4. (a) Show that the points A (4, −1, −8) and B (2, 1, −4) lie on the line l with equation $r = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

(b) Find the coordinates of the point C on the line l such that $AB = BC$. *[6 marks]*

5. (a) Find the vector equation of line l through points P (7, 1, 2) and Q (3, −1, 5).

(b) Point R lies on l and $PR = 3PQ$. Find the possible coordinates of R. *[6 marks]*

6. (a) Write down the vector equation of the line l through the point A (2, 1, 4) parallel to the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

(b) Calculate the magnitude of the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

(c) Find the possible coordinates of point P on l such that $AP = 35$. *[8 marks]*

14B Solving problems with lines

In this section we will use vector equations of lines to solve problems involving angles and intersections.

We will also see how vector equations of lines can be used to describe paths of moving objects in mechanics.

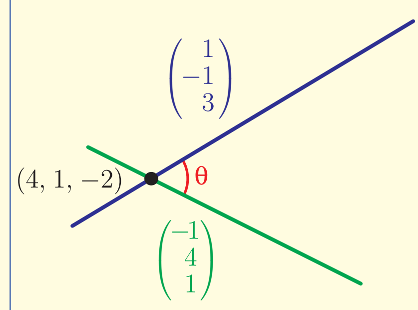
Worked example 14.5

Find the acute angle between lines with equations $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

We know the formula for the angle between two vectors (see Section 13D)

$$\cos\theta = \frac{\underline{a}\bullet\underline{b}}{|\underline{a}||\underline{b}|}$$

Draw a diagram to identify which two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ make the required angle



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The two vectors are in the directions of the two lines. So we take $\boldsymbol{a}$ and $\boldsymbol{b}$ to be the direction vectors of the two lines

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

We can now use the formula to calculate the angle

$$\therefore \quad \cos\theta = \frac{-1-4+3}{\sqrt{(1+1+9)}\sqrt{1+16+1}}$$

$$= -\frac{2}{\sqrt{11}\sqrt{18}}$$

$$\theta = 98.2°$$

The angle found is obtuse; the question asked for the acute angle

$$\text{acute angle} = 180° - 98.2° = 81.8°$$

The example above illustrates the general method for finding an angle between two lines.

KEY POINT 14.2

The angle between two lines is equal to the angle between their direction vectors.

Now that we know that the angle between two lines is the angle between their direction vectors, it is easy to identify parallel and perpendicular lines.

KEY POINT 14.3

Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are:

- parallel if $\boldsymbol{d}_1 = k\,\boldsymbol{d}_2$
- perpendicular if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.

Parallel and perpendicular vectors were covered in Sections 13B and 13E.

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Worked example 14.6

Decide whether the following pairs of lines are parallel, perpendicular, or neither:

(a) $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

(b) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

(c) $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(a) If $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ then

$$\begin{cases} 4 = k \times 1 \Rightarrow k = 4 \\ -1 = k \times (-2) \Rightarrow k = \frac{1}{2} \end{cases}$$

$4 \neq \frac{1}{2}$

∴ They are not parallel.

Is $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$?

$$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 4 + 2 - 6 = 0$$

∴ The lines are perpendicular.

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(b) If $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ then

$$\begin{cases} 2 = k \times 1 \Rightarrow k = 2 \\ 1 = k \times 0 \text{ impossible} \end{cases}$$

∴ They are not parallel.

Is $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$?

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 2 + 0 + 6 = 8 \neq 0$$

∴ The lines are neither parallel nor perpendicular.

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Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(c) If $\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} = k\begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$ then

$$\begin{cases} 4 = k\times(-10) \Rightarrow k = -\frac{2}{5} \\ -6 = k\times 15 \Rightarrow k = -\frac{2}{5} \\ 2 = k\times(-5) \Rightarrow k = -\frac{2}{5} \end{cases}$$

∴ The lines have parallel directions.

Check to see if they are the same line

Point $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ does not lie on the first line:

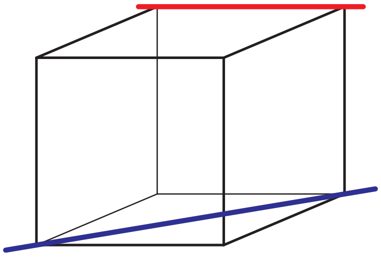
$$\begin{cases} 2 + 4t = -2 \Rightarrow t = -1 \\ -1 - 6t = 0 \Rightarrow t = -\frac{1}{6} \end{cases}$$

They are not the same line.

∴ The lines are parallel.

