} \sin(3x)\sin(5x)dx$.