Divisibility

Things you need to know

- Definition of prime and composite numbers.
- Decomposition of composite numbers into a product of prime numbers.
- Divisibility rules for 2,3,4,5,6, 8, 9 and 11.

Some notation before we begin

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Rational numbers $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$. In other words rational numbers are numbers that **can be** written as a fraction where both the numerator and the denominator are integers and the denominator is not 0.

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We will write $m \not| n$ to indicate that n is not divisible by m. So for instance we have $8 \not| 42$, because there is no integer k such that $42 = k \cdot 8$.

Definitions

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A prime number is a natural number, which has exactly two positive divisors (1 and itself).

Try and prove (or at least convince yourself) that these two definitions are indeed equivalent.

Prime numbers

There are 25 prime numbers less than 100:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,$$
$$43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

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There are infinitely many prime numbers.

Can you prove this? Think about it!

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The question is how large can the gap between two consecutive prime numbers be?

Composite numbers

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Note that 1 is neither prime nor composite.

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Every natural number greater than 1 is either prime or can be uniquely written as a product of prime numbers.

Example: $20 = 2 \times 2 \times 5$.

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For example we have $20 = 4 \times 5$, but also $20 = 2 \times 10$, so 20 can be written as a product of two natural numbers in more than one way.

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For example we have $20 = 4 \times 5$, but also $20 = 2 \times 10$, so 20 can be written as a product of two natural numbers in more than one way.

However if we wanted to express 20 as a product of prime numbers then it can be done and it can be done in only one way: $20 = 2 \times 2 \times 5$

Prime decomposition

In order to decompose a number into its prime factors we do the following. We divide the given number by the least possible prime number that divides it. We then repeat this for the quotient (result of the division) until our quotient is 1.

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Lets decompose 20 into prime. The least prime number that divides 20 is **2**. $20 \div 2 = 10$. Now we work with 10. The least prime number that divides 10 is again **2**. $10 \div 2 = 5$. Now 5, the least prime number that divides 5 is **5** itself. $5 \div 5 = 1$. We got to 1, we're done.

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Finally we have: $20 = 2 \times 2 \times 5$.

Prime decomposition

The whole thing can be written as:

20	2
10	2
5	5
1	

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The whole thing can be written as:

$$\begin{array}{c|c} 20 & 2 \\ 10 & 2 \\ 5 & 5 \\ 1 & \end{array}$$

So (for the fourth time): $20 = 2 \times 2 \times 5$.

Prime decomposition - examples

Write 378 as a product of prime numbers.

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378	2
189	3
63	3
21	3
7	7
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Prime decomposition - examples

Write 378 as a product of prime numbers.

378	2
189	3
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So: $378 = 2 \times 3 \times 3 \times 3 \times 7$.

Prime decomposition - examples

Write 14300 as a product of prime numbers.

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$$\begin{array}{c|c} 14300 & 2 \\ 7150 & 2 \\ 3575 & 5 \\ 715 & 5 \\ 143 & 11 \\ 13 & 13 \\ 1 & \end{array}$$

Prime decomposition - examples

Write 14300 as a product of prime numbers.

14300	2
7150	2
3575	5
715	5
143	11
13	13
1	

So: $14300 = 2 \times 2 \times 5 \times 5 \times 11 \times 13$.

Now we move on to divisibility rules. In other words - how do we know that some number is divisible by, say, 3?

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Given any integer n we have:

- $2|n$ iff the units digit of n is 0,2,4,6 or 8.

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- $6|n$ iff the n is divisible by 2 and 3.

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- $6|n$ iff the n is divisible by 2 and 3.
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- $9|n$ iff the sum of the digits of n is divisible by 9.

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- $9|n$ iff the sum of the digits of n is divisible by 9.
- $11|n$ iff the difference between the sums of every second digit and the sum of the remaining digits is divisible by 11.

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Note that iff stands for *if and only if*.

Divisibility rules - example

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- not divisible by 3, the sum of all its digits is $1+2+3+4+3+2+1=16$, which is not divisible by 3;

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- not divisible by 4, the last two digits make up a number 21, which is not divisible by 4.

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- not divisible by 5, last digit not 0 or 5;
- not divisible by 6, because it's not divisible by 2 (and 3);

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- not divisible by 4, the last two digits make up a number 21, which is not divisible by 4. Of course it cannot be divisible by 4 also because it's not divisible by 2;
- not divisible by 5, last digit not 0 or 5;
- not divisible by 6, because it's not divisible by 2 (and 3);
- not divisible by 8, 321 is not divisible by 8 and also the number is not divisible by 2 (and 4);

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- not divisible by 4, the last two digits make up a number 21, which is not divisible by 4. Of course it cannot be divisible by 4 also because it's not divisible by 2;
- not divisible by 5, last digit not 0 or 5;
- not divisible by 6, because it's not divisible by 2 (and 3);
- not divisible by 8, 321 is not divisible by 8 and also the number is not divisible by 2 (and 4);
- not divisible by 9, the sum of its digits is 16, which is not divisible by 9. And also the number is not divisible by 3;

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- not divisible by 4, the last two digits make up a number 21, which is not divisible by 4. Of course it cannot be divisible by 4 also because it's not divisible by 2;
- not divisible by 5, last digit not 0 or 5;
- not divisible by 6, because it's not divisible by 2 (and 3);
- not divisible by 8, 321 is not divisible by 8 and also the number is not divisible by 2 (and 4);
- not divisible by 9, the sum of its digits is 16, which is not divisible by 9. And also the number is not divisible by 3;
- divisible by 11, we have $(1 + 3 + 3 + 1) - (2 + 4 + 2) = 0$, which is divisible by 11.

Divisibility rules - example

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Divisibility rules - example

The number 659442 is:

- divisible by 2, the last digits 2;
- divisible by 3, the sum of all its digits is $6 + 5 + 9 + 4 + 4 + 2 = 30$, which is divisible by 3;

Divisibility rules - example

The number 659442 is:

- divisible by 2, the last digits 2;
- divisible by 3, the sum of all its digits is $6 + 5 + 9 + 4 + 4 + 2 = 30$, which is divisible by 3;
- not divisible by 4, the last two digits make up a number 42, which is not divisible by 4.

Divisibility rules - example

The number 659442 is:

- divisible by 2, the last digits 2;
- divisible by 3, the sum of all its digits is $6 + 5 + 9 + 4 + 4 + 2 = 30$, which is divisible by 3;
- not divisible by 4, the last two digits make up a number 42, which is not divisible by 4.
- not divisible by 5, last digit not 0 or 5;

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- not divisible by 4, the last two digits make up a number 42, which is not divisible by 4.
- not divisible by 5, last digit not 0 or 5;
- divisible by 6, because it's divisible by both 2 and 3;

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- not divisible by 4, the last two digits make up a number 42, which is not divisible by 4.
- not divisible by 5, last digit not 0 or 5;
- divisible by 6, because it's divisible by both 2 and 3;
- not divisible by 8, 442 is not divisible by 8 and also the number is not divisible by 4;

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- divisible by 6, because it's divisible by both 2 and 3;
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- not divisible by 9, the sum of its digits is 30, which is not divisible by 9.

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- not divisible by 4, the last two digits make up a number 42, which is not divisible by 4.
- not divisible by 5, last digit not 0 or 5;
- divisible by 6, because it's divisible by both 2 and 3;
- not divisible by 8, 442 is not divisible by 8 and also the number is not divisible by 4;
- not divisible by 9, the sum of its digits is 30, which is not divisible by 9.
- not divisible by 11, we have $(6 + 9 + 4) - (5 + 4 + 2) = 8$, which is not divisible by 11.

Short test

The class will begin with a short test. You may be asked to write a number as a product of primes or check whether a given number is divisible by 2,3,4 and so on.

If there are any questions or doubts, you can email me at T.J.Lechowski@gmail.com