(c) Let $n = 2$ in part (b). Then

$$\sum_{r=1}^{2} \sin x \sin(2r-1)x = \frac{1-\cos 4x}{2}$$

$$\Leftrightarrow \sin x \sin x + \sin x \sin 3x = \frac{1-\cos 4x}{2}$$

$$\Leftrightarrow \sin^2 x + \sin x \sin 3x = \frac{1-\cos 4x}{2}$$

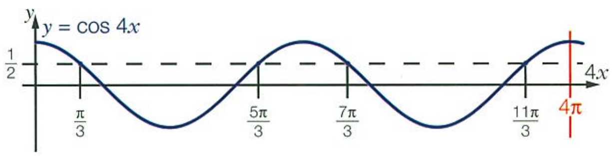
Therefore, solve

$$\frac{1-\cos 4x}{2} = \frac{1}{4}$$

$$\Leftrightarrow \cos 4x = \frac{1}{2}$$

$0 < x < \pi \Rightarrow 0 < 4x < 4\pi$

The graph shows there to be four solutions:



One solution is $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Another is $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Then, adding 2π to these gives two more solutions in $[0, 4\pi]$:

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$$

$$\frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$$

$$\therefore 4x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

(d) Let $n = 5$ and $x = \frac{\pi}{5}$ in part (b). Then

$$\sum_{r=1}^{5} \sin\left(\frac{\pi}{5}\right) \sin\left[(2r-1)\frac{\pi}{5}\right] = \frac{1-\cos\left[2\times 5\left(\frac{\pi}{5}\right)\right]}{2}$$

$$\Rightarrow \sin\frac{\pi}{5}\sin\frac{\pi}{5} + \sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \sin\frac{\pi}{5}\sin\frac{5\pi}{5} + \sin\frac{\pi}{5}\sin\frac{7\pi}{5} + \sin\frac{\pi}{5}\sin\frac{9\pi}{5} = \frac{1-\cos 2\pi}{2}$$

$$\Rightarrow \sin\frac{\pi}{5}\left(\sin\frac{\pi}{5} + \sin\frac{3\pi}{5} + \sin\frac{5\pi}{5} + \sin\frac{7\pi}{5} + \sin\frac{9\pi}{5}\right) = 0$$

$$\Rightarrow \sin\frac{\pi}{5} + \sin\frac{3\pi}{5} + \sin\frac{5\pi}{5} + \sin\frac{7\pi}{5} + \sin\frac{9\pi}{5} = 0$$

7 VECTORS

Mixed practice 7

1. (a) $$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix} (-1)\times p - 2\times 1 \\ 2\times 1 - 3\times p \\ 3\times 1 - (-1)\times 1 \end{pmatrix} = \begin{pmatrix} -p-2 \\ 2-3p \\ 4 \end{pmatrix}$$

(or $-(p+2)\boldsymbol{i} + (2-3p)\boldsymbol{j} + 4\boldsymbol{k}$)

(b) If $\boldsymbol{a}\times\boldsymbol{b}$ is parallel to $\boldsymbol{c}$, then $\boldsymbol{a}\times\boldsymbol{b} = \lambda\boldsymbol{c}$ for some constant λ.

So $\begin{pmatrix} -p-2 \\ 2-3p \\ 4 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 22 \\ 8 \end{pmatrix}$

The third component gives $4 = 8\lambda \Rightarrow \lambda = \frac{1}{2}$

And the first component gives

$-p - 2 = 2\lambda$

$\therefore p = -3$

Check these values in the second equation: $2 - 3p = 11 = 22\lambda$, as stated.

2. If point D lies on the line, its position vector $\boldsymbol{d}$ satisfies

$$\boldsymbol{d}=\begin{pmatrix}1\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix}=\begin{pmatrix}1-\lambda\\-1+\lambda\\5-2\lambda\end{pmatrix} \text{ for some value of } \lambda.$$

$$\therefore \overrightarrow{AD}=\boldsymbol{d}-\boldsymbol{a}=\begin{pmatrix}1-\lambda\\-1+\lambda\\5-2\lambda\end{pmatrix}-\begin{pmatrix}3\\2\\-1\end{pmatrix}=\begin{pmatrix}-2-\lambda\\-3+\lambda\\6-2\lambda\end{pmatrix}$$

For $\overrightarrow{AD}$ to be parallel to the x-axis,

$$\overrightarrow{AD}=\mu\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{for some scalar } \mu$$

$$\text{i.e. } \begin{pmatrix}-2-\lambda\\-3+\lambda\\6-2\lambda\end{pmatrix}=\begin{pmatrix}\mu\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases}-3+\lambda=0\\6-2\lambda=0\end{cases}$$

Both equations are satisfied when $\lambda=3$.

3. Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if

$$\boldsymbol{a}\cdot\boldsymbol{b}=0$$
$$\sin\theta\cos\theta+\cos\theta\sin 2\theta=0$$
$$\Leftrightarrow \sin\theta\cos\theta+\cos\theta(2\sin\theta\cos\theta)=0$$
$$\Leftrightarrow \sin\theta\cos\theta(1+2\cos\theta)=0$$
$$\Leftrightarrow \sin\theta=0 \text{ or } \cos\theta=0 \text{ or } \cos\theta=-\frac{1}{2}$$
$$\therefore \theta=0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

4. (a) $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}=\begin{pmatrix}4-3\\1-1\\3-1\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$

$$\overrightarrow{AC}=\boldsymbol{c}-\boldsymbol{a}=\begin{pmatrix}3-3\\q+1-1\\q+1-1\end{pmatrix}=\begin{pmatrix}0\\q\\q\end{pmatrix}$$

A vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}0\times q-2q\\2\times 0-1\times q\\1\times q-0\times 0\end{pmatrix}=\begin{pmatrix}-2q\\-q\\q\end{pmatrix}$$

(b) The area of triangle ABC is

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=6\sqrt{2}$$
$$\therefore \sqrt{4q^2+q^2+q^2}=12\sqrt{2}$$
$$\Leftrightarrow 6q^2=\left(12\sqrt{2}\right)^2$$
$$\Leftrightarrow q^2=\frac{144\times 2}{6}=\frac{144}{3}=48$$
$$\therefore q=4\sqrt{3}$$

(c) n is parallel to $\overrightarrow{AB}\times\overrightarrow{AC}$

$$\therefore n=\begin{pmatrix}-2\\-1\\1\end{pmatrix}$$

So the equation of the plane is

$$\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$$
$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$
$$-2x-y+z=-6-1+1$$
$$2x+y-z=6$$

5. (a) $\boldsymbol{d}=\overrightarrow{PQ}=\boldsymbol{q}-\boldsymbol{p}=\begin{pmatrix}-4\\2\\5\end{pmatrix}$

Then, using $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{d}$ with $\boldsymbol{a}=\boldsymbol{p}$ and the $\boldsymbol{d}$ found above, we get the vector equation

$$\boldsymbol{r}=\begin{pmatrix}3\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}-4\\2\\5\end{pmatrix}$$

(b) $\overrightarrow{PM}=\boldsymbol{m}-\boldsymbol{p}=\begin{pmatrix}0\\-3\\-1\end{pmatrix}$

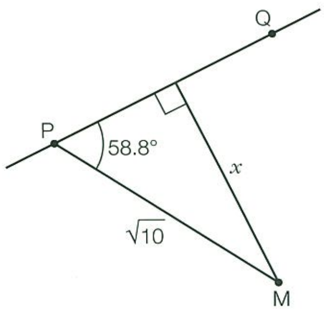
So the angle θ between l and (PM) satisfies

$$\cos\theta=\frac{\boldsymbol{d}\cdot\overrightarrow{PM}}{|\boldsymbol{d}|\left|\overrightarrow{PM}\right|}=\frac{\begin{pmatrix}-4\\2\\5\end{pmatrix}\cdot\begin{pmatrix}0\\-3\\-1\end{pmatrix}}{\sqrt{16+4+25}\sqrt{0+9+1}}$$
$$=\frac{-11}{\sqrt{45}\sqrt{10}}$$
$$\therefore \theta=\cos^{-1}\left(\frac{-11}{\sqrt{45}\sqrt{10}}\right)=121.2°$$

But since we are asked for the acute angle, it is $180° - 121.2° = 58.8°$.

(c) Let the shortest distance from M to the line be x.

From the diagram,

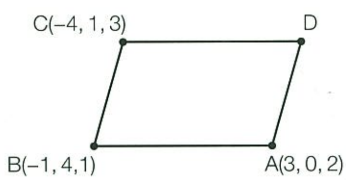


$$\sin 58.8° = \frac{x}{|\overrightarrow{PM}|} = \frac{x}{\sqrt{10}}$$

$$\therefore\ x = \sqrt{10}\sin 58.8° = 2.70$$

6. $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} -4-(-1) \\ 1-4 \\ 3-1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix}$

For ABCD to be a parallelogram, we must have $\overrightarrow{AD} = \overrightarrow{BC}$.



$$\overrightarrow{AD} = \boldsymbol{d} - \boldsymbol{a}$$
$$\Rightarrow \boldsymbol{d} = \overrightarrow{AD} + \boldsymbol{a}$$
$$= \overrightarrow{BC} + \boldsymbol{a}$$
$$= \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$

i.e. coordinates of D are (0, −3, 4).