We will now see how to find the point of intersection of two lines. Suppose two lines have vector equations $\boldsymbol{r}_1 = \boldsymbol{a} + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r}_2 = \boldsymbol{b} + \mu\boldsymbol{d}_2$. If they intersect, then there must be a point which lies on both lines. Remembering that the position vector of a point on the line is given by the vector $\boldsymbol{r}$, this means that we need to find the values of λ and μ which make $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

In two dimensions, two straight lines either intersect or are parallel. However, in three dimensions it is possible to have two lines which are not parallel but do not intersect, as illustrated by the red and blue lines in the diagram. Such lines are called **skew lines**.



With skew lines we will see that we cannot find values of λ and μ such that $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

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Worked example 14.7

Find the coordinates of the point of intersection of the following pairs of lines.

(a) $\boldsymbol{r}=\begin{pmatrix}0\\-4\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\mu\begin{pmatrix}4\\-2\\-2\end{pmatrix}$ (b) $\boldsymbol{r}=\begin{pmatrix}-4\\1\\3\end{pmatrix}+t\begin{pmatrix}1\\1\\4\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\2\end{pmatrix}$

We need to make $\boldsymbol{r}_1 = \boldsymbol{r}_2$

(a) $\begin{pmatrix}0\\-4\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\mu\begin{pmatrix}4\\-2\\-2\end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix}0+\lambda\\-4+2\lambda\\1+\lambda\end{pmatrix}=\begin{pmatrix}1+4\mu\\3-2\mu\\5-2\mu\end{pmatrix}$$

If two vectors are equal, then all their components are equal

$$\Rightarrow \begin{cases}0+\lambda=1+4\mu\\-4+2\lambda=3-2\mu\\1+\lambda=5-2\mu\end{cases}$$

$$\Rightarrow \begin{cases}\lambda-4\mu=1 & (1)\\2\lambda+2\mu=7 & (2)\\\lambda+2\mu=4 & (3)\end{cases}$$

Solve two simultaneous equations in two variables. Use eqn (1) and (3) (as subtracting them eliminates λ)

$(3)-(1)\quad 6\mu=3$

$\therefore \mu=\frac{1}{2}, \lambda=3$

We need to check that the values of λ and μ also satisfy the second equation otherwise the lines do not actually meet

$(2): 2\times3+2\times\frac{1}{2}=7$

∴ the lines intersect

The position of the intersection point is given by the vector $\boldsymbol{r}_1$ (or $\boldsymbol{r}_2$ - they should be the same)

$$\underline{r}_1=\begin{pmatrix}0\\-4\\1\end{pmatrix}+3\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}3\\2\\4\end{pmatrix}$$

The lines intersect at the point (3, 2, 4)

Make $\boldsymbol{r}_1 = \boldsymbol{r}_2$

(b) $\begin{pmatrix}-4\\3\\3\end{pmatrix}+t\begin{pmatrix}1\\1\\4\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\2\end{pmatrix}$

$$\Rightarrow \begin{cases}t-2\lambda=6 & (1)\\t+3\lambda=-2 & (2)\\4t-2\lambda=-2 & (3)\end{cases}$$

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We can find t and λ from eqn (1) and (2)	(1) and (2) $\Rightarrow \lambda = -\frac{8}{5}, t = \frac{14}{5}$
We need to check that the values found also satisfy the third equation	(3) $4 \times \frac{14}{5} - 2 \times \left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$
This tells us that it is impossible to find t and λ to make $\boldsymbol{r}_1 = \boldsymbol{r}_2$	The two lines do not intersect.

EXAM HINT

You can use your calculator to solve simultaneous equations.
See Calculator sheet 6 on the CD-ROM.

Vector questions often ask you to find a point on a given line which satisfies certain conditions. We have already seen how we can use the position vector $\boldsymbol{r}$ for a general point on the line, and then use the condition to write an equation for λ.

See Worked example 14.4.

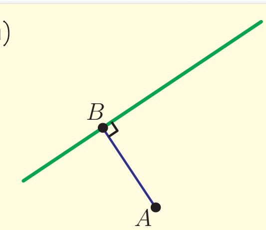
Worked example 14.8

Line l has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and point A has coordinates (3, 9, –2).

(a) Find the coordinates of point B on l so that AB is perpendicular to l.
(b) Hence find the shortest distance from A to l.
(c) Find the coordinates of the reflection of the point A in l.

Draw a diagram. The line AB should be perpendicular to the direction vector of l

(a)



$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$ (1)

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We know that B lies on l, so its position vector is given by the equation for $\boldsymbol{r}$

$$\overrightarrow{OB} = \boldsymbol{r} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix}$$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

Now find the value of λ for which the two lines are perpendicular

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow (\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\Rightarrow \lambda = -4$$

Use value of λ in the equation of the line to give the position vector of B

$$r = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

$\therefore$ B has coordinates $(-1, 3, -4)$

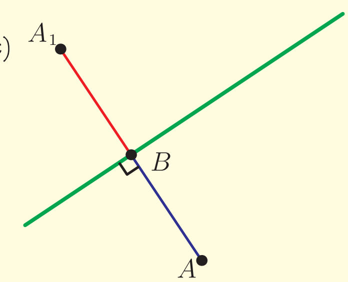
The shortest distance from a point to a line is the perpendicular distance AB. Again, use $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{16+36+4} = 2\sqrt{14}$$

The reflection A_1 lies on the line (AB). Since $BA_1 = AB$ and they are also in the same direction, $\overrightarrow{BA_1} = \overrightarrow{AB}$

(c)



$$\overrightarrow{BA_1} = \overrightarrow{AB}$$

$$\Rightarrow \underline{a}_1 - \underline{b} = \overrightarrow{AB}$$

$$\therefore \underline{a}_1 = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

So A_1 has coordinates $(-5, -3, -6)$

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Worked example 14.8(c) illustrates the power of vectors. As vectors contain both distance and direction information, just one equation ($\overrightarrow{BA_1} = \overrightarrow{AB}$) was needed to express both the fact that A_1 lies on the line (AB) and that $BA_1 = AB$.

We have already mentioned that vectors have many applications, particularly in physics. One such application is describing positions, displacements and velocities. These are all vector quantities, since they have both magnitude and direction.

You are probably familiar with the rule that, for an object moving with constant velocity, displacement = velocity × time. If we are working in two or three dimensions, the positions of points also need to be described by vectors. Suppose an object has constant velocity $\boldsymbol{v}$ and in time t moves from the point with position $\boldsymbol{a}$ to the point with position $\boldsymbol{r}$. Then its displacement is $\boldsymbol{r} - \boldsymbol{a}$, so we can write:

$$\boldsymbol{r} - \boldsymbol{a} = \boldsymbol{v}t$$

This equation can be rearranged to $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$, which looks very much like a vector equation of a line with direction vector $\boldsymbol{v}$. This makes sense, as the object will move in the direction given by its velocity vector. As t changes, $\boldsymbol{r}$ gives position vectors of different points along the object's path.

Note that the speed is the magnitude of the velocity, $|\boldsymbol{v}|$, and the distance travelled is the magnitude of the displacement, $|\boldsymbol{r} - \boldsymbol{a}|$.

KEY POINT 14.4

For an object moving with constant velocity $\boldsymbol{v}$ from an initial position $\boldsymbol{a}$, the position at time t is given by $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$.

The object moves along the straight line with equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$.

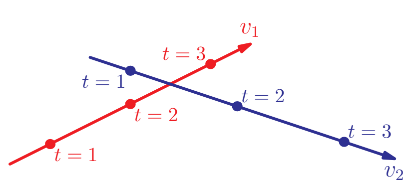
The speed of the object is equal to $|\boldsymbol{v}|$.

In 1896 the physicist Lord Kelvin wrote: "'vector' is a useless survival, or offshoot from quaternions, and has never been of the slightest use to any creature".

(Quaternions are a special type of number linked to complex numbers.)

They are now one of the most important tools in physics. Even great mathematicians cannot always predict what will be useful!

When we wanted to find the intersection of two lines, we had to use different parameters (for example, λ and μ) in the two equations. If we have two objects, we can write an equation for $\boldsymbol{r}(t)$ for each of them. In this case, we should use the same t in both equations, as both objects are moving at the same time. For the two objects to meet, they need to be at the same place at the same time. Notice that it is possible for the objects' paths to cross without the objects themselves meeting, if they pass through the intersection point at different times.



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Worked example 14.9

Two objects, A and B, have velocities $\boldsymbol{v}_A = 6\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{v}_B = -2\boldsymbol{i}+\boldsymbol{j}+7\boldsymbol{k}$. Object A starts from the origin and object B from the point with position vector $13\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$. Distance is measured in kilometres and time in hours.

(a) What is the speed of object B?

(b) Find the distance between the two objects after 5 hours.

(c) Show that the two objects do not meet.

Speed is the magnitude of velocity

(a)

$$|v_B| = \sqrt{2^2+1^2+7^2} = \sqrt{54}$$

So the speed of B is 7.35 km/h.

