# 7.1 EXPONENTS

In §5.3.3 and §5.3.4 we looked at the exponential function, $f(x) = a^x, a > 0$ and the logarithmic function, $f(x) = \log_a x, a > 0$ and considered their general behaviour. In this chapter we will look in more detail at how to solve exponential and logarithmic equations as well as applications of both the exponential and logarithmic functions.

## 7.1.1 BASIC RULES OF INDICES

We start by looking at the notation involved when dealing with indices (or exponents).
The expression

$$\underbrace{a \times a \times a \times \ldots \times a}_{n \text{ times}}$$

can be written in index form, $a^n$ , where ***n* is the index** (or power or **exponent**) and $a$ is the **base**. This expression is read as "*a to the power of n*." or more briefly as "*a to the n*".

For example, we have that $3^5 = 3 \times 3 \times 3 \times 3 \times 3$ so that 3 is the base and 5 the exponent (or index).

The laws for positive integral indices are summarised below.

If $a$ and $b$ are real numbers and $m$ and $n$ are positive integers, we have that

| | Law | Rule | Example |
|---|---|---|---|
| **1.** | Multiplication [same base] | : $a^m \times a^n = a^{m+n}$ | $3^4 \times 3^6 = 3^{4+6} = 3^{10}$ |
| **2.** | Division [same base] | : $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$ | $7^9 \div 7^5 = 7^{9-5} = 7^4$ |
| **3.** | Power of a power [same base] | : $(a^m)^n = a^{m \times n}$ | $(2^3)^5 = 2^{3 \times 5} = 2^{15}$ |
| **4.** | Power of a power [same power] | : $a^m \times b^m = (ab)^m$ | $3^4 \times 7^4 = (3 \times 7)^4 = 21^4$ |
| **5.** | Division [same power] | : $a^m \div b^m = \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$ | $5^3 \div 7^3 = \left(\frac{5}{7}\right)^3$ |
| **6.** | Negative one to a power | : $(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ | $(-1)^3 = (-1)^5 = \ldots = -1$<br>$(-1)^2 = (-1)^4 = \ldots = 1$ |

There are more laws of indices that are based on rational indices, negative indices and the zero index. A summary of these laws is provided next.

| Law | Rule | Example |
|---|---|---|
| **1.** Fractional index Type 1 [nth root] | : $a^{1/n} = \sqrt[n]{a}, n \in \mathbb{N}$. Note: if $n$ is even, then $a \geq 0$. if $n$ is odd, then $a \in \mathbb{R}$. | $8^{1/3} = \sqrt[3]{8} = 2$ <br> $(-27)^{1/3} = \sqrt[3]{-27} = -3$ |
| **2.** Fractional index Type 2 | : $a^{m/n} = \sqrt[n]{a^m}$ Note: if $n$ is even, then $a^m \geq 0$ if $n$ is odd, then $a \in \mathbb{R}$. | $16^{3/4} = \sqrt[4]{16^3} = 8$ |
| **3.** Negative index | : $a^{-1} = \dfrac{1}{a}, a \neq 0$ <br> $a^{-n} = \dfrac{1}{a^n}, a \neq 0, n \in \mathbb{N}$ | $2^{-1} = \dfrac{1}{2} = 0.5$ <br> $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$ |
| **4.** Zero index | : $a^0 = 1, a \neq 0$ Note: $0^n = 0, n \neq 0$ | $12^0 = 1$ |

We make the following note about fractional indices:

As $\dfrac{m}{n} = m \times \dfrac{1}{n} = \dfrac{1}{n} \times m$, we have that for $b \geq 0$ i. $b^{\frac{m}{n}} = b^{m \times \frac{1}{n}} = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$

ii. $b^{\frac{m}{n}} = b^{\frac{1}{n} \times m} = \left(b^{\frac{1}{n}}\right)^m = (\sqrt[n]{b})^m$

Then, If $b \geq 0$, then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m, m \in \mathbb{Z}, n \in \mathbb{N}$

If $b < 0$, then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m, m \in \mathbb{Z}, n \in \{1, 3, 5, \ldots\}$

**EXAMPLE 7.1**

Simplify the following

(a) $\left(\dfrac{4x^2}{5y^4}\right)^2 \times (2x^3y)^3$ (b) $\dfrac{3^{n+1} + 3^2}{3}$

**Solution**

(a) $\left(\dfrac{4x^2}{5y^4}\right)^2 \times (2x^3y)^3 = \dfrac{4^2x^{2\times 2}}{5^2y^{4\times 2}} \times 2^3x^{3\times 3}y^{1\times 3}$

$= \dfrac{16x^4}{25y^8} \times 8x^9y^3$

$= \dfrac{128}{25}x^{4+9}y^{3-8}$

$= \dfrac{128}{25}x^{13}y^{-5}$

$= \dfrac{128x^{13}}{25y^5}$

(b) $\dfrac{3^{n+1} + 3^2}{3} = \dfrac{3(3^n + 3)}{3}$

$= 3^n + 3$

**EXAMPLE 7.2**

Simplify the following

(a) $\dfrac{4x^2(-y^{-1})^{-2}}{(-2x^2)^3(y^{-2})^2}$ (b) $\dfrac{x^{-1}+y^{-1}}{x^{-1}y^{-1}}$

**Solution**

(a) $$\frac{4x^2(-y^{-1})^{-2}}{(-2x^2)^3(y^{-2})^2} = \frac{4x^2 \times y^{-1\times -2}}{-8x^{2\times 3}\times y^{-2\times 2}} = -\frac{x^2y^2}{2x^6y^{-4}}$$
$$= -\frac{y^{2-(-4)}}{2x^{6-(2)}}$$
$$= -\frac{y^6}{2x^4}$$