7. (a) (i) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 2-1 \\ 1-(-1) \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 0-1 \\ 1-(-1) \\ 5-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2\times 2-(-1)2 \\ (-1)(-1)-1\times 2 \\ 1\times 2-2(-1) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$$

(ii) For the plane ABC:

$$\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$$

$$\Rightarrow 6x - y + 4z = 19$$

(b) (i) $\boldsymbol{r} = \begin{pmatrix} 9 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$

(ii) Writing $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the equation of part (b)(i):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} \Rightarrow \begin{cases} x = 9 + 6\lambda \\ y = 2 - \lambda \\ z = 5 + 4\lambda \end{cases}$$

Substituting these expressions into the equation of the plane from part (a)(ii):

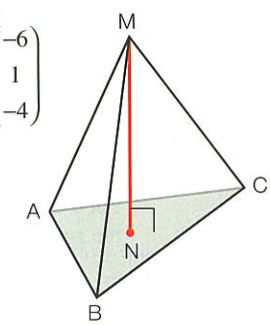
$$6(9+6\lambda) - (2-\lambda) + 4(5+4\lambda) = 19$$
$$\Leftrightarrow 54 + 36\lambda - 2 + \lambda + 20 + 16\lambda = 19$$
$$\Leftrightarrow 53\lambda = -53$$
$$\Leftrightarrow \lambda = -1$$

Substituting $\lambda = -1$ back into the equation of the line:

$$x = 9 + 6\lambda = 3$$
$$y = 2 - \lambda = 3$$
$$z = 5 + 4\lambda = 1$$

i.e. the foot of the perpendicular, N, has coordinates (3, 3, 1).

(iii) $\overrightarrow{MN} = \boldsymbol{n} - \boldsymbol{m} = \begin{pmatrix} 3-9 \\ 3-2 \\ 1-5 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ -4 \end{pmatrix}$



So the height, h, of the pyramid is

$$h = |\overrightarrow{MN}|$$
$$= \sqrt{(-6)^2 + 1^2 + (-4)^2} = \sqrt{53}$$

The area, A, of the (triangular) base is

$$A = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$
$$= \frac{1}{2}\sqrt{6^2 + (-1)^2 + 4^2} \quad \text{(using part (a)(i))}$$
$$= \frac{1}{2}\sqrt{53}$$

Therefore the volume, V, of the tetrahedron is

$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\left(\frac{1}{2}\sqrt{53}\right)\left(\sqrt{53}\right)$$
$$= \frac{53}{6}\left(\approx 8.83\right)$$

(c) The angle, θ, can be found from the right-angled triangle MNB.

$$\overrightarrow{BM} = \boldsymbol{m} - \boldsymbol{b} = \begin{pmatrix} 9-2 \\ 2-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$$\sin\theta = \frac{\left|\overrightarrow{MN}\right|}{\left|\overrightarrow{BM}\right|} = \frac{\sqrt{53}}{\sqrt{7^2+1^2+3^2}}$$

$$\therefore\ \theta = \sin^{-1}\left(\frac{\sqrt{53}}{\sqrt{59}}\right) = 71.4°$$

8. (a) $\boldsymbol{p}\cdot\boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|\cos 60°$

$$\therefore\ x + 2x + 4 = \sqrt{1^2+2^2+2^2}\sqrt{x^2+x^2+4} \times \frac{1}{2}$$
$$\Rightarrow 3x + 4 = \sqrt{9}\sqrt{2x^2+4} \times \frac{1}{2}$$
$$\Rightarrow 3x + 4 = \frac{3}{2}\sqrt{2x^2+4}$$
$$\Rightarrow (3x+4)^2 = \frac{9}{4}(2x^2+4)$$
$$\Rightarrow 9x^2 + 24x + 16 = \frac{9}{4}(2x^2+4)$$
$$\Rightarrow 18x^2 + 48x + 32 = 9x^2 + 18$$
$$\Rightarrow 9x^2 + 48x + 14 = 0$$

So $a = 9$, $b = 48$, $c = 14$.

(b) From GDC, $x = -0.3096...$ or $-5.0236...$

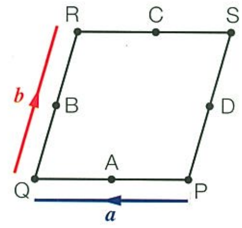
Let θ be the (acute) angle between vector $\boldsymbol{q}$ and the z-axis. Then

$$\cos\theta = \frac{\begin{pmatrix} x \\ x \\ 2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{x^2+x^2+4}\ \sqrt{0^2+0^2+1^2}}$$
$$= \frac{2}{\sqrt{2x^2+4}}$$
$$= 0.977 \text{ or } 0.271$$

$$\therefore\ \theta = 12.3° \text{ or } 74.3°$$

9. (a) $\overrightarrow{AB} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

$\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$



(b) (i) $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$ and $\boldsymbol{b}\cdot\boldsymbol{b} = |\boldsymbol{b}|^2$. Since PQRS is a rhombus, $|\boldsymbol{a}| = |\boldsymbol{b}|$ and therefore $\boldsymbol{a}\cdot\boldsymbol{a} = \boldsymbol{b}\cdot\boldsymbol{b}$.

(ii) $\overrightarrow{AB}\cdot\overrightarrow{BC} = \left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)\cdot\left(\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right)$

$$= \frac{1}{4}\boldsymbol{a}\cdot\boldsymbol{b} - \frac{1}{4}\boldsymbol{a}\cdot\boldsymbol{a} + \frac{1}{4}\boldsymbol{b}\cdot\boldsymbol{b} - \frac{1}{4}\boldsymbol{b}\cdot\boldsymbol{a}$$
$$= \frac{1}{4}(\boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{a}) + \frac{1}{4}(\boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{a})$$
$$= \frac{1}{4}(\boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b}) \quad \text{since } \boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a} \text{ and } \boldsymbol{b}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{a}$$
$$= 0$$

Therefore $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular.

