Chapter 22

Inequalities

Contents:

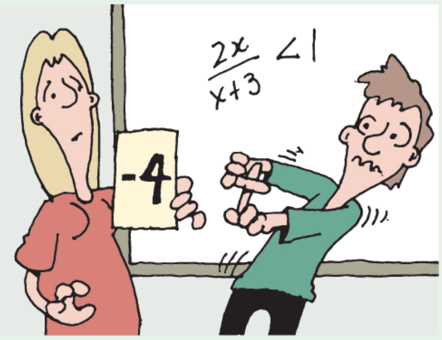
In this course so far, we have mostly dealt with **equations** in which two expressions are separated by the equality sign $=$.

In this chapter we consider **inequalities** in which two expressions are separated by one of the four inequality signs $<$, $\leqslant$, $>$, or $\geqslant$.

OPENING PROBLEM

To solve the inequality $\frac{2x}{x+3} < 1$, Trent performs these steps:

$$\frac{2x}{x+3} < 1$$
$$\therefore\ 2x < x+3 \quad \{\text{multiplying both sides by } (x+3)\}$$
$$\therefore\ x < 3 \quad \{\text{subtracting } x \text{ from both sides}\}$$



His friend Donna notices that $x = -4$ does not satisfy the inequality, since $\frac{2(-4)}{-4+3} = \frac{-8}{-1} = 8$, which is not < 1.

They concluded that there is something wrong with Trent's solution.

Things to think about:

a At what step was Trent's method wrong?

b Is there an algebraic method which gives the correct solution to this inequality?

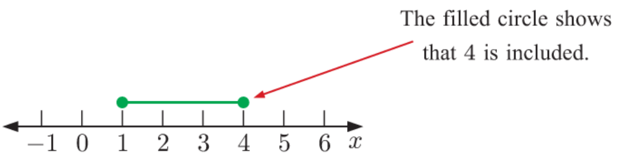
A INTERVAL NOTATION

In **Chapter 2**, we used **interval notation** to describe a set of numbers.

For example, the set of real numbers from 1 to 4 inclusive can be represented by

$$\{x \mid 1 \leqslant x \leqslant 4,\ \ x \in \mathbb{R}\}$$

the set of all — such that — x is real.



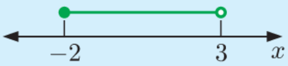
If it is not stated otherwise, we assume we are dealing with real x. So, the set can be represented simply as $\{x \mid 1 \leqslant x \leqslant 4\}$.

Example 1 — Self Tutor

Draw a number line graph to display:

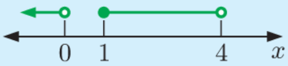
a $\{x \mid -2 \leqslant x < 3\}$ **b** $\{x \mid x < 2 \text{ or } x \geqslant 7\}$ **c** $\{x \mid x < 0 \text{ or } 1 \leqslant x < 4\}$

a



b

c



SQUARE BRACKET NOTATION

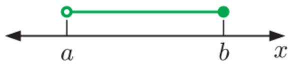
An alternative to using inequality signs is to use **square bracket notation**.

The endpoints of the interval are written within square brackets. The bracket is reversed if the endpoint is not included.

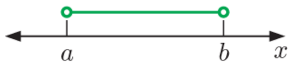
- $[a,\ b]$ represents the interval $\{x \mid a \leqslant x \leqslant b\}$

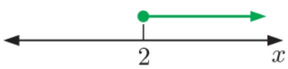
- $[a,\ b[$ represents the interval $\{x \mid a \leqslant x < b\}$

- $]a,\ b]$ represents the interval $\{x \mid a < x \leqslant b\}$



- $]a,\ b[$ represents the interval $\{x \mid a < x < b\}$





For intervals which extend to infinity, we use the symbol ∞. We always use an 'outwards' bracket for infinity. So, $[2,\ \infty[$ represents the interval $\{x \mid x \geqslant 2\}$.

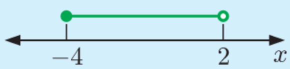
In square bracket notation, we use the union symbol $\cup$ to replace 'or'.

So, for $\{x \mid 1 \leqslant x < 3 \text{ or } x \geqslant 5\}$ we would write $[1,\ 3[\ \cup\ [5,\ \infty[$.

Example 2 — Self Tutor

Use square bracket notation to describe:

a



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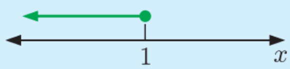

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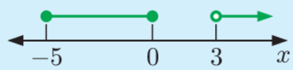


a $[-4,\ 2[$ **b** $]-\infty,\ 1]$ **c** $[-5,\ 0]\ \cup\ ]3,\ \infty[$

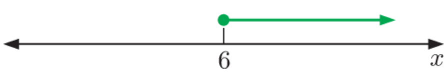
EXERCISE 22A

1 Draw a number line graph to display:

a $\{x \mid x > 4\}$ **b** $\{x \mid x \leqslant -5\}$ **c** $\{x \mid -2 \leqslant x \leqslant 3\}$
d $\{x \mid 0 < x \leqslant 7\}$ **e** $\{x \mid x < 1 \text{ or } x > 3\}$ **f** $\{x \mid x \leqslant 2 \text{ or } x > 6\}$
g $[-1,\ 6]$ **h** $]4,\ 9]$ **i** $[-5,\ 0[$
j $]2,\ 8[$ **k** $[5,\ \infty[$ **l** $]-\infty,\ 6]$
m $]-3,\ \infty[$ **n** $[0,\ 4]\ \cup\ [7,\ \infty[$ **o** $]-\infty,\ 2[\ \cup\ [5,\ 11[$

2 Use interval notation to describe:

a



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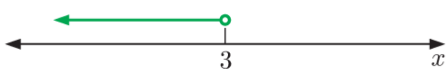

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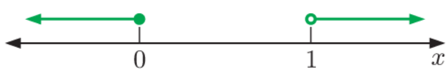
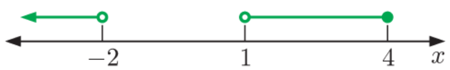
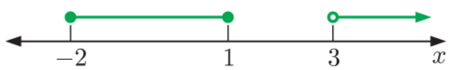


e 

f 

g 

h 

3 Write these number sets using square bracket notation:

a $\{x \mid -1 \leqslant x \leqslant 6\}$

b $\{x \mid 0 < x < 5\}$

c $\{x \mid -4 < x \leqslant 7\}$

d $\{x \mid 4 \leqslant x < 8\}$

e $\{x \mid x > -7\}$

f $\{x \mid x \leqslant 0\}$

g $\{x \mid x \leqslant 2 \text{ or } x \geqslant 5\}$

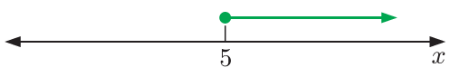
h $\{x \mid x < -3 \text{ or } x > 4\}$

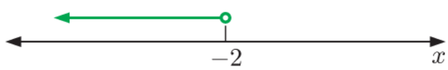
i $\{x \mid -1 < x \leqslant 1 \text{ or } x \geqslant 2\}$

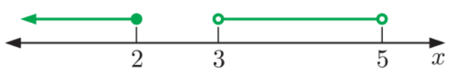
j $\{x \mid x < -4 \text{ or } 2 \leqslant x < 7\}$

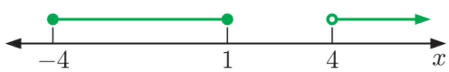
k 

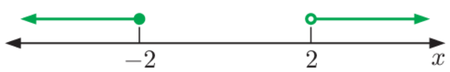
l 

m 

n 

o 

p 

q 

r 

4 Display each set on a number line graph, and hence describe using a single interval:

a $\{x \mid x > 3 \cup 0 < x < 7\}$

b $\{x \mid \frac{7}{2} \leqslant x < 9 \cup -1 \leqslant x < 4\}$

c $\{x \mid x < 3 \cap 0 < x < 7\}$

d $\{x \mid \frac{7}{2} \leqslant x < 9 \cap -1 \leqslant x < 4\}$

B LINEAR INEQUALITIES

Linear inequalities take the same form as linear equations, except they contain an inequality sign instead of an 'equals' sign.

$2x < 7$ and $3x + 5 \geqslant -10$ are examples of linear inequalities.

RULES FOR SOLVING LINEAR INEQUALITIES

Notice that $5 > 3$ and $3 < 5$,
and that $-3 < 2$ and $2 > -3$.

This suggests that if we **interchange** the LHS and RHS of an inequality, then we must **reverse** the inequality sign. $>$ is the reverse of $<$, and $\geqslant$ is the reverse of $\leqslant$.

You may also remember from previous years that:

- If we **add** or **subtract** the same number to both sides, the inequality sign is *maintained*.
  For example, if $5 > 3$, then $5 + 2 > 3 + 2$.
- If we **multiply** or **divide** both sides by a **positive** number, the inequality sign is *maintained*.
  For example, if $5 > 3$, then $5 \times 2 > 3 \times 2$.
- If we **multiply** or **divide** both sides by a **negative** number, the inequality sign is *reversed*.
  For example, if $5 > 3$, then $5 \times -1 < 3 \times -1$.

The method of solving linear inequalities is thus identical to that of linear equations, with the exceptions that:

- **interchanging** the sides **reverses** the inequality sign
- **multiplying** or **dividing** both sides by a **negative** number **reverses** the inequality sign.

We should not multiply or divide both sides of an inequality by an expression involving the variable, unless we are certain that the expression is always positive or always negative. This is the mistake which Trent made in the **Opening Problem**.

Example 3 Self Tutor

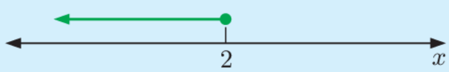
Solve for x and graph the solution:

a $3x - 4 \leqslant 2$ **b** $3 - 2x < 7$

a $3x - 4 \leqslant 2$

$\therefore\ 3x \leqslant 6$ {adding 4 to both sides}

$\therefore\ x \leqslant 2$ {dividing both sides by 3}

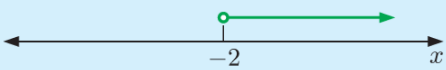


Check: If $x = 1$ then $3x - 4 = 3 \times 1 - 4 = -1$ and $-1 \leqslant 2$ ✓

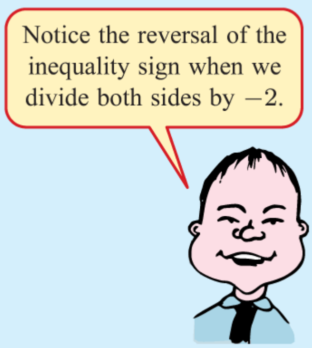
b $3 - 2x < 7$

$\therefore\ -2x < 4$ {subtracting 3 from both sides}

$\therefore\ x > -2$ {dividing both sides by -2, so reverse the sign}



Check: If $x = 3$ then $3 - 2x = 3 - 2 \times 3 = -3$ and $-3 < 7$ ✓



EXERCISE 22B

1 Solve for x and graph the solution:

a $3x + 2 < 0$ **b** $5x - 7 > 2$ **c** $2 - 3x \geqslant 1$

d $5 - 2x \leqslant 11$ **e** $2(3x - 1) < 4$ **f** $5(1 - 3x) \geqslant 8$

2 Solve for x and graph the solution:

a $7 \geqslant 2x - 1$ **b** $-13 < 3x + 2$ **c** $20 > -5x$

d $-3 \geqslant 4 - 3x$ **e** $3 < 5 - 2x$ **f** $2 \leqslant 5(1 - x)$

Example 4 **Self Tutor**

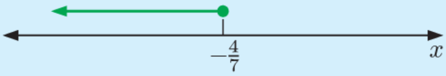
Solve for x and graph the solution: $3-5x \geqslant 2x+7$

$3-5x \geqslant 2x+7$

$\therefore\ 3-7x \geqslant 7$ {subtracting $2x$ from both sides}

$\therefore\ -7x \geqslant 4$ {subtracting 3 from both sides}

$\therefore\ x \leqslant -\frac{4}{7}$ {dividing both sides by -7, so reverse the sign}



3 Solve for x and graph the solution:

a $3x+2 > x-5$ **b** $2x-3 < 5x-7$ **c** $5-2x \geqslant x+4$

d $7-3x \leqslant 5-x$ **e** $3x-2 > 2(x-1)+5x$ **f** $1-(x-3) \geqslant 2(x+5)-1$

DISCUSSION

- Try to solve the quadratic inequality $x^2+5x < 14$.
- Can you solve quadratic inequalities in the same way you solve quadratic equations?

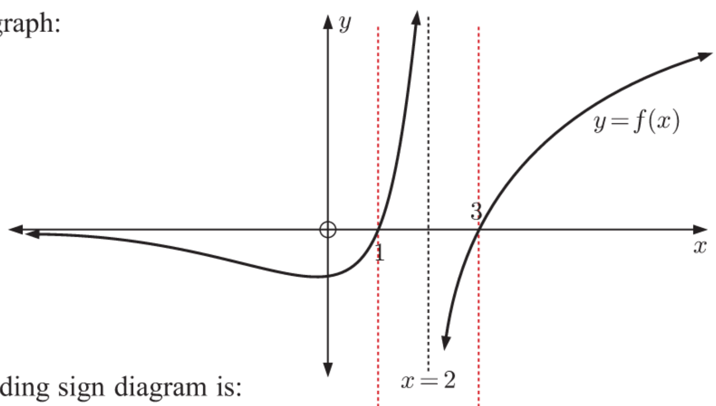
C SIGN DIAGRAMS

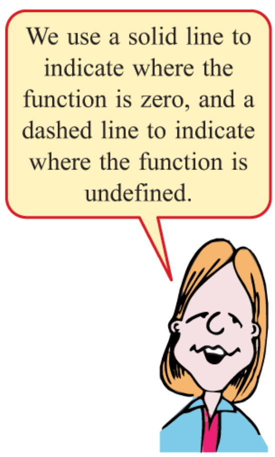
To solve non-linear inequalities, we do not usually need a complete graph of a function. We only need to know when the function is positive, negative, zero, or undefined. A **sign diagram** enables us to do this.

A sign diagram consists of:

- a **horizontal line** which represents the x-axis
- **positive** (+) and **negative** (−) signs indicating where the graph is **above** and **below** the x-axis respectively
- **critical values**, which are the graph's x-intercepts, or where it is undefined.

Consider the graph:





The corresponding sign diagram is: