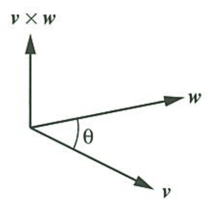
7 VECTORS

WHAT YOU NEED TO KNOW

- A vector can represent the position of a point relative to the origin (position vector) or the displacement from one point to another.
  - The components of a position vector are the coordinates of the point.
  - If points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the displacement vector $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.
  - The position vector of the midpoint of [AB] is $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.
- A vector (in 3-dimensional space) can be represented by its components: $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$
  - The magnitude of $\boldsymbol{v}$ is $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
  - The unit vector in the same direction as $\boldsymbol{v}$ is $\hat{\boldsymbol{v}} = \frac{1}{|\boldsymbol{v}|}\boldsymbol{v}$.
  - The distance between two points A and B, with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, is $AB = |\boldsymbol{b} - \boldsymbol{a}|$.
- The scalar product (or dot product) can be used to calculate the angle between two vectors:

  $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{v}||\boldsymbol{w}|}$ where $\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$
  - If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, $\boldsymbol{a}\cdot\boldsymbol{b} = 0$.
  - If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$; in particular, $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$.
  - The scalar product has many properties similar to those of multiplication:
    - $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$
    - $(k\boldsymbol{a})\cdot\boldsymbol{b} = k(\boldsymbol{a}\cdot\boldsymbol{b})$
    - $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$
- The vector product (or cross product) of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted by $\boldsymbol{v}\times\boldsymbol{w}$:

  $$\boldsymbol{v}\times\boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$$



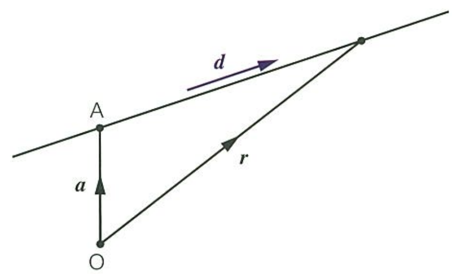
  - The magnitude of the vector product is:

    $|\boldsymbol{v}\times\boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|\sin\theta$
  - The vector product can be used to find the area of some geometrical figures:
    - Area of parallelogram $= |\boldsymbol{v}\times\boldsymbol{w}|$
    - Area of triangle $= \frac{1}{2}|\boldsymbol{v}\times\boldsymbol{w}|$

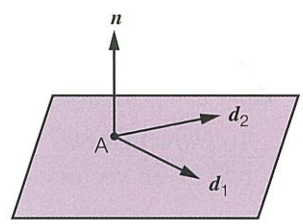
    where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of the shape.

- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{0}$; in particular, $\boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{0}$.
- The vector product has many properties similar to those of multiplication:
  - $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$
  - $(k\boldsymbol{a})\times\boldsymbol{b} = k(\boldsymbol{a}\times\boldsymbol{b})$
  - $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$

- A straight line can be described by:
  - a Cartesian equation $\dfrac{x-a_1}{d_1} = \dfrac{y-a_2}{d_2} = \dfrac{z-a_3}{d_3}$
  - a vector equation $\boldsymbol{r} = \boldsymbol{a}+\lambda\boldsymbol{d}$, where:
    - $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ is the direction vector of the line
    - $A(a_1, a_2, a_3)$ is one point on the line.



- A plane can be described by:
  - a Cartesian equation $n_1x+n_2y+n_3z = \boldsymbol{a}\cdot\boldsymbol{n}$ (which can also be written in scalar product form $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$)
  - a vector equation $\boldsymbol{r} = \boldsymbol{a}+\lambda\boldsymbol{d}_1+\mu\boldsymbol{d}_2$, where:
    - $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are direction vectors of two lines in the plane
    - $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is the normal vector of the plane: $\boldsymbol{n} = \boldsymbol{d}_1\times\boldsymbol{d}_2$
    - $A(a_1, a_2, a_3)$ is one point in the plane.



- To find the intersection:
  - of two lines, set the two position vectors equal to each other and use two of the equations to find the unknown parameters (checking that the parameter values satisfy the third equation)
  - between a line and a plane, substitute x, y and z from the equation of the line into the Cartesian equation of the plane, and solve to find the parameter
  - of two or three planes, solve the system of equations given by the Cartesian equations of the planes.
- To find the angle between two lines, two planes, or a line and a plane, identify two direction vectors which make the required angle, and then use $\cos\theta = \dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{v}||\boldsymbol{w}|}$.

- The shortest distance between:
  - a point and a line occurs when the connecting vector is perpendicular to the direction vector of the line
  - a point and a plane occurs along a normal vector to the plane
  - two lines occurs when the connecting vector is perpendicular to the direction vectors of both lines.

  Although not required by the syllabus, the following formulae may be useful for finding shortest distances.

  - The shortest distance between the line $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$ and the point with position vector $\boldsymbol{b}$ is $\frac{|(\boldsymbol{a} - \boldsymbol{b}) \times \boldsymbol{d}|}{|\boldsymbol{d}|}$.
  - The shortest distance between the lines $\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + \lambda \boldsymbol{d}_2$ is $\frac{|(\boldsymbol{a}_2 - \boldsymbol{a}_1) \cdot \boldsymbol{n}|}{|\boldsymbol{n}|}$, where $\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2$.

- An object starting at the point with position vector $\boldsymbol{a}$ and moving with (constant) velocity vector $\boldsymbol{v}$ moves along a path given by $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$, where t is the time after the start of the motion.

EXAM TIPS AND COMMON ERRORS

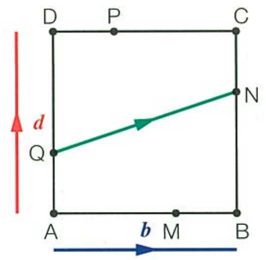
- Use a vector diagram to show all the given information and add to it as you work through a question.
- To *show* that two vectors are perpendicular, use $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. To *find* a vector perpendicular to two given vectors, use $\boldsymbol{a} \times \boldsymbol{b}$.
- Be careful with the vector product: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$. (Note the negative sign; it **is not** true that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{a}$.)
- In long questions, it may be possible to answer a later part without having done all the previous parts.
- Be clear on the difference between the coordinates of a point, (a_1, a_2, a_3), and the position vector of the point, $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

7.1 PROVING GEOMETRICAL PROPERTIES USING VECTORS

WORKED EXAMPLE 7.1

ABCD is a square. Points M, N, P and Q lie on the sides [AB], [BC], [CD] and [DA] so that $AM:MB = BN:NC = CP:PD = DQ:QA = 2:1$. Let $\overrightarrow{AB} = \boldsymbol{b}$ and $\overrightarrow{AD} = \boldsymbol{d}$.

Prove that [QN] and [MP] are perpendicular.



Always draw a diagram for questions involving geometrical figures.

$$\overrightarrow{QN} = \overrightarrow{QD} + \overrightarrow{DC} + \overrightarrow{CN}$$
$$= \frac{2}{3}\boldsymbol{d} + \boldsymbol{b} + \frac{1}{3}(-\boldsymbol{d})$$
$$= \boldsymbol{b} + \frac{1}{3}\boldsymbol{d}$$

We need an expression for $\overrightarrow{QN}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$. Note that a ratio of 2 : 1 means $\frac{2}{3}$ and $\frac{1}{3}$, respectively, of the total length, and that moving backwards along a vector (in this case $\boldsymbol{d}$) means a negative sign.

$$\overrightarrow{MP} = \overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CP}$$
$$= \frac{1}{3}\boldsymbol{b} + \boldsymbol{d} + \frac{2}{3}(-\boldsymbol{b}) = -\frac{1}{3}\boldsymbol{b} + \boldsymbol{d}$$

Similarly, we find an expression for $\overrightarrow{MP}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{d}$.

$$\overrightarrow{MP} \cdot \overrightarrow{QN} = \left(-\frac{1}{3}\boldsymbol{b} + \boldsymbol{d}\right) \cdot \left(\boldsymbol{b} + \frac{1}{3}\boldsymbol{d}\right)$$

To prove that $\overrightarrow{MP}$ and $\overrightarrow{QN}$ are perpendicular, we need to show that $\overrightarrow{MP} \cdot \overrightarrow{QN} = 0$.

$$= -\frac{1}{3}\boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{d} \cdot \boldsymbol{b} - \frac{1}{9}\boldsymbol{b} \cdot \boldsymbol{d} + \frac{1}{3}\boldsymbol{d} \cdot \boldsymbol{d}$$

The brackets can be expanded with the scalar product just as with normal multiplication.

$$= -\frac{1}{3}|\boldsymbol{b}|^2 + \boldsymbol{b} \cdot \boldsymbol{d} - \frac{1}{9}\boldsymbol{b} \cdot \boldsymbol{d} + \frac{1}{3}|\boldsymbol{d}|^2$$

Use the facts that $\boldsymbol{b} \cdot \boldsymbol{d} = \boldsymbol{d} \cdot \boldsymbol{b}$ and $\boldsymbol{b} \cdot \boldsymbol{b} = |\boldsymbol{b}|^2$.

$$= \frac{8}{9}\boldsymbol{b} \cdot \boldsymbol{d} = 0$$

Hence [QN] and [MP] are perpendicular.

Since the sides of a square have equal length and are perpendicular, $|\boldsymbol{b}| = |\boldsymbol{d}|$ and $\boldsymbol{b} \cdot \boldsymbol{d} = 0$.

Practice questions 7.1

1. Points A, B, C and D have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$, respectively. M is the midpoint of [AB] and N is the midpoint of [BC].

   (a) Express $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

   P is the midpoint of [CD] and Q is the midpoint of [DA].

   (b) By finding an expression for $\overrightarrow{QP}$ in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$, show that MNPQ is a parallelogram.

7.2 APPLICATIONS OF THE VECTOR PRODUCT

WORKED EXAMPLE 7.2

Points A, B and C have coordinates (–3, 1, 5), (1, 1, 2) and (1, –2, 7), respectively.

(a) Find a vector perpendicular to both (AB) and (AC).

(b) Find the area of the triangle ABC.

> ⚠ Remember that (AB) represents the line through A and B (infinite), while [AB] is the line segment (finite) and AB means the length of [AB].

(a)

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}4\\0\\-3\end{pmatrix}\times\begin{pmatrix}4\\-3\\2\end{pmatrix}$$

$$=\begin{pmatrix}0-9\\-12-8\\-12-0\end{pmatrix}=\begin{pmatrix}-9\\-20\\-12\end{pmatrix}$$

The vector product of two vectors is perpendicular to both of them, and we calculate this using the formula $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$.

(b) $\text{Area}=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

$$=\frac{1}{2}\sqrt{(-9)^2+(-20)^2+(-12)^2}=12.5$$

We can find the area of a triangle with sides $\boldsymbol{a}$ and $\boldsymbol{b}$ using the formula $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$, but note that we have to use the actual vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ rather than any other parallel vectors (that is, we must not 'simplify' either vector by dividing through by a common factor).

Practice questions 7.2

2. Three points A, B and C are such that $\overrightarrow{AB}=3\boldsymbol{i}-\boldsymbol{j}+5\boldsymbol{k}$ and $\overrightarrow{AC}=-\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$.

(a) Calculate $\overrightarrow{AB}\times\overrightarrow{AC}$.

(b) Find the area of the triangle ABC.

3. Let $\boldsymbol{p}=\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{q}=\boldsymbol{i}-3\boldsymbol{k}$.

(a) Calculate $\boldsymbol{p}\times\boldsymbol{q}$.

(b) Find a unit vector perpendicular to both $\boldsymbol{p}$ and $\boldsymbol{q}$.

7.3 EQUATION OF A LINE AND THE INTERSECTION OF LINES

WORKED EXAMPLE 7.3

(a) Find a vector equation of the line l_1 passing through points A(2, 1, 3) and B(−1, 1, 2).

(b) Show that l_1 does not intersect the line with equation $\frac{x-1}{2} = y+1 = \frac{z}{3}$.

(a) $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$

$$\boldsymbol{d} = \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$$

We need the position vector of one point on the line and the direction vector of the line: $\boldsymbol{a}$ is the position vector of point A on the line, and $\overrightarrow{AB}$ is a vector in the direction of the line.

So a vector equation is $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$

We could also have used $\boldsymbol{b}$ as the point on the line and/or taken $\overrightarrow{BA}$ to be our $\boldsymbol{d}$.