(b) $$\frac{x^{-1}+y^{-1}}{x^{-1}y^{-1}} = \frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{xy}} = \left(\frac{1}{x}+\frac{1}{y}\right)\times\frac{xy}{1} = \frac{xy}{x}+\frac{xy}{y}$$
$$= y + x$$

**EXAMPLE 7.3**

Simplify the following

(a) $\dfrac{2^{n-3}\times 8^{n+1}}{2^{2n-1}\times 4^{2-n}}$ (b) $\dfrac{(a^{1/3}\times b^{1/2})^{-6}}{\sqrt[4]{a^8b^9}}$

**Solution**

(a) $$\frac{2^{n-3}\times 8^{n+1}}{2^{2n-1}\times 4^{2-n}} = \frac{2^{n-3}\times(2^3)^{n+1}}{2^{2n-1}\times(2^2)^{2-n}} = \frac{2^{n-3}\times 2^{3n+3}}{2^{2n-1}\times 2^{4-2n}}$$
$$= \frac{2^{n-3+(3n+3)}}{2^{2n-1+(4-2n)}}$$
$$= \frac{2^{4n}}{2^3}$$
$$= 2^{4n-3}$$

(b) $$\frac{(a^{1/3}\times b^{1/2})^{-6}}{\sqrt[4]{a^8b^9}} = \frac{a^{\frac{1}{3}\times -6}\times b^{\frac{1}{2}\times -6}}{(a^8b^9)^{\frac{1}{4}}} = \frac{a^{-2}\times b^{-3}}{a^2b^{\frac{9}{4}}}$$
$$= a^{-2-2}\times b^{-3-\frac{9}{4}}$$
$$= a^{-4}b^{-\frac{21}{4}}$$
$$= \frac{1}{a^4b^{\frac{21}{4}}}$$

## EXERCISES 7.1.1

**1.** Simplify the following

(a) $\left(\frac{3y^2}{4x^3}\right)^3 \times (2x^2y^3)^3$ (b) $\left(\frac{2}{3a^2}\right)^3 + \frac{1}{8a^6}$ (c) $\frac{2^{n+1}+2^2}{2}$

(d) $\left(\frac{2x^3}{3y^2}\right)^3 \times (xy^2)^2$ (e) $\left(\frac{2x^3}{4y^2}\right)^2 \times \frac{12y^6}{8x^4}$ (f) $\frac{3^{n+2}+9}{3}$

(g) $\frac{4^{n+2}-16}{4}$ (h) $\frac{4^{n+2}-16}{2}$ (i) $\left(\frac{1}{2b}\right)^4 - \frac{b^2}{16}$

**2.** Simplify the following

(a) $\frac{20^6}{10^6}$ (b) $\frac{12^{2x}}{(6^3)^x}$ (c) $\frac{16^{2y+1}}{8^{2y+1}}$

(d) $\frac{(ab)^{2x}}{a^{2x}b^{4x}}$ (e) $\frac{(xy)^6}{64x^6}$ (f) $\frac{27^{n+2}}{6^{n+2}}$

**3.** Simplify the following

(a) $\left(\frac{x}{y}\right)^3 \times \left(\frac{y}{z}\right)^2 \times \left(\frac{z}{x}\right)^4$ (b) $3^{2n} \times 27 \times 243^{n-1}$ (c) $\frac{25^{2n} \times 5^{1-n}}{(5^2)^n}$

(d) $\frac{9^n \times 3^{n+2}}{27^n}$ (e) $\frac{2^n \times 4^{2n+1}}{2^{1-n}}$ (f) $\frac{2^{2n+1} \times 4^{-n}}{(2^n)^3}$

(g) $\frac{x^{4n+1}}{(x^{n+1})^{(n-1)}}$ (h) $\frac{x^{4n^2+n}}{(x^{n+1})^{(n-1)}}$ (i) $\frac{(3^x)(3^{x+1})(3^2)}{(3^x)^2}$

**4.** Simplify $\frac{(x^m)^n(y^2)^m}{(x^m)^{(n+1)}y^2}$.

**5.** Simplify the following, leaving your answer in positive power form

(a) $\frac{(-3^4) \times 3^{-2}}{(-3)^{-2}}$ (b) $\frac{9y^2(-x^{-1})^{-2}}{(-2y^2)^3(x^{-2})^3}$ (c) $\frac{x^{-1}-y^{-1}}{x^{-1}y^{-1}}$

(d) $\frac{x^{-2}+2x^{-1}}{x^{-1}+x^{-2}}$ (e) $\frac{(-2)^3 \times 2^{-3}}{(x^{-1})^2 \times x^2}$ (f) $\frac{(-a)^3 \times a^{-3}}{(b^{-1})^{-2}b^{-3}}$

**6.** Simplify the following

(a) $\frac{(x^{-1})^2+(y^2)^{-1}}{x^2+y^2}$ (b) $\frac{(x^2)^{-2}+2y}{1+2yx^4}$ (c) $\frac{(x+h)^{-1}-x^{-1}}{h}$

(d) $(x^2-1)^{-1} \times (x+1)$ (e) $\frac{(x-1)^{-3}}{(x+1)^{-1}(x^2-1)^2}$ (f) $\frac{y(x^{-1})^2+x^{-1}}{x+y}$

**7.** Simplify the following

(a) $5^{n+1} - 5^{n-1} - 2 \times 5^{n-2}$ (b) $a^{x-y} \times a^{y-z} \times a^{z-x}$ (c) $\left(\frac{a^{-\frac{1}{2}}b^3}{ab^{-1}}\right)^2 \times \frac{1}{ab}$