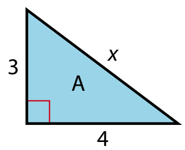
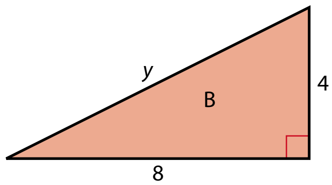


$$x^2 = 3^2 + 4^2$$
$$x^2 = 9 + 16$$
$$x^2 = 25$$
$$\sqrt{x^2} = \sqrt{25}$$
$$x = \sqrt{25}$$
$$x = 5$$



$$y^2 = 4^2 + 8^2$$
$$y^2 = 16 + 64$$
$$y^2 = 80$$
$$\sqrt{y^2} = \sqrt{80}$$
$$y = \sqrt{80}$$

• **Hint:** The solution for the hypotenuse of triangle A involves the equation $x^2 = 25$. Because x represents a length that must be positive, we want only the positive square root when taking the square root of both sides of the equation – i.e. $\sqrt{25}$. However, if there were no constraints on the value of x, we must remember that a positive number will have two square roots and we would write $\sqrt{x^2} = |x| = 5 \Rightarrow x = \pm 5$.

Radicals (surds)

Some roots are rational and some are irrational. Consider the two right triangles on the left. By applying Pythagoras' theorem, we find the length of the hypotenuse for triangle A to be exactly 5 (an integer and rational number) and the hypotenuse for triangle B to be exactly $\sqrt{80}$ (an irrational number). An irrational root – e.g. $\sqrt{80}, \sqrt{3}, \sqrt{10}, \sqrt[3]{4}$ – is called a **radical** or **surd**. The only way to express irrational roots exactly is in radical, or surd, form.

It is not immediately obvious that the following expressions are all equivalent.

$$\sqrt{80},\ 2\sqrt{20},\ \frac{16\sqrt{5}}{\sqrt{16}},\ 2\sqrt{2}\sqrt{10},\ \frac{10\sqrt{8}}{\sqrt{10}},\ 4\sqrt{5},\ 5\sqrt{\frac{16}{5}}$$

Square roots occur frequently in several of the topics in this course, so it will be useful for us to be able to simplify radicals and recognise equivalent radicals. Two useful rules for manipulating expressions with radicals are given below.

Simplifying radicals

For $a \geqslant 0$, $b \geqslant 0$ and $n \in \mathbb{Z}^+$, the following rules can be applied:

1 $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ **2** $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

Note: Each rule can be applied in either direction.

Example 7 – Simplifying radicals I

Simplify completely:

a) $\sqrt{5} \times \sqrt{5}$ b) $\sqrt{12} \times \sqrt{21}$ c) $\frac{\sqrt{48}}{\sqrt{3}}$

d) $\sqrt[3]{12} \times \sqrt[3]{18}$ e) $7\sqrt{2} - 3\sqrt{2}$ f) $\sqrt{5} + 2\sqrt{25} - 3\sqrt{5}$

g) $\sqrt{3}(2 - 2\sqrt{3})$ h) $(1 + \sqrt{2})(1 - \sqrt{2})$

Solution

a) $\sqrt{5} \times \sqrt{5} = \sqrt{5 \cdot 5} = \sqrt{25} = 5$

Note: A special case of the rule $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ when $n = 2$ is $\sqrt{a} \times \sqrt{a} = a$.

b) $\sqrt{12} \times \sqrt{21} = \sqrt{4} \times \sqrt{3} \times \sqrt{3} \times \sqrt{7} = \sqrt{4} \times (\sqrt{3} \times \sqrt{3}) \times \sqrt{7}$

$= 2 \times 3 \times \sqrt{7} = 6\sqrt{7}$

c) $\frac{\sqrt{48}}{\sqrt{3}} = \sqrt{\frac{48}{3}} = \sqrt{16} = 4$

d) $\sqrt[3]{12} \times \sqrt[3]{18} = \sqrt[3]{12 \cdot 18} = \sqrt[3]{216} = 6$

e) $7\sqrt{2} - 3\sqrt{2} = 4\sqrt{2}$

f) $\sqrt{5} + 2\sqrt{25} - 3\sqrt{5} = 10 - 2\sqrt{5}$

g) $\sqrt{3}(2 - 2\sqrt{3}) = 2\sqrt{3} - 2\sqrt{3}\sqrt{3} = 2\sqrt{3} - 2 \cdot 3 = 2\sqrt{3} - 6$ or $-6 + 2\sqrt{3}$

h) $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - \sqrt{2} + \sqrt{2} - \sqrt{2}\sqrt{2} = 1 - 2 = -1$

The radical $\sqrt{24}$ can be simplified because one of the factors of 24 is 4, and the square root of 4 is rational (i.e. 4 is a perfect square).

$$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$$

Rewriting 24 as the product of 3 and 8 (rather than 4 and 6) would not help simplify $\sqrt{24}$ because neither 3 nor 8 are perfect squares, i.e. there is no integer whose square is 3 or 8.

