12 A conical tank with vertex pointed downward has a radius of 10 m at its top and is 24 m high. Water flows out of the tank at a rate of $20\text{m}^3/\text{min}$. How fast is the depth of the water in the tank decreasing when it reaches a depth of 16m?

Review exercise

EXAM-STYLE QUESTION

1 Find the limits, if they exist, of the following.

a $\lim_{x\to 1}\frac{x^2-3}{x+1}$ **b** $\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{x}$ **c** $\lim_{x\to 2}\frac{3^x-1}{x}$

d $\lim_{x\to 0}\frac{3x^2+x^2}{x^2}$ **e** $\lim_{x\to\infty}\frac{5x^2}{2x^3+1}$ **f** $\lim_{x\to-\infty}\frac{7}{x^3+1}$

2 Determine if $y=\begin{cases}x^2+2x, & x\le 2\\ x^3-6x, & x>2\end{cases}$ is continuous at $x = 2$.

3 Determine if the sequence $a_n=\frac{2n^2-3}{n^3-2}$ converges as n tends to $+\infty$.

4 Determine if the series $\sum_{n=0}^{\infty}3\left(\frac{(-1)^n}{5^n}\right)$ converges, and if it does, find its sum.

EXAM-STYLE QUESTIONS

5 Find the values of a for which the series $a^2+\frac{a^2}{1+a^2}+\frac{a^2}{(1+a^2)^2}+\ldots$ is convergent, and find its sum.

6 Given $y=\frac{x^3-2x^2+5}{x^2-x^3}$, find

a its horizontal asymptote

b the points where the curve intersects its horizontal asymptote, for small values of x.

7 Find the equation of the tangent and normal to the curve $y=\frac{2x+1}{x^2+1}$ at $x = 0$

EXAM-STYLE QUESTION

8 Let f be an even function with domain $(-a, a)$, $a > 0$. f is differentiable throughout its domain. Show that the tangent to the graph of f at $x=0$ is parallel to the x-axis.

9 Find any points on the curve $y = x\sqrt{x+1}$ those tangents are parallel to the line $x + y = -3$

10 The normal to the curve $y = \frac{1}{2}(2x^4 - 5x^3 - 5x^2 + 3x)$ at the point where $x = 1$ meets the curve again at point P. Find the coordinates of P.

EXAM-STYLE QUESTION

11 If f is a function such that $f(x) = [g(x)]^3$, $g(0) = -\frac{1}{2}$, $g'(0) = \frac{8}{3}$, find the equation of the tangent to $f(x)$ at $x = 0$.

12 Differentiate y with respect to x.

a $y = (1 - 3x)^7(3x + 5)^3$ **b** $y = \sqrt{(4x^2 - 3x + 1)^5}$

c $y = \frac{x^2 - 3}{\sqrt{x + 1}}, x \neq -1$ **d** $y = \sqrt{x + \sqrt{x^2 + 1}}$

e $(x + 2 + (x - 3)^8)^3$

13 Consider the polynomial function $f(x) = ax^3 + 6x^2 - bx$. Determine the values of a and b if f has a minimum at $x = -1$, and a point of inflexion at $x = 1$.

14 Consider the function $y = x - \sqrt[3]{x}$

a Find the intercepts of the function.

b Find any stationary points and distinguish between them.

c Find any points of inflexion.

d Determine the intervals where

i the function increases **ii** the function decreases.

EXAM-STYLE QUESTION

15 Consider the function $y = \frac{2x}{x^2 - 1}$

a Find the vertical and horizontal asymptotes.

b Show that the function is an odd function.

c Show that $\frac{dy}{dx} < 0$ for all x in the domain.

d Sketch the function.

16 Consider the function $f(x) = \frac{(x - 3)^2}{x^2 - 3}$

a Find any zeros, intercepts, and asymptotes of f.

b Find any stationary points, and justify your answers.

c Find any points of inflexion.

d Find the intervals where f is

i increasing, **ii** decreasing.

e Sketch the function showing all features found.

