# Divisibility

## Things you need to learn

- Write down a general form of a number given the information about its division with remainder.
- Use these form to solve simple problems and write simple proofs.

# Example 1 - even and odd numbers

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d) $x$ is a product of three consecutive odd numbers.
$x = (2k - 1) \times (2k + 1) \times (2k + 3) \qquad k \in \mathbb{Z}$

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e) $x$ is divisible by 4 and 6

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d) $x$ is divisible by 2 and 6
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e) $x$ is divisible by 4 and 6
$x = 12k \qquad k \in \mathbb{Z}$

# Reminder

Remember, if a number is divisible by $m$ and $n$, then it is divisible by $lcm(m, n)$, but not necessarily by $mn$.

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c) the remainder when $x$ is divided 7 is equal to 6.
$x = 7k + 6 \qquad k \in \mathbb{N}$

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Note: you could have also written for example:
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b) the remainder when they are divided by 13 is 5 .

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b) the remainder when they are divided by 13 is 5 .
$13k+5, 13k+18, 13k+31 \qquad k \in \mathbb{Z}$

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Note: you could have also written for example:
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b) the remainder when they are divided by 13 is 5 .
$13k + 5, 13k + 18, 13k + 31 \qquad k \in \mathbb{Z}$

Note: again, another possible way would be:
$13k - 8, 13k + 5, 13k + 18 \qquad k \in \mathbb{Z}$

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Find three consecutive odd numbers whose sum is 159

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$$(2k-1)+(2k+1)+(2k+3)=159$$

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$$2k-1=2\times 26-1=51$$

The numbers are 51, 53 and 55.

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Find four consecutive numbers, which when divide by 4 give remainder equal to 3, and their sum is 116.

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$$(4k-1)+(4k+3)+(4k+7)+(4k+11)=116$$

$$16k=96$$

$$k=6$$

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$$(4k-1)+(4k+3)+(4k+7)+(4k+11)=116$$

$$16k=96$$

$$k=6$$

$$4k-1=4\times 6-1=23$$

The numbers are 23, 27, 31 and 35.

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$$(2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$$

where $m = 2k^2 + 2k$, and so $m$ is a integer.

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where $m = 2k^2 + 2k$, and so $m$ is a integer. So $(2k+1)^2$ is in the form $2m+1$, with $m$ integer, so it is odd.

□

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$$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 = 3m + 1$$

where $m = 3k^2 + 4k + 1$, so it's an integer.

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where $m = 3k^2 + 4k + 1$, so it's an integer. So $n^2$ is of the form $3m + 1$, with $m$ integer, so it has a remainder of 1, when divided by 3.

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where $n = 2km + k + m$, so it's an integer.

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where $n = 2km + k + m$, so it's an integer. So we got the form $2n+1$, with $n$ being an integer, which means that we got an odd number.

$\square$

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$$(4k)^2 = 16k^2 = 4 \times 4k^2 = 4m$$
$$(4k+1)^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1 = 4n + 1$$
$$(4k+2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) = 4s$$
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in each case we have a number of the form $4m$ or $4m+1$, so we either have a number divisible by 4 or that gives remainder 1 when divided by 1.

The short test at the beggining of next class may include some simple questions/proofs similar to the ones on the presentation.