18 Show that $5^n - 1$ is divisible by 4 for all integers n.

19 Show that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ for every positive integer n and where a and b are real numbers.

20 Prove each of the following statements.

a) $\sum_{i=1}^{n} (2i + 4) = n^2 + 5n$ for each positive integer n.

b) $\sum_{i=1}^{n} (2 \cdot 3^{i-1}) = 3^{n-1}$ for each positive integer n.

c) $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$ for each positive integer n.

Practice questions

1 In an arithmetic sequence, the first term is 4, the 4th term is 19 and the nth term is 99. Find the common difference and the number of terms n.

2 How much money should you invest now if you wish to have an amount of €3000 in your account after 6 years if interest is compounded quarterly at an annual rate of 6%?

3 Two students, Nick and Charlotte, decide to start preparing for their IB exams 15 weeks ahead of the exams. Nick starts by studying for 12 hours in the first week and plans to increase the amount by 2 hours per week. Charlotte starts with 12 hours in the first week and decides to increase her time by 10% every week.

a) How many hours did each student study in week 5?

b) How many hours in total does each student study for the 15 weeks?

c) In which week will Charlotte exceed 40 hours per week?

d) In which week does Charlotte catch up with Nick in the number of hours spent on studying per week?

4 Two diet schemes are available for relatively overweight people to lose weight. Plan A promises the patient an initial weight loss of 1000 g the first month, with a steady loss of an additional 80 g every month after the first. So, the second month the patient will lose 1080 g and so on for a maximum duration of 12 months.

Plan B starts with a weight loss of 1000 g the first month and an increase in weight loss by 6% more every following month.

a) Write down the amount of grams lost under Plan B in the second and third months.

b) Find the weight lost in the 12th month for each plan.

c) Find the total weight loss during a 12-month period under

(i) Plan A **(ii)** Plan B.

5 Planning on buying your first car in 10 years, you start a savings plan where you invest €500 at the beginning of the year for 10 years. Your investment scheme offers a fixed rate of 6% per year compounded annually.

Calculate, giving your answers to the nearest euro (€),

(a) how much the first €500 is worth at the end of 10 years

(b) the total value your investment will give you at the end of the 10 years.

6 The first three terms of an arithmetic sequence are 6, 9.5, 13.

a) What is the 40th term of the sequence?

b) What is the sum of the first 103 terms of the sequence?

7 $\{a^n\}$ is defined as follows

$a_n = \sqrt[3]{(8 - a^3{}_{n-1})}$

a) Given that $a_1 = 1$, evaluate a_2, a_3, a_4. Describe $\{a_n\}$.

b) Given that $a_1 = 2$, evaluate a_2, a_3, a_4. Describe $\{a_n\}$.

8 A marathon runner plans her training programme for a 20 km race. On the first day she plans to run 2 km, and then she wants to increase her distance by 500 m on each subsequent training day.

a) On which day of her training does she first run a distance of 20 km?

b) By the time she manages to run the 20 km distance, what is the total distance she would have run for the whole training programme?

9 In the nation of Telefonica, cellular phones were first introduced in the year 2000. During the first year, the number of people who bought a cellular phone was 1600. In 2001, the number of new participants was 2400, and in 2002 the new participants numbered 3600.

a) You notice that the trend is a geometric sequence; find the common ratio.

Assuming that the trend continues,

b) how many participants will join in 2012?

c) in what year would the number of new participants first exceed 50 000?

Between 2000 and 2002, the total number of participants reaches 7600.

d) What is the total number of participants between 2000 and 2012?

During this period, the total adult population of Telefonica remains at approximately 800 000.

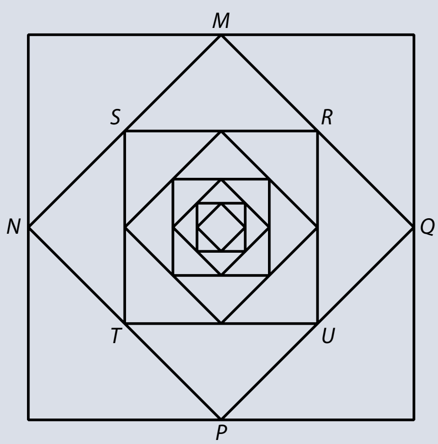
e) Use this information to suggest a reason why this trend in growth would not continue.

10 In an arithmetic sequence, the first term is 25, the fourth term is 13 and the nth term is $-11\,995$. Find the common difference d and the number of terms n.

11 The midpoints M, N, P, Q of the sides of a square of side 1 cm are joined to form a new square.

a) Show that the side of the second square $MNPQ$ is $\frac{\sqrt{2}}{2}$.

b) Find the area of square $MNPQ$.



A new third square $RSTU$ is constructed in the same manner.

c) **(i)** Find the area of the third square just constructed.

(ii) Show that the areas of the squares are in a geometric sequence and find its common ratio.

The procedure continues indefinitely.

d) **(i)** Find the area of the tenth square.