(c) ABCD is a rectangle because $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular. (But since

$\left|\overrightarrow{AB}\right|^2 = \left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)\cdot\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$ and

$\left|\overrightarrow{BC}\right|^2 = \left(\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right)\cdot\left(\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}\right)$ are not necessarily equal, ABCD is not necessarily a square.)

10. (a) $\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} 4-1 \\ 1-5 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} \lambda-1 \\ \lambda-5 \\ 3-1 \end{pmatrix} = \begin{pmatrix} \lambda-1 \\ \lambda-5 \\ 2 \end{pmatrix}$

If there is a right angle at B, $\overrightarrow{BA}\cdot\overrightarrow{BC} = 0$.

i.e. $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} \lambda-1 \\ \lambda-5 \\ 2 \end{pmatrix} = 0$

$$\Leftrightarrow 3(\lambda-1) - 4(\lambda-5) + 1\times 2 = 0$$
$$\Leftrightarrow 3\lambda - 3 - 4\lambda + 20 + 2 = 0$$
$$\Leftrightarrow \lambda = 19$$

(b) $\overrightarrow{AD} = 2\overrightarrow{DC}$

$\Rightarrow \boldsymbol{d} - \boldsymbol{a} = 2(\boldsymbol{c} - \boldsymbol{d})$

$\Rightarrow 3\boldsymbol{d} = 2\boldsymbol{c} + \boldsymbol{a}$

$$= 2\begin{pmatrix} 19 \\ 19 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 42 \\ 39 \\ 8 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{d} = \begin{pmatrix} 14 \\ 13 \\ \frac{8}{3} \end{pmatrix}$$

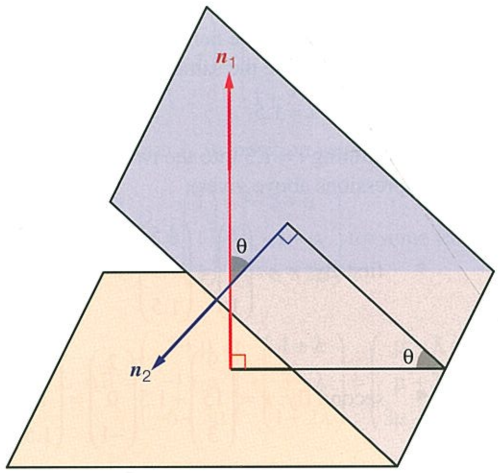
So the coordinates of D are $(14, 13, \frac{8}{3})$.

11. A normal to the plane $2x + 3y - 4z = 5$ is $\boldsymbol{n}_1 = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$.

A normal to the plane $6x - 2y - 3z = 4$ is $\boldsymbol{n}_2 = \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}$.

The angle between the planes, θ, is the same as the angle between the normals.

So



$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

$$= \frac{\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}}{\sqrt{4+9+16}\sqrt{36+4+9}}$$

$$= \frac{18}{\sqrt{29}\sqrt{49}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{18}{7\sqrt{29}}\right) = 61.5°$$

12. (a) If $(-1, 3, 5)$ lies on l, then there exists a λ such that

$$\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} -1 = -2 + \lambda & \cdots (1) \\ 5 = 3 + 2\lambda & \cdots (2) \\ 3 = 1 - \lambda & \cdots (3) \end{cases}$$

$(1) \Rightarrow \lambda = 1;\ \ (2) \Rightarrow \lambda = 1;\ \ (3) \Rightarrow \lambda = -2.$

There is no consistent value of λ, so A does not lie on l.

(b) Since B lies on l, its position vector is

$$\boldsymbol{b} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2+\lambda \\ 3+2\lambda \\ 1-\lambda \end{pmatrix} \text{ for some value of } \lambda.$$

Therefore $\overrightarrow{AB} = \begin{pmatrix} -2+\lambda \\ 3+2\lambda \\ 1-\lambda \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1+\lambda \\ -2+2\lambda \\ -2-\lambda \end{pmatrix}$

Since $\overrightarrow{AB}$ is perpendicular to l,

$$\begin{pmatrix} -1+\lambda \\ -2+2\lambda \\ -2-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$$

$\Leftrightarrow -1 + \lambda - 4 + 4\lambda + 2 + \lambda = 0$

$\Leftrightarrow 6\lambda = 3$

$\Leftrightarrow \lambda = \frac{1}{2}$

Substituting back into the expression for $\boldsymbol{b}$ gives

$$\boldsymbol{b} = \begin{pmatrix} -2+\lambda \\ 3+2\lambda \\ 1-\lambda \end{pmatrix} = \begin{pmatrix} -1.5 \\ 4 \\ 0.5 \end{pmatrix}$$

So the coordinates of B are $(-1.5, 4, 0.5)$.

13. (a) Putting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the equation of the line:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} x = 1 + t \\ y = -1 + t \\ z = 3 + 3t \end{cases}$$

Substituting each of these expressions into the equation of the plane:

$(1+t) - 3(-1+t) + (3+3t) = 17$

$\Leftrightarrow 1 + t + 3 - 3t + 3 + 3t = 17$

$\Leftrightarrow t = 10$

Then, substituting $t = 10$ back into the equation of the line gives:

$$x = 1 + t = 11$$
$$y = -1 + t = 9$$
$$z = 3 + 3t = 33$$

so Q has coordinates (11, 9, 33).

(b) A direction vector of the line $\frac{x+1}{3} = \frac{2-y}{7} = \frac{z}{3}$ is $\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix}$.

So the equation of the line through Q in the same direction is $\boldsymbol{r} = \begin{pmatrix} 11 \\ 9 \\ 33 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix}$

(c) The angle between l_1 and l_2 is the angle between their direction vectors:

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix}}{\sqrt{1+1+9}\sqrt{9+49+9}}$$
$$= \frac{19}{\sqrt{11}\sqrt{67}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{19}{\sqrt{11}\sqrt{67}}\right) = 45.6°$$

14. (a) Equating the x, y and z components of l_1 and l_2:

$$\begin{cases} 3\lambda = 9 + 2\mu \\ 5\lambda = 15 \\ \lambda = 3 - \mu \end{cases}$$

From the second equation, $\lambda = 3$; then, from the third equation, $\mu = 0$.

Checking these values in the first equation: $3\lambda = 9 = 9 + 2\mu$, so the lines do intersect.

To find the coordinates of the point of intersection, substitute $\lambda = 3$ into the equation for l_1 (or substitute $\mu = 0$ into the equation for l_2) to get (9, 15, 3).

(b) The angle between l_1 and l_2 is the angle between their direction vectors:

$$\cos\theta = \frac{\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{9+25+1}\sqrt{4+0+1}}$$
$$= \frac{5}{\sqrt{35}\sqrt{5}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{5}{\sqrt{35}\sqrt{5}}\right) = 67.8°$$

(c) Velocity vector is $\boldsymbol{v} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \text{cm s}^{-1}$

Speed is the magnitude of velocity, so

$$\text{speed} = \left\| \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \right\| = \sqrt{9+25+1} = 5.92 \text{ cm s}^{-1}$$

(d) At time t, the position of the

- first fly is $\boldsymbol{r} = t\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$
- second fly is $\boldsymbol{r} = \begin{pmatrix} 9 \\ 15 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

From part (a), both flies pass through the point (9, 15, 3), but the first fly is there when $t = 3$ and the second one when $t = 0$. As there can be only one possible intersection point of these two straight line paths, the flies do not meet.

(e) They are at the same height when their z-coordinates are the same:

$$t = 3 - t \Leftrightarrow t = 1.5$$

Substituting $t = 1.5$ into the two position vector expressions above gives:

- first fly: $\boldsymbol{r} = t\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 7.5 \\ 1.5 \end{pmatrix}$
- second fly: $\boldsymbol{r} = \begin{pmatrix} 9 \\ 15 \\ 3 \end{pmatrix} + 1.5\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ 15 \\ 1.5 \end{pmatrix}$

i.e. the coordinates of the two flies are (4.5, 7.5, 1.5) and (12, 15, 1.5) when they are at the same height.

The distance between them is

$$d = \sqrt{(4.5-12)^2 + (7.5-15)^2 + (1.5-1.5)^2}$$
$$= \sqrt{112.5} = 10.6 \text{ cm}$$

(f) The displacement vector from the first fly to the second fly is

$$\begin{pmatrix} 9+2t \\ 15 \\ 3-t \end{pmatrix} - \begin{pmatrix} 3t \\ 5t \\ t \end{pmatrix} = \begin{pmatrix} 9-t \\ 15-5t \\ 3-2t \end{pmatrix}$$

The distance d between the flies at time t therefore satisfies

$$d^2 = \left|\begin{pmatrix} 9-t \\ 15-5t \\ 3-2t \end{pmatrix}\right|^2$$
$$= (9-t)^2 + (15-5t)^2 + (3-2t)^2$$
$$= 315 - 180t + 30t^2$$

Using a graph on the GDC (or by differentiation), the minimum value is $d^2 = 45$, i.e. $d = 6.71$ cm.

Going for the top 7

1. (a) $|\boldsymbol{a}+2\boldsymbol{b}|^2 = |\boldsymbol{b}-2\boldsymbol{a}|^2$

$\Leftrightarrow (\boldsymbol{a}+2\boldsymbol{b})\cdot(\boldsymbol{a}+2\boldsymbol{b}) = (\boldsymbol{b}-2\boldsymbol{a})\cdot(\boldsymbol{b}-2\boldsymbol{a})$

$\Leftrightarrow |\boldsymbol{a}|^2 + 4\boldsymbol{a}\cdot\boldsymbol{b} + 4|\boldsymbol{b}|^2 = |\boldsymbol{b}|^2 - 4\boldsymbol{a}\cdot\boldsymbol{b} + 4|\boldsymbol{a}|^2$

$\Leftrightarrow 1 + 4\boldsymbol{a}\cdot\boldsymbol{b} + 4 = 1 - 4\boldsymbol{a}\cdot\boldsymbol{b} + 4$ (using $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$)

$\Leftrightarrow 8\boldsymbol{a}\cdot\boldsymbol{b} = 0$

$\Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b} = 0$

(b) Because $\boldsymbol{a}\cdot\boldsymbol{b} = 0$, the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, so the angle between them is 90°.