17 Given $x = y^5 - y$, find $\frac{dy}{dx}$, if it exists, at the points where $x = 0$

Review exercise

EXAM-STYLE QUESTIONS

1 Find the shortest distance between the point $(1.5, 0)$ and the curve $y = \sqrt{x}$

2 A piece of wire 80 cm in length is cut into three parts: two equal circles and a square. Find the radius of the circles if the sum of the three areas is to be minimized.

3 The radius of a right circular cylinder is increasing at a rate of 3 cm min^{-1} and the height is decreasing at a rate of 4 cm min^{-1}. Find the rate at which the volume is changing when the radius is 9 cm and the height is 12 cm, and determine if the volume is increasing or decreasing.

4 A poster has a total area of 180 cm^2 with a 1 cm margin at the bottom and sides, and a 2 cm margin at the top. Find the dimensions that will give the largest printing area.

5 A particle travels along the x-axis. Its velocity at any point x is $\frac{dx}{dt} = \frac{1}{1+2x}$. Find the particle's acceleration at $x = 2$ in terms of x.

CHAPTER 4 SUMMARY

Continuous function

- A function $y = f(x)$ is **continuous** at $x = c$, if $\lim_{x \to c} f(x) = f(c)$. The three necessary conditions for f to be continuous at $x = c$ are:

  1 f is defined at c, i.e., c is an element of the domain of f.

  2 the limit of f at c exists.

  3 the limit of f at c is equal to the value of the function at c.

  A function that is not continuous at a point $x = c$ is said to be **discontinuous** at $x = c$.

Properties of limits

- **Properties of limits as $x \to \pm\infty$**

  Let L_1, L_2, and k be real numbers and $\lim_{x \to \pm\infty} f(x) = L_1$ and $\lim_{x \to \pm\infty} g(x) = L_2$. Then,

  1 $\lim_{x \to \pm\infty}(f(x) \pm g(x)) = \lim_{x \to \pm\infty} f(x) \pm \lim_{x \to \pm\infty} g(x) = L_1 \pm L_2$

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Exercise 7M

1 Find the volume of the solid formed when the region between the graphs of the functions $y = x$ and $y = \frac{x}{2}$ is rotated through 2π radians about the x-axis between $x = 2$ and $x = 5$.

2 Find the volume of the solid formed when the region between the graphs of the functions $y = x - 4$ and $y = x^2 - 4x$ is revolved 2π radians about the x-axis.

3 Find the volume of the solid formed when the region between the graphs of $y = x$ and $y^2 = 2x$ is revolved 2π radians about the y-axis.

4 Find the volume of the solid formed when the region between the graphs of the functions $y = 2x - 1$, $y = x^{\frac{1}{2}}$, and $x = 0$ is revolved 2π radians about the y-axis.

Review exercise

EXAM-STYLE QUESTION

1 The gradient function of a curve is $\frac{dy}{dx} = ax + \frac{b}{x^2}$. The curve passes through the point $(-1, 2)$, and has a point whose gradient is 0 at $(-2, 0)$. Find the equation of the curve.

2 Calculate the area enclosed by the graphs of $y = x^2$ and $y^2 = x$

3 The region enclosed by $y = 1 + 3x - x^2$ and $y = \frac{2}{x}$ for $x > 0$ is rotated 2π radians about the x-axis. Find the volume of the solid formed.

4 Evaluate

a $\int_1^2 \left(x + \frac{1}{x^2} - \frac{1}{x^4}\right) dx$

b $\int_1^4 \frac{5x^2 - 4}{\sqrt{x}} dx$

c $\int_1^2 \frac{1}{x - 3} dx$

d $\int_1^e \frac{1}{1 - 4x} dx$

Review exercise

1 A particle moves in a straight line so that its velocity after t seconds is $v(t) = t^3 - 4t\ \text{m s}^{-1}$
Find the total distance traveled in the first 3 seconds.