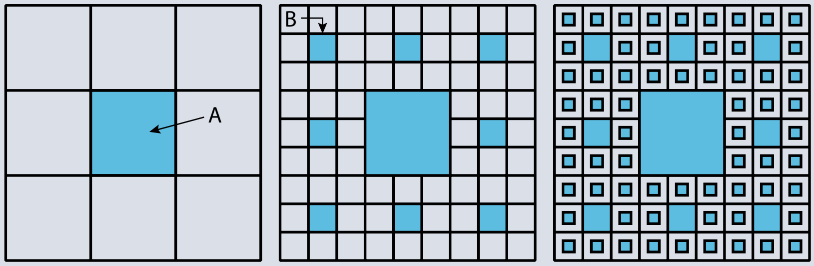
(ii) Find the sum of the areas of all the squares.

12 Tim is a dedicated swimmer. He goes swimming once every week. He starts the first week of the year by swimming 200 metres. Each week after that he swims 20 m more than the previous week. He does that all year long (52 weeks).

a) How far does he swim in the final week?

b) How far does he swim altogether?

13 The diagram below shows three iterations of constructing squares in the following manner: A square of side 3 units is given, then it is divided into nine smaller squares as shown and the middle square is shaded. Each of the unshaded squares is in turn divided into nine squares and the process is repeated. The area of the first shaded square is 1 unit.



a) Find the area of each of the squares A and B.

b) Find the area of any small square in the third diagram.

c) Find the area of the shaded regions in the second and third iterations.

d) If the process is continued indefinitely, find the area left unshaded.

14 The table below shows four series of numbers. One series is an arithmetic one, one is a converging geometric series, one is a diverging geometric series and the fourth is neither geometric nor arithmetic.

Series		Type of series
(i)	$2 + 22 + 222 + 2222 + \ldots$	
(ii)	$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$	
(iii)	$0.8 + 0.78 + 0.76 + 0.74 + \ldots$	
(iv)	$2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \ldots$	

a) Complete the table by stating the type of each series.

b) Find the sum of the infinite geometric series above.

15 Two IT companies offer 'apparently' similar salary schemes for their new appointees. Kell offers a starting salary of €18 000 per year and then an annual increase of €400 every year after the first. YBO offers a starting salary of €17 000 per year and an annual increase of 7% for the rest of the years after the first.

a) **(i)** Write down the salary paid during the second and third years for each company.

(ii) Calculate the total amount that an employee working for 10 years will accumulate in each company.

(iii) Calculate the salary paid during the tenth year for each company.

b) Tim works at Kell and Merijayne works at YBO.

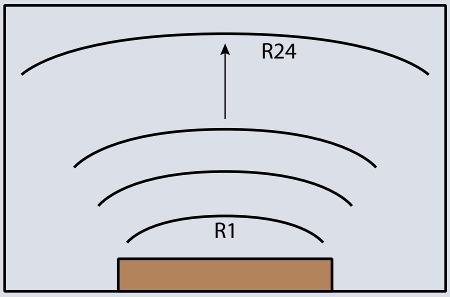
(i) When would Merijayne start earning more than Tim?

(ii) What is the minimum number of years that Merijayne requires so that her total earnings exceed Tim's total earnings?

16 A theatre has 24 rows of seats. There are 16 seats in the first row and each successive row increases by 2 seats, 1 on each side.

a) Calculate the number of seats in the 24th row.

b) Calculate the number of seats in the whole theatre.



17 The amount of €7000 is invested at 5.25% annual compound interest.

a) Write down an expression for the value of this investment after t full years.

b) Calculate the minimum number of years required for this amount to become €10 000.

c) For the same number of years as in part b), would an investment of the same amount be better if it were at a 5% rate compounded quarterly?

18 With S_n denoting the sum of the first n terms of an arithmetic sequence, we are given that $S_1 = 9$ and $S_2 = 20$.

a) Find the second term.

b) Calculate the common difference of the sequence.

c) Find the fourth term.

19 The second term of an arithmetic sequence is 7. The sum of the first four terms of the arithmetic sequence is 12. Find the first term, a, and the common difference, d, of the sequence.

20 Given that

$(1 + x)^5 (1 + ax)^6 \equiv 1 + bx + 10x^2 + \dots\dots\dots + a^6 x^{11}$,

find the values of $a, b \in \mathbb{Z}$, where $a \neq 0$.

21 The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$. If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of this sequence.

22 Using mathematical induction, prove that the number $2^{2n} - 3n - 1$ is divisible by 9, for $n = 1, 2, \dots$.

23 An arithmetic sequence has 5 and 13 as its first two terms respectively.

a) Write down, in terms of n, an expression for the nth term, an.

b) Find the number of terms of the sequence which are less than 400.

24 Find the coefficient of x^7 in the expansion of $(2 + 3x)^{10}$, giving your answer as a whole number.

25 The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term u_n.

26 Mr Blue, Mr Black, Mr Green, Mrs White, Mrs Yellow and Mrs Red sit around a circular table for a meeting. Mr Black and Mrs White must not sit together.