2. (a) As A lies on l_1, its position vector satisfies

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix} \text{ for some value of } \lambda.$$

As B lies on l_2, its position vector satisfies

$$\boldsymbol{b} = \begin{pmatrix} 5 \\ -1 \\ -6 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5+\mu \\ -1+\mu \\ -6+3\mu \end{pmatrix} \text{ for some value of } \mu.$$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 5+\mu \\ -1+\mu \\ -6+3\mu \end{pmatrix} - \begin{pmatrix} 1+\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix} = \begin{pmatrix} \mu-\lambda+4 \\ \mu+\lambda-4 \\ 3\mu-2\lambda-7 \end{pmatrix}$$

Since $\overrightarrow{AB}$ is perpendicular to l_1,

$$\begin{pmatrix} \mu-\lambda+4 \\ \mu+\lambda-4 \\ 3\mu-2\lambda-7 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$$

$\Leftrightarrow (\mu-\lambda+4) - (\mu+\lambda-4) + 2(3\mu-2\lambda-7) = 0$

$\Leftrightarrow 6\mu - 6\lambda - 6 = 0$

$\Leftrightarrow \mu - \lambda = 1$

(b) Since $\overrightarrow{AB}$ is perpendicular to l_2,

$$\begin{pmatrix} \mu-\lambda+4 \\ \mu+\lambda-4 \\ 3\mu-2\lambda-7 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$$

$\Leftrightarrow (\mu-\lambda+4) + (\mu+\lambda-4) + 3(3\mu-2\lambda-7) = 0$

$\Leftrightarrow 11\mu - 6\lambda - 21 = 0$

$\Leftrightarrow 11\mu - 6\lambda = 21$

(c) Solving the two equations in (a) and (b) simultaneously using the GDC gives $\mu = 3, \lambda = 2$.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3-2+4 \\ 3+2-4 \\ 9-4-7 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$

The shortest distance is $|\overrightarrow{AB}| = \sqrt{25+1+4} = \sqrt{30}$.

3. (a) Eliminating x and y from the last equation:

$$\begin{cases} x-2y+z=0 & \cdots(1) \\ 3x-z=4 & \cdots(2) \\ x+y-z=k & \cdots(3) \end{cases}$$

$$\begin{matrix} \\ (2)-3\times(1) \\ (3)-(1) \end{matrix}\begin{cases} x-2y+z=0 & \cdots(1) \\ 6y-4z=4 & \cdots(4) \\ 3y-2z=k & \cdots(5) \end{cases}$$

$$\begin{matrix} \\ \\ 2\times(5)-(4) \end{matrix}\begin{cases} x-2y+z=0 & \cdots(1) \\ 6y-4z=4 & \cdots(4) \\ 0=2k-4 & \cdots(6) \end{cases}$$

Hence the system does not have a unique solution (as the last equation has no variables left in it).

(b) There are no solutions unless $2k-4=0$, i.e. $k=2$.

When $k=2$, there are infinitely many solutions:

Let $z=t$. Then, from equation (4):

$6y - 4t = 4$

$\Rightarrow y = \dfrac{4+4t}{6} = \dfrac{2}{3} + \dfrac{2}{3}t$

From equation (1):

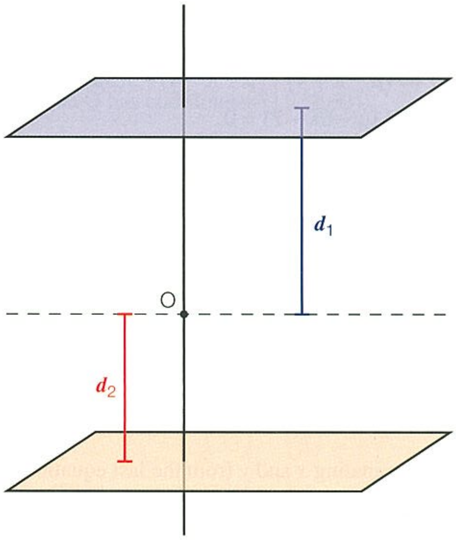
$x - 2y + t = 0$

$\therefore x = -t + 2\left(\dfrac{2}{3} + \dfrac{2}{3}t\right) = \dfrac{4}{3} + \dfrac{1}{3}t$

Therefore the equation of the line is $\boldsymbol{r} = \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$.

4. The two planes share a common normal, $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, and so are parallel.

To find the (perpendicular) distance between the two parallel planes, consider the distance of each from the origin along this normal.