(b) For l_1: $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-3\lambda \\ 1 \\ 3-\lambda \end{pmatrix}$

$\therefore x = 2-3\lambda,\ y = 1,\ z = 3-\lambda$

By letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the equation for l_1 and combining the two vectors on the RHS into one, we can get expressions for x, y and z.

Substitute into the equation of l_2:

$$\frac{(2-3\lambda)-1}{2} = (1)+1 = \frac{(3-\lambda)}{3}$$

If l_1 and l_2 intersect, there must be a solution for λ when the equations of the two lines are solved simultaneously.

$$\therefore \begin{cases} \frac{1-3\lambda}{2} = 2 \Rightarrow \lambda = -1 \\ 2 = \frac{3-\lambda}{3} \Rightarrow \lambda = -3 \end{cases}$$

λ value not consistent, so the lines do not intersect.

We can split the equation for l_2 into two separate equations.

Because these two equations do not have a common solution for λ, the lines do not intersect.

Practice questions 7.3

4. (a) Find a vector equation of the line passing through the points A(−3, 1, 2) and B(−3, 3, 7).

(b) Find the coordinates of the point where this line meets the line with equation $\frac{x-1}{-2} = \frac{y+1}{3} = \frac{z+2}{7}$.

5. Determine whether the lines $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 10 \\ 2 \end{pmatrix}$ intersect. If they do, find the point of intersection.

7.4 EQUATION OF A PLANE

WORKED EXAMPLE 7.4

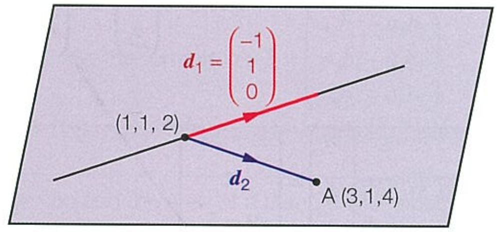
Find the Cartesian equation of the plane containing the point A(3, 1, 4) and the line with equation $\boldsymbol{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\end{pmatrix}$.

$$\boldsymbol{d}_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\boldsymbol{d}_2=\begin{pmatrix}3\\1\\4\end{pmatrix}-\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}2\\0\\2\end{pmatrix}$$

$$\therefore\ \boldsymbol{n}=\boldsymbol{d}_1\times\boldsymbol{d}_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}\times\begin{pmatrix}2\\0\\2\end{pmatrix}=\begin{pmatrix}2\\2\\-2\end{pmatrix}$$

We need a normal vector to the plane; this will be perpendicular to two vectors that lie in the plane. One such vector is the direction vector of the line and the other can be found by noting that (1, 1, 2) from the equation of the line (as well as point A) lies in the plane.



$$\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-2\end{pmatrix}=\begin{pmatrix}3\\1\\4\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-2\end{pmatrix}$$

For the point in the plane, we can use A.

To get the Cartesian equation, write $\boldsymbol{r}$ as $\begin{pmatrix}x\\y\\z\end{pmatrix}$.

$\Leftrightarrow 2x+2y-2z=6+2-8$

$\Leftrightarrow 2x+2y-2z=0$

$\Leftrightarrow x+y-z=0$

Practice questions 7.4

6. Find the Cartesian equation of the plane containing the line with equation $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+t\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and the point (4, 1, 2).

7. Find the Cartesian equation of the plane containing the points A(−1, 1, 2), B(3, 2, 1) and C(0, 0, 5).

8. (a) Find the coordinates of the point of intersection of the lines with equations $\boldsymbol{r}=(2\boldsymbol{i}+\boldsymbol{j}+7\boldsymbol{k})+\lambda(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})$ and $\boldsymbol{r}=(2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})+\mu(-2\boldsymbol{i}+3\boldsymbol{j}+12\boldsymbol{k})$.

(b) Find the Cartesian equation of the plane containing the two lines.