EXAM-STYLE QUESTION

2 The velocity of a particle moving in a straight line is $v(t) = t^3 - 3t^2 + 2\ \text{m s}^{-1}$
Find the total distance traveled between the maximum and minimum velocities.

3 Find the total area of the region enclosed by the graph of $y = x^2 - 4 + \frac{3}{x^2}$ and the x-axis.

4 Integrate these where possible with respect to x.

a $\frac{3x^4+6}{x^2}$ **b** $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$

c $\frac{1}{2-3x}$ **d** $\frac{2}{\sqrt{1-4x}}$

e $2e^{-3x}+\sqrt[3]{e^x}$

5 Find the quotient when $2x^2 + 3x$ is divided by $2x - 1$.
Hence, evaluate $\int_1^2 \left(\frac{2x^2+3x}{2x-1}\right)dx$

6 Find the area enclosed by the graph of $y = \frac{1}{(x+1)}$, the y-axis, and the line $y = 5$.

7 Find the area enclosed by the graph of $y=\sqrt{x+1}$, and the x- and y-axes.

EXAM-STYLE QUESTION

8 The area enclosed by the curve $y = 3x(a - x)$ and the x-axis is 4 units^2. Find the value of a.

9 The region between the graphs of $y = 3^x$, $y = 3^{-x}$, and the lines $x = -1$ and $x = 1$ is rotated 2π radians about the x-axis. Find the volume of the solid formed.

CHAPTER 7 SUMMARY

Integration

- $\int f(x)\mathrm{d}x = F(x) + c,\ c \in \mathbb{R}$
- $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1},\ n \neq -1$
- $\int [f(x) \pm g(x)]\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$
- $\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ a \neq 0$
- $\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c$
- $\int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + c,\ a \neq 0$
- $\int m^{ax+b}\,\mathrm{d}x = \frac{1}{a\ln(m)}m^{ax+b} + c$, where m is a positive real number, $a \neq 0$.
- $\int \frac{1}{x}\mathrm{d}x = \ln|x| + c$
- $\int \frac{1}{(ax+b)}\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c,\ a \neq 0$

Definite integration

- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\int_b^a f(x)\,\mathrm{d}x = 0$
- $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$
- $\int_a^b (f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_b^c f(x)\mathrm{d}x$

The fundamental theorem of calculus

- If f is continuous in $[a, b]$ and if F is any anti-derivative of f on $[a, b]$

  then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

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Areas between graphs of functions and the axes

- If the integral of f exists in the interval $[a, b]$, and f is non-negative in this interval, then the area A under the curve $y = f(x)$ from a to b is $A = \int_a^b f(x)\,dx$.
- When f is negative for all $x \in [a, b]$, then the area bounded by the curve and the lines $x = a$ and $x = b$ is $\left|\int_a^b f(x)\,dx\right|$
- If functions f and g are continuous in the interval $[a, b]$, and $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area between the graphs of f and g is $A = \int_a^b (f(x) - g(x))dx$

Kinematics

- If v is a velocity function in terms of t, then the total distance traveled between times t_1 and t_2 is $\int_{t_1}^{t_2} |v|\,dt$

Volumes of revolution

- The volume of a solid formed when a function $y = f(x)$, continuous in the interval $[a, b]$, is rotated 2π radians about the x-axis is $V = \pi\int_a^b y^2 dx$.
- The volume of a solid of revolution formed when $x = f(y)$ in the interval $y = c$ to $y = d$ is rotated 2π radians about the y-axis is $V = \pi\int_c^d x^2 dy$
- If $f(x) \geq g(x)$ for all x in the interval $[a,b]$, then the volume formed when rotating the region between the two curves 2π radians about the x-axis in the interval $[a, b]$ is
  $V = \pi\int_a^b (f(x))^2 dx - \pi\int_a^b (g(x))^2 dx$, or $V = \pi\int_a^b ([f(x)]^2 - [g(x)]^2)\,dx$.
- If x_1 and x_2 are relations in y such that $x_1 \geq x_2$ for all y in the interval $[c, d]$, then the volume formed when rotating the region between the two curves 2π radians about the y-axis in the interval $[c, d]$ is $V = \pi\int_d^c x_1^2 dy - \pi\int_d^c x_2^2 dy$
  or $V = \pi\int_c^d (x_1^2 - x_2^2)dy$