The equation of the line through (0, 0, 0) along the normal is $r = \lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}$,

i.e. any point on this line has $\begin{cases}x=\lambda\\y=-2\lambda\\z=3\lambda\end{cases}$

This line intersects the plane $x-2y+3z=7$ when

$$\lambda-2(-2\lambda)+3(3\lambda)=7$$
$$\Leftrightarrow 14\lambda=7$$
$$\Leftrightarrow \lambda=\frac{1}{2}$$

Substituting $\lambda=\frac{1}{2}$ back into the equation of the line:

$$r=\frac{1}{2}\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-1\\\frac{3}{2}\end{pmatrix}$$

so the coordinates of the point of intersection are

$$\left(\frac{1}{2},-1,\frac{3}{2}\right).$$

Hence the distance between plane Π_1 and the origin is

$$d_1=\sqrt{\left(\frac{1}{2}-0\right)^2+(-1-0)^2+\left(\frac{3}{2}-0\right)^2}=\frac{1}{2}\sqrt{14}$$

Now we repeat the same process for the other plane (which is on the opposite side of the origin as the RHS is negative). The equation of the line through (0, 0, 0) along the normal intersects the plane $x-2y+3z=-21$ when

$$\lambda-2(-2\lambda)+3(3\lambda)=-21$$
$$\Leftrightarrow 14\lambda=-21$$
$$\Leftrightarrow \lambda=-\frac{3}{2}$$

Substituting $\lambda=-\frac{3}{2}$ back into the equation of the line:

$$r=-\frac{3}{2}\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}-\frac{3}{2}\\3\\-\frac{9}{2}\end{pmatrix}$$

so the coordinates of the point of intersection are

$$\left(-\frac{3}{2},3,-\frac{9}{2}\right).$$

Hence the distance between plane Π_2 and the origin is

$$d_2=\sqrt{\left(-\frac{3}{2}-0\right)^2+(3-0)^2+\left(-\frac{9}{2}-0\right)^2}=\frac{3}{2}\sqrt{14}$$

Therefore, the distance between the two planes is

$$d_1+d_2=\frac{1}{2}\sqrt{14}+\frac{3}{2}\sqrt{14}=2\sqrt{14}$$

8 COMPLEX NUMBERS

Mixed practice 8

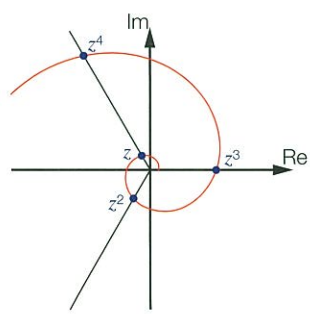
1. (a) In polar form, $z=2e^{i\frac{2\pi}{3}}$, so

$$z^2=\left(2e^{i\frac{2\pi}{3}}\right)^2=4e^{i\frac{4\pi}{3}}=4e^{-i\frac{2\pi}{3}}$$

$$z^3=\left(2e^{i\frac{2\pi}{3}}\right)^3=8\,e^{i\frac{6\pi}{3}}=8\,e^{i2\pi}=8$$

$$z^4=\left(2e^{i\frac{2\pi}{3}}\right)^4=16e^{i\frac{8\pi}{3}}=16e^{i\frac{2\pi}{3}}$$

(b)



21. (a) Unique solution: a is any real number; $x = 14 - 2a$, $y = 4a - 15$, $z = 9 - 2a$

(b) Unique solution: $a \neq 3$; $x = -7$, $y = 17$, $z = 0$. Infinitely many solutions: $a = 3$;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$

5 SEQUENCES AND SERIES

1. (a) -86 (b) 660

2. (a) 6.5 (b) 33rd term

(c) 24 or 41

3. -0.669

4. 128 or 384

5. $a = 8$, $r = \frac{3}{2}$

6. (a) $a = \frac{1}{2}$, $r = 4$

(b) 9th term (c) 10

7. (a) -3, 7 (b) 54

8. (a) \$32 619 (b) \$1 511 552

(c) 13th year (d) 27

9. (a) \$185 (c) 56

10. (a) 1.77147 m

(b) 11th (c) 57 m

11. (b) $k = 2.5$ (c) 28 years

6 TRIGONOMETRY

1. $a = 3$, $b = 2$

2. $f(x) \in \left[\frac{2}{7}, \frac{2}{3}\right]$

3. $\frac{3\pi}{4}$

10. $-\sqrt{\frac{5}{6}}$

11. $-\frac{\sqrt{7}}{4}$

12. $\frac{2}{\sqrt{5}}$

13. $x = \frac{\pi}{8}, \frac{7\pi}{8}$

14. $x = -\frac{8\pi}{9}, -\frac{5\pi}{9}, -\frac{2\pi}{9}, \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$

15. $x = \frac{\pi}{36}, \frac{5\pi}{36}, \frac{25\pi}{36}, \frac{29\pi}{36}, -\frac{23\pi}{36}, -\frac{19\pi}{36}$

16. $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$

17. $\theta = -1.89, -0.464, 1.25, 2.68$

18. $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

19. $x = \pm\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

20. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

21. $R = 2$, $\theta = \frac{\pi}{6}$; minimum value $\frac{2}{5}$

22. $R = \sqrt{34}$, $\alpha = 0.540$, maximum point $\left(0.540, \sqrt{34}\right)$