7.5 INTERSECTIONS INVOLVING PLANES

WORKED EXAMPLE 7.5

Find a vector equation of the line of intersection of the planes $2x - y + z = 3$ and $x - y - 2z = 7$.

$$\begin{cases} 2x - y + z = 3 & \cdots (1) \\ x - y - 2z = 7 & \cdots (2) \end{cases}$$

$$\begin{matrix} \\ (1)-2\times(2) \end{matrix} \begin{cases} 2x - y + z = 3 & \cdots (1) \\ y + 5z = -11 & \cdots (3) \end{cases}$$

The points on the line of intersection satisfy both equations, so we need to solve the pair of equations using elimination.

We start by eliminating x.

Systems of equations are covered in Chapter 4.

Let $z = t$.

From (3): $y = -11 - 5t$. Substitute into (1):

$$x = \frac{3 + y - z}{2} = \frac{3 + (-11 - 5t) - t}{2} = \frac{-8 - 6t}{2} = -4 - 3t$$

Since there are three unknowns but only two equations, we cannot find a unique solution; however, we can express both x and y in terms of z.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 - 3t \\ -11 - 5t \\ 0 + 1t \end{pmatrix}$$

Let (x, y, z) be the coordinates of any point on the line of intersection. To find the vector equation of the line, write the coordinates as a position vector.

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} -4 \\ -11 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}$$

Finally, write the equation of the line in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$.

Practice questions 7.5

9. Find a vector equation of the line of intersection of the planes with equations $3x - y + z = 16$ and $x - 2y + 2z = 12$.

10. Find the coordinates of the point where the line with equation $\boldsymbol{r} = (5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + \lambda(\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$ meets the plane with equation $x + 2y + z = 11$.

11. Find the coordinates of the point of intersection of the planes $\Pi_1: x + y - 3z = -9$, $\Pi_2: 3x + y = 2$ and $\Pi_3: x + y + z = 3$.

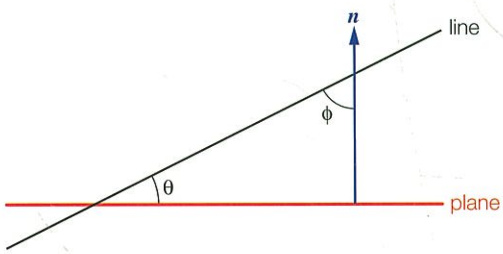
7.6 ANGLES BETWEEN LINES AND PLANES

WORKED EXAMPLE 7.6

Find the angle between the line with equation $\boldsymbol{r} = (-\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$ and the plane with equation $2x + 3y - 7z = 5$.

A normal to the plane $2x+3y-7z=5$ is $\boldsymbol{n} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$

Since the Cartesian equation of a plane is derived from the vector equation $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$ with $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we can see that $\boldsymbol{n} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$.



A diagram clarifies the relationship between the normal, the line and the angle (θ) between the line and the plane.

$$\cos\phi = \frac{\boldsymbol{d}\cdot\boldsymbol{n}}{|\boldsymbol{d}||\boldsymbol{n}|}$$

$$= \frac{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}}{\sqrt{2^2+1^2+(-2)^2}\sqrt{2^2+3^2+(-7)^2}}$$

$$= \frac{21}{\sqrt{9}\sqrt{62}}$$

$\therefore \phi = 27.3°$

We cannot find θ directly, but we can find the angle φ between the line and the normal by using the scalar product formula.

⚠ When finding the angle between two lines, always use the direction vectors of the lines in the scalar product formula.

So, $\theta = 90° - \phi = 62.7°$

Now we can find the required angle θ.

Practice questions 7.6

12. Find the angle between the lines $\boldsymbol{r} = (\boldsymbol{i} + 2\boldsymbol{k}) + \lambda(3\boldsymbol{i} + 4\boldsymbol{j} + 12\boldsymbol{k})$ and $\boldsymbol{r} = -2\boldsymbol{j} + \mu(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$.

13. Find the acute angle that the line with equation $\boldsymbol{r} = 4\boldsymbol{j} + \lambda(2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$ makes with the y-axis.

14. Find the acute angle between the line $\frac{x+6}{-2} = \frac{y-2}{3} = \frac{z+4}{-6}$ and the plane $2x - 6y + 5z = 63$.

15. (a) Write down a normal to each of the planes $\Pi_1: x - 2y - 5z = 7$ and $\Pi_2: 3x - 7y - z = 4$.

(b) Hence find the angle between the planes Π_1 and Π_2.

7.7 DISTANCES FROM LINES AND PLANES

WORKED EXAMPLE 7.7

Find the shortest distance between the lines $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{k} + \lambda(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ and $\boldsymbol{r} = 11\boldsymbol{j} + \mu(2\boldsymbol{i} - \boldsymbol{k})$.

Consider a general point P on the line l_1 and a general point Q on the line l_2. Then

$$\boldsymbol{p} = \begin{pmatrix} 1+\lambda \\ -\lambda \\ 2+\lambda \end{pmatrix} \text{ and } \boldsymbol{q} = \begin{pmatrix} 2\mu \\ 11 \\ -\mu \end{pmatrix}$$

The position vector of any point on a line is given by the equation of that line.

$$\overrightarrow{QP} = \boldsymbol{p} - \boldsymbol{q} = \begin{pmatrix} 1+\lambda \\ -\lambda \\ 2+\lambda \end{pmatrix} - \begin{pmatrix} 2\mu \\ 11 \\ -\mu \end{pmatrix} = \begin{pmatrix} 1+\lambda-2\mu \\ -\lambda-11 \\ 2+\lambda+\mu \end{pmatrix}$$

We can now find a general vector, $\overrightarrow{QP}$, connecting the two lines.

$$\begin{pmatrix} 1+\lambda-2\mu \\ -\lambda-11 \\ 2+\lambda+\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 1+\lambda-2\mu \\ -\lambda-11 \\ 2+\lambda+\mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = 0$$

This connecting vector will be shortest when it is perpendicular to the direction vectors of both lines.

In any question involving the shortest distance, a perpendicular vector is always needed.

$$\Leftrightarrow \begin{cases} (1+\lambda-2\mu)+(\lambda+11)+(2+\lambda+\mu)=0 \\ (2+2\lambda-4\mu)+0-(2+\lambda+\mu)=0 \end{cases}$$

$$\Leftrightarrow \lambda = -5,\ \mu = -1$$

Use a GDC to solve the simultaneous equations.

$$\therefore \overrightarrow{QP} = \begin{pmatrix} 1-5+2 \\ 5-11 \\ 2-5-1 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix}$$

Substitute back to find $\overrightarrow{QP}$.

$$d = |\overrightarrow{QP}| = \sqrt{(-2)^2 + (-6)^2 + (-4)^2} = 2\sqrt{14} \approx 7.48$$

The distance is the length of $\overrightarrow{QP}$.

Practice questions 7.7

16. Find the perpendicular distance from A(1, 1, 2) to the line $\boldsymbol{r} = (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) + t(\boldsymbol{i} + 2\boldsymbol{k})$.

17. The point M has position vector $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and the plane Π has equation $3x - y + z = 17$.

(a) Write down the equation of the line through M which is perpendicular to Π.

(b) Hence find the distance from the point M to the plane Π.

18. Find the shortest distance between the lines $\boldsymbol{r} = (\boldsymbol{i} - \boldsymbol{k}) + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$ and $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-3}{5}$.

7.8 APPLYING VECTORS TO MOTION

WORKED EXAMPLE 7.8

An aircraft taking off from $0\boldsymbol{i} + 10\boldsymbol{j} + 0\boldsymbol{k}$ km moves with velocity $150\boldsymbol{i} + 150\boldsymbol{j} + 40\boldsymbol{k}$ km h^{-1}, where the vector $\boldsymbol{i}$ represents East, $\boldsymbol{j}$ represents North and $\boldsymbol{k}$ represents vertically up.

(a) If t is the time in hours, write down a vector equation for the motion of the aircraft.

(b) Find the speed of the aircraft.

(c) Find the angle of elevation of the aircraft.

(a) $\boldsymbol{r} = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix} + t\begin{pmatrix} 150 \\ 150 \\ 40 \end{pmatrix}$

We can use the standard form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$ to describe the path of the aircraft.

You can use either $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ notation or column vector notation, whichever you are more comfortable with, but do not forget that an equation needs '$\boldsymbol{r} =$'.

(b) Speed $= \sqrt{150^2 + 150^2 + 40^2} \approx 216$ km h^{-1}

Speed is the modulus of the velocity vector.