For any real number a, it would first appear that the rule $\sqrt{a^2} = a$ would be correct, but it is not. What if $a = -3$? Then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3. The correct rule that is true for any real number a is $\sqrt{a^2} = |a|$. Generalizing for any index where n is a positive integer, we need to consider whether n is even or odd. If n is even, then $\sqrt[n]{a^n} = |a|$; and if n is odd, then $\sqrt[n]{a^n} = a$.
For example,
$\sqrt[6]{(-3)^6} = \sqrt[6]{729} = \sqrt[6]{3^6} = 3$;
and
$\sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5$.

Example 8 – Simplifying radicals II

Express each in terms of the simplest possible radical.

a) $\sqrt{80}$ b) $\sqrt{\dfrac{14}{81}}$ c) $\sqrt[3]{24}$ d) $5\sqrt{128}$

e) $\sqrt{x^2}$ f) $\sqrt{20a^4b^2}$ g) $\sqrt[3]{81}$ h) $\sqrt{4+9}$

Solution

a) $\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$

Note: 4 is a factor of 80 and is a perfect square, but 16 is the *largest* factor that is a perfect square

b) $\sqrt{\dfrac{14}{81}} = \dfrac{\sqrt{14}}{\sqrt{81}} = \dfrac{\sqrt{14}}{9}$

c) $\sqrt[3]{24} = \sqrt[3]{8} \times \sqrt[3]{3} = 2\sqrt[3]{3}$

d) $5\sqrt{128} = 5\sqrt{64}\sqrt{2} = 5 \cdot 8\sqrt{2} = 40\sqrt{2}$

e) $\sqrt{x^2} = |x|$

f) $\sqrt{20a^4b^2} = \sqrt{4}\sqrt{5}\sqrt{a^4}\sqrt{b^2} = 2a^2|b|\sqrt{5}$

g) $\sqrt[3]{81} = \sqrt[3]{27}\sqrt[3]{3} = 3\sqrt[3]{3}$

h) $\sqrt{4+9} = \sqrt{13}$

● **Hint:** Note that in Example 8 h) the square root of a sum is *not* equal to the sum of the square roots. That is, avoid the error $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

In many cases we prefer not to have radicals in the denominator of a fraction. Recall from Example 7, part a), the special case of the rule $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ when $n = 2$ is $\sqrt{a} \times \sqrt{a} = a$, assuming $a > 0$. The process of eliminating irrational numbers from the denominator is called **rationalizing the denominator**.

Example 9 – Rationalizing the denominator I

Rationalize the denominator of each expression.

a) $\dfrac{2}{\sqrt{3}}$ b) $\dfrac{\sqrt{7}}{4\sqrt{10}}$

Solution

a) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

b) $\dfrac{\sqrt{7}}{4\sqrt{10}} = \dfrac{\sqrt{7}}{4\sqrt{10}} \cdot \dfrac{\sqrt{10}}{\sqrt{10}} = \dfrac{\sqrt{70}}{4 \cdot 10} = \dfrac{\sqrt{70}}{40}$

Changing a fraction from having a denominator that is irrational to an equivalent fraction where the denominator is rational (rationalizing the denominator) is not always a necessity. For example, expressing the cosine ratio of 45° as $\frac{1}{\sqrt{2}}$ rather than the equivalent value of $\frac{\sqrt{2}}{2}$ is mathematically correct. However, there will be instances where a fraction with a rational denominator will be preferred. It is a useful skill for simplifying some more complex fractions and for recognizing that two expressions are equivalent. For example, $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, or a little less obvious, $\frac{3}{2 + \sqrt{5}} = -6 + 3\sqrt{5}$. There are even situations where it might be useful to rationalize the numerator (see Example 11 below).

Recall the algebraic rule $(a + b)(a - b) = a^2 - b^2$. Any pair of expressions fitting the form of $a + b$ and $a - b$ are called a pair of **conjugates**. The result of multiplying a pair of conjugates is always a **difference of two squares**, $a^2 - b^2$, and this can be helpful in some algebraic manipulations – as we will see in the next example.

Example 10 – Rationalizing the denominator II

Express the quotient $\frac{2}{4 - \sqrt{3}}$ so that the denominator is a rational number.

Solution

Multiply numerator and denominator by the conjugate of the denominator, $4 + \sqrt{3}$, and simplify:

$$\frac{2}{4 - \sqrt{3}} \cdot \frac{4 + \sqrt{3}}{4 + \sqrt{3}} = \frac{8 + 2\sqrt{3}}{4^2 - (\sqrt{3})^2} = \frac{8 + 2\sqrt{3}}{16 - 3} = \frac{8 + 2\sqrt{3}}{13} \text{ or } \frac{8}{13} + \frac{2\sqrt{3}}{13}$$

Example 11 – Rationalizing the numerator

We will encounter the following situation in our study of calculus.

We are interested to analyze the behaviour of the quotient $\frac{\sqrt{x + h} - \sqrt{x}}{h}$ as the value of h approaches zero. It is not possible to directly substitute zero in for h in the present form of the quotient because that will give an undefined result of $\frac{0}{0}$. Perhaps we can perform the substitution if we rationalize the numerator. We will assume that x and $x + h$ are positive.

Solution

Multiplying numerator and denominator by the conjugate of the numerator and simplifying:

$$\frac{(\sqrt{x + h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x + h} + \sqrt{x})}{(\sqrt{x + h} + \sqrt{x})} = \frac{(\sqrt{x + h})^2 - (\sqrt{x})^2}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Substituting zero for h into this expression causes no problems. Therefore, as h approaches zero, the expression $\frac{\sqrt{x+h} - \sqrt{x}}{h}$ would appear to approach the expression $\frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Exercise 1.2

In questions 1–15, express each in terms of the simplest possible radical.

1 $\sqrt{h^2} \times \sqrt{h^2}$

2 $\frac{\sqrt{45}}{\sqrt{5}}$

3 $\sqrt{18} \times \sqrt{10}$

4 $\sqrt{\frac{28}{49}}$

5 $\sqrt[3]{4} \times \sqrt[3]{16}$

6 $\sqrt{\frac{15}{20}}$

7 $\sqrt{5}(3 + 4\sqrt{5})$

8 $(2 + \sqrt{6})(2 - \sqrt{6})$

9 $\sqrt{98}$

10 $4\sqrt{1000}$

11 $\sqrt[3]{48}$

12 $\sqrt{12x^3y^3}$

13 $\sqrt[5]{m^5}$

14 $\sqrt{\frac{27}{6}}$

15 $\sqrt{x^{16}(1 + x)^2}$

In questions 16–18, completely simplify the expression.

16 $13\sqrt{7} - 10\sqrt{7}$

17 $\sqrt{72} - 8\sqrt{3} + 3\sqrt{48}$

18 $\sqrt{500} + 5\sqrt{20} - \sqrt{45}$

In questions 19–30, rationalize the denominator, simplifying if possible.

19 $\frac{1}{\sqrt{5}}$

20 $\frac{2}{5\sqrt{2}}$

21 $\frac{6\sqrt{7}}{\sqrt{3}}$

22 $\frac{4}{\sqrt{32}}$

23 $\frac{2}{1 + \sqrt{5}}$

24 $\frac{1}{3 + 2\sqrt{5}}$

25 $\frac{\sqrt{3}}{2 - \sqrt{3}}$

26 $\frac{4}{\sqrt{2} + \sqrt{5}}$

27 $\frac{x - y}{\sqrt{x} + \sqrt{y}}$

28 $\frac{1 + \sqrt{3}}{2 + \sqrt{3}}$

29 $\sqrt{\frac{1}{x^2} - 1}$

30 $\frac{h}{\sqrt{x+h} - \sqrt{x}}$

In questions 31–33, rationalize the numerator, simplifying if possible.

31 $\frac{\sqrt{a} - 3}{a - 9}$

32 $\frac{\sqrt{x} - \sqrt{y}}{x - y}$

33 $\frac{\sqrt{m} - \sqrt{7}}{7 - x}$