23. 6.75 cm

24. 23.2

25. 21.2 m

7 VECTORS

1. (a) $\overrightarrow{MN} = \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ (b) $\overrightarrow{QP} = \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

2. (a) $\begin{pmatrix} -16 \\ -8 \\ 8 \end{pmatrix}$ (b) $4\sqrt{6} \approx 9.80$

3. (a) $\begin{pmatrix} -3 \\ 6 \\ -1 \end{pmatrix}$ (b) $\frac{1}{\sqrt{46}}\begin{pmatrix} -3 \\ 6 \\ -1 \end{pmatrix}$

4. (a) $\boldsymbol{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$ (b) $(-3, 5, 12)$

5. The lines do not intersect (they are skew).

6. $x + 5y - 2z = 5$

7. $2x - 13y - 5z = -25$

8. (a) $(0, 5, 13)$ (b) $15x + 6y + z = 43$

9. $\boldsymbol{r} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

10. $\left(\frac{9}{2}, \frac{5}{2}, \frac{3}{2}\right)$

11. $(1, -1, 3)$

12. 48.2°

13. 79.5°

14. 67.1°

15. (a) $\Pi_1 : \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \Pi_2 : \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}$

    (b) 58.5°

16. $\frac{\sqrt{230}}{5} \approx 3.03$

17. (a) $\boldsymbol{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ (b) $\frac{7\sqrt{11}}{11} \approx 2.11$

18. $\frac{7}{6}\sqrt{6} = 2.86$

19. (a) $\sqrt{53} \approx 7.28\,\text{m s}^{-1}$

    (b) 7.90° (c) $12\sqrt{53} \approx 87.4\,\text{m}$

20. Speed $= 5\sqrt{17} \approx 20.6\,\text{km h}^{-1}$

8 COMPLEX NUMBERS

1. $z = -3 - 4\text{i}$
2. $z = 1 - 2\text{i}$
3. $\text{Im}(z) = -\frac{1}{2}$
4. (b) $z = 1 + \text{i}$ or $z = \frac{7}{5} - \frac{1}{5}\text{i}$
5. $z = 1 - \text{i}$ or $z = -\frac{5}{3}$
6. (b) $z = \frac{25}{13} + \frac{60}{13}\text{i}$
7. $4\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right) = -4\text{i}$
8. (a) $z_1 = \sqrt{2}\,\text{e}^{-\text{i}\frac{\pi}{4}},\ z_2 = 2\text{e}^{\text{i}\frac{2\pi}{3}}$ (b) $-\frac{1}{4}\text{i}$
9. $-2, -3\text{i}$
10. $a = 2, b = -3, c = 20$
11. (b) $-2\text{i}, 3 + \text{i}, 3 - \text{i}$
12. $4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$
13. (a) $\cos^5\theta + 5\text{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta -$ $10\text{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \text{i}\sin^5\theta$

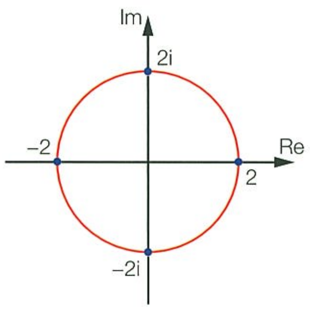
    (b) (i) $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

    (ii) $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$
14. (a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

    $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
15. $\frac{1}{4}(\cos 3\theta + 3\cos\theta)$
17. (a) $A = 5, B = 10$

    (b) $\frac{1}{16}\left(\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta\right) + c$
18. (a) $|z| = 8,\ \arg z = -\frac{\pi}{3}$

    (b) $2\text{e}^{-\text{i}\frac{\pi}{9}}, 2\text{e}^{-\text{i}\frac{7\pi}{9}}, 2\text{e}^{\text{i}\frac{5\pi}{9}}$
19. $2, -2, 2\text{i}, -2\text{i}$



20. $\text{e}^{-\text{i}\frac{\pi}{12}}, \text{e}^{\text{i}\frac{\pi}{4}}, \text{e}^{\text{i}\frac{7\pi}{12}}, \text{e}^{\text{i}\frac{11\pi}{12}}, \text{e}^{-\text{i}\frac{3\pi}{4}}, \text{e}^{-\text{i}\frac{5\pi}{12}}$

9 DIFFERENTIATION

5. $2x\text{e}^{x^2} + \frac{3x\cos 3x - \sin 3x}{2x^2}$
6. $x = -4, 3$
7. $\frac{12 - 18x^2}{(x^2+2)^3}$
8. (c) $\frac{1}{2}\ln 3$
9. $y - \text{e}^{-12} = \frac{\text{e}^{12}}{12}(x-2)$ or $\text{e}^{12}x - 12y + 12\text{e}^{-12} - 2\text{e}^{12} = 0$
10. -1.26
11. $-\frac{1}{2}$
12. $(0, 5), (0, -2)$
13. $\frac{y - 3 - 2y^2(y-3)^2}{4xy(y-3)^2 + x + 1}$