Calculus

Name:

1. (6 points) Let $y = \dfrac{1}{2x+1}$. Use the principle of mathematical induction to prove that

$$\frac{d^n y}{dx^n} = \frac{(-2)^n n!}{(2x+1)^{n+1}}$$

for $n \in \mathbb{Z}^+$.

2. (6 points) Consider the function $f(x) = \arcsin x$.
   (a) State the domain and range of $f(x)$.
   (b) Sketch the graph of $y = f(x)$.
   (c) The graph of $y = f(x)$ has been rotated about y-axis by 2π, find the exact value of the volume of the solid formed.

3. (10 points) Consider the curve $x^2y - y^2 = 4$.
   (a) Find $\frac{dy}{dx}$.
   (b) Show that there are no points on the curve where the tangent line to the curve is horizontal.
   (c) Find the coordinates of all points where the tangent line to the curve is vertical.

4. (13 points) Consider the curve $f(x) = x \sin x$ for $x \in [-2\pi, 2\pi]$
   (a) Find the equation of the tangent line to the curve at $x = -\frac{3\pi}{2}$.
   (b) The tangent line meets the curve again at two more points $P(a, b)$ and $Q(c, d)$ with $a < c$. Find a, b, c and d.
   (c) Calculate $\int x \sin x dx$
   (d) Find the area enclosed by the graph of $y = f(x)$ and the tangent line found in part (a) between $x = -\frac{3\pi}{2}$ and $x = c$.

5. (4 points) The displacement of a moving body is given by the formula:

$$s(t) = t^3 - 2t^2 + 5t$$

where s is measured in meters and time t is measured in seconds with $0 \leq t \leq 20$,

(a) Show that the velocity of the body is always positive.

(b) Find the initial acceleration of the body.

6. (15 points) Three cities A, B and C are connected with roads in such a way that the road AC is 2km long, the road BC is 3km long and they intersect at C at the angle of 50°. A person walks from B to C with a constant speed of 4km/h.

   (a) Find the rate at which the distance from that person to A is changing when she is midway between B and C.

   (b) Let P be the point where the person is at a given time. Find the rate at which the angle APC is changing when she is 1km away from C.

   20 minutes after the first person began to walk, another person starts to walk from A to C with a constant speed of 3km/h.

   (c) Find the rate at which the distance between them is changing when the second person began to walk.

7. (16 points) A particle starts to move from a fixed origin along a straight line. It's velocity v (in meters per second) at time t (in seconds), where $0 \leq t \leq 10$, is given by the equation:

$$v = \frac{2^t \sin t}{t^2 + 1}$$

   (a) Find the exact times when the particle is at rest.

   (b) Sketch the graph of $v(t)$.

   (c) Find the maximum speed of the particle in the first 2π seconds.

   (d) Describe in words the motion of the particle during the 10 seconds.

   (e) Find the total distance travelled during the 10 seconds.

   (f) Find the furthest distance from the origin the particle is in during the 10 second journey.