Review exercise

1 Differentiate with respect to x:

a $f(x) = (2x + 3)\sin x$

b $g(x) = e^x \cos 3x$

c $h(x) = \dfrac{\tan x}{2x^2}$

2 Find the equation of a tangent to the curve $\sin y + e^{2x} = 1$ at the origin.

3 Find the value of m that satisfies this equation

$$\int_{\frac{\pi}{4}}^{m} \sec^2 x \, dx = 2\left(\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\right).$$

4 Use the method of integration by parts to solve:

a $\int (2x - 5)e^{2x}\, dx$;

b $\int (x^2 - 5x)\cos x\, dx$;

c $\int e^x \cos 3x\, dx$.

5 The diagonal of a square is increasing at a rate of 0.2 cm s^{-1}. Find the rate of change of the area of the square when the side has a length of 5 cm.

6 The curve $y = e^{2x-1}$ is given.

a Find the equation of the tangent to the curve that passes through the origin.

b Find the area, in terms of e, of the region bounded by the curve, the tangent and the y-axis.

c Find the volume of the revolution, in terms of π, obtained by rotating the region in part **b** about the x-axis.

7 Use the substitution $x = 3\cos\theta$ to find $\int \sqrt{9 - x^2}\, dx$.

8 The region bounded by the curve $y = \ln(2x)$, the vertical line $x = 1$ and the x-axis is rotated through 2π radians about the y-axis.

a Sketch the region in the coordinate system.

b Find the exact value of the volume of revolution obtained by this rotation.

9 The velocity, v, of an object, at a time t, is given by $v = 5e^{-\frac{2t}{3}}$, where t is in seconds and v is in m s^{-1}.

a Find the distance travelled in the first k seconds, $k > 0$.

b What is the total distance travelled by the object?

10 Find the equation of the normal to the curve $x^2y^3 = \cos(\pi x)$ at the point $(1, -1)$.

Review exercise

1 Find the points of inflection of the curve $y = x^2 \sin 2x$, $-1 \le x \le 1$.

2 Given the curve $y^3 = \cos x$, find the equation of the tangent at the point where $x = 1$.

3 Find the value of a, $0 < a < 1$, such that $\int_{a^2}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = 0.2709$

4 An airplane is flying at a constant speed at a constant altitude of 10 km in a straight line directly over an observer. At a given moment the observer notes that the angle of elevation θ to the plane is 54° and is increasing at 1° per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.

5 The region in the first quadrant bounded by the curves $y = \cos x$ and $y = e^x - 1$ is rotated by the x-axis by 2π radians. Find the volume of revolution of the solid generated.

CHAPTER 9 SUMMARY

Derivatives of trigonometric functions

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivatives of inverse trigonometric functions

If $y = \arcsin x$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

If $y = \arcsin \frac{x}{a}$ then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$

Basic integrals of trigonometric functions

$\int \cos x\,dx = \sin x + c,\ c \in \mathbb{R}$ since $\frac{d(\sin x)}{dx} = \cos x$

$\int \sin x\,dx = -\cos x + c$ since $\frac{d(-\cos x)}{dx} = \sin x$

$\int \sec^2 x\,dx = \tan x + c$ since $\frac{d(\tan x)}{dx} = \sec^2 x$

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$

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Definite integrals

$$\int f(x)\,\mathrm{d}x = F(x) + c \Rightarrow \int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Integration by parts

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Trigonometric substitutions

If an integral contains a quadratic radical expression use one of the following substitutions.

If the form is $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$.

If the form is $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec\theta$.

If the form is $\sqrt{x^2 + a^2}$ use the substitution $x = a\tan\theta$.