We need an equation for the position of each object in terms of t

(b)

Using $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{v}$:

$$\boldsymbol{r}_A(t) = t(6\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})$$

$$\boldsymbol{r}_B(t) = 13\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}+t(-2\boldsymbol{i}+\boldsymbol{j}+7\boldsymbol{k})$$

We can then find the position of each object when $t = 5$

When $t = 5$:

$$\boldsymbol{r}_A = 30\boldsymbol{i}+15\boldsymbol{j}+5\boldsymbol{k}$$

$$\boldsymbol{r}_B = 3\boldsymbol{i}+4\boldsymbol{j}+38\boldsymbol{k}$$

The distance is the magnitude of $\boldsymbol{r}_A - \boldsymbol{r}_B$

$$|\boldsymbol{r}_A - \boldsymbol{r}_B| = \sqrt{27^2+11^2+33^2}$$

$$= 44.0 \text{ km}$$

If the two objects meet then $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$

(c)

If $\boldsymbol{r}_A(t) = \boldsymbol{r}_B(t)$:

$$\begin{cases} 6t = 13-2t \Rightarrow t = \frac{13}{8} \\ 3t = -1+t \Rightarrow t = -\frac{1}{2} \\ t = 3+7t \Rightarrow t = -\frac{1}{2} \end{cases}$$

The three coordinates are not equal at the same time, so the objects do not meet.

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Exercise 14B

1. Find the acute angle between the following pairs of lines, giving your answer in degrees.

(a) (i) $\boldsymbol{r} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + s \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 6 \\ 6 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

2. For each pair of lines, state whether they are parallel, perpendicular, the same line, or none of the above.

(a) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \text{»} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

(b) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$

(c) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

(d) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 5 \\ -1 \\ 10 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

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3. Determine whether the following pairs of lines intersect and, if they do, find the coordinates of the intersection point.

(a) (i) $\boldsymbol{r} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -4 \\ -4 \\ -11 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

(ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

4. Line l has equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates (7, 2, 3).

Point C lies on l and PC is perpendicular to l. Find the coordinates of C. *[6 marks]*

5. Find the shortest distance from the point (−1, 1, 2) to the line with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. *[6 marks]*

6. Two lines are given by $l_1 : \boldsymbol{r} = \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and $l_2 : \boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

(a) l_1 and l_2 intersect at P, find the coordinates of P.

(b) Show that the point Q (5, 2, 5) lies on l_2.

(c) Find the coordinates of point M on l_1 such that QM is perpendicular to l_1.

(d) Find the area of the triangle PQM. *[10 marks]*

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7. In this question, unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ point due East and North, respectively.

A port is located at the origin. One ship starts from the port and moves with velocity $\boldsymbol{v}_1 = (3\boldsymbol{i} + 4\boldsymbol{j})$ kmh^{-1}.

(a) Write down the position vector at time t hours.

At the same time, a second ship starts 18 km north of the port and moves with velocity $\boldsymbol{v}_2 = (3\boldsymbol{i} - 5\boldsymbol{j})$ kmh^{-1}.

(b) Write down the position vector of the second ship at time t hours.

(c) Show that after half an hour, the distance between the two ships is 13.5 km.

(d) Show that the ships meet, and find the time when this happens.

(e) How long after the meeting are the ships 18 km apart?

[12 marks]

8. At time $t = 0$, two aircraft have position vectors $5\boldsymbol{j}$ and $7\boldsymbol{k}$. The first moves with velocity $3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and the second with velocity $5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

(a) Write down the position vector of the first aircraft at time t.

(b) Show that at time t the distance, d, between the two aircraft is given by $d^2 = 44t^2 - 88t + 74$.

(c) Show that the two aircraft will not collide.

(d) Find the minimum distance between the two aircraft.

[12 marks]

9. Find the distance of the line with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ from the origin.

[7 marks]

10. Two lines with equations $l_1 : \boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and

$l_2 : \boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ intersect at point P.

(a) Find the coordinates of P.

(b) Find, in degrees, the acute angle between the two lines.

Point Q has coordinates (−1, 5, 10).

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(c) Show that Q lies on l_2.

(d) Find the distance PQ.

(e) Hence find the shortest distance from Q to the line l_1. *[12 marks]*

11. Given line $l: \boldsymbol{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ and point P (21, 5, 10):

(a) Find the coordinates of point M on l such that PM is perpendicular to l.

(b) Show that the point Q (15, −14, 17) lies on l.

(c) Find the coordinates of point R on l_1 such that $|PR| = |PQ|$. *[10 marks]*

12. Two lines have equations $l_1: \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and

$l_2: \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and intersect at point P.

(a) Show that Q (5, 2, 6) lies on l_2.

(b) R is a point on l_1 such that $|PR| = |PQ|$. Find the possible coordinates of R. *[8 marks]*

14C Other forms of equation of a line

You know that in two dimensions, a straight line has equation of the form $y = mx + c$ or $ax + by = c$. How is this related to the vector equation of the line we introduced in this chapter?

Let us look at an example of a vector equation of a line in two dimensions. A line with direction vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ passing through the point (1, 4) has vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Vector $\boldsymbol{r}$ is the position vector of a point on the line; in other words, it gives

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