Calculate the number of different ways these six people can sit at the table without Mr Black and Mrs White sitting together.

27 Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, ….

28 The coefficient of x in the expansion of $\left(x + \frac{1}{a(x)^2}\right)^7$ is $\frac{7}{3}$. Find the possible values of a.

29 The sum of an infinite geometric sequence is $\frac{27}{2}$, and the sum of the first three terms is 13. Find the first term.

30 In how many ways can six different coins be divided between two students so that each student receives at least one coin?

31 Find the sum to infinity of the geometric series $-12 + 8 - \frac{16}{3}$.

32 The nth term, u_n, of a geometric sequence is given by $u_n = 3(4)^{n+1}$, $n \in \mathbb{Z}^+$.

a) Find the common ratio r.

b) Hence, or otherwise, find S_n, the sum of the first n terms of this sequence.

33 Consider the infinite geometric series

$$1 + \left(\frac{2x}{3}\right) + \left(\frac{2x}{3}\right)^2 + \left(\frac{2x}{3}\right)^3 + \dots$$

a) For what values of x does the series converge?

b) Find the sum of the series if $x = 1.2$.

34 How many four-digit numbers are there which contain at least one digit 3?

35 Consider the arithmetic series $2 + 5 + 8 + \dots$.

a) Find an expression for S_n, the sum of the first n terms.

b) Find the value of n for which $S_n = 1365$.

36 Find the coefficient of x^3 in the binomial expansion of $\left(1 - \frac{1}{2}x\right)^8$.

37 Find $\sum_{r=1}^{50} \ln(2^r)$, giving the answer in the form $a \ln 2$, where $a \in \mathbb{Q}$.

38 A sequence $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = 2$, $u_{n+1} = 3u_n - 2u_{n-1}$ where $n \in \mathbb{Z}^+$.

a) Find u_2, u_3, and u_4.

b) **(i)** Express u_n in terms of n.

(ii) Verify that your answer to part b)(i) satisfies the equation $u_{n+1} = 3u_n - 2u_{n-1}$.

39 A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

a) the common ratio;

b) the first term.

40 The first four terms of an arithmetic sequence are 2, $a - b$, $2a + b + 7$, and $a - 3b$, where a and b are constants. Find a and b.

41 A committee of four children is chosen from eight children. The two oldest children cannot both be chosen. Find the number of ways the committee may be chosen.

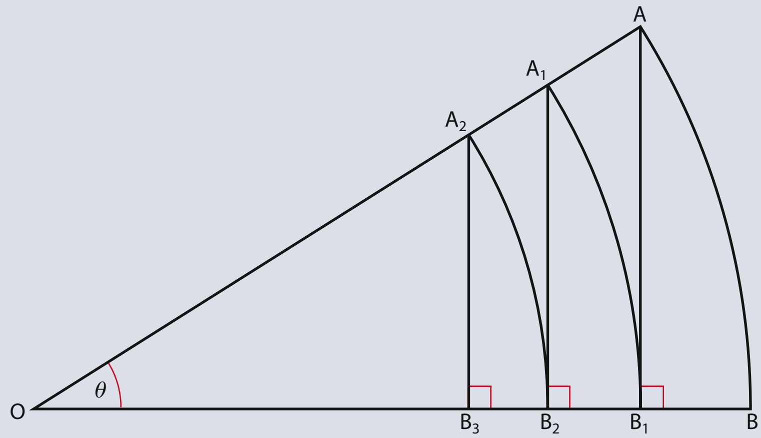
42 The three terms a, 1, b are in arithmetic progression. The three terms 1, a, b are in geometric progression. Find the value of a and of b given that $a \neq b$.

43 The diagram on the following page shows a sector AOB of a circle of radius 1 and centre O, where $\text{A}\hat{\text{O}}\text{B} = \theta$.

The lines (AB_1), (A_1B_2), (A_2B_3) are perpendicular to OB. A_1B_1, A_2B_2 are all arcs of circles with centre O.

Calculate the sum to infinity of the arc lengths

$\text{AB} + \text{A}_1\text{B}_1 + \text{A}_2\text{B}_2 + \text{A}_3\text{B}_3 + \dots$



44 The sum of the first n terms of a series is given by

$S_n = 2n^2 - n$, where $n \in \mathbb{Z}^+$.

a) Find the first three terms of the series.

b) Find an expression for the nth term of the series, giving your answer in terms of n.

45 **a)** Find the expansion of $(2 + x)^5$, giving your answer in ascending powers of x.

b) By letting $x = 0.01$ or otherwise, find the exact value of 2.01^5.

46 A sum of \$5000 is invested at a compound interest rate of 6.3% per annum.

a) Write down an expression for the value of the investment after n full years.

b) What will be the value of the investment at the end of five years?

c) The value of the investment will exceed \$10 000 after n full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of n.

47 Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

48 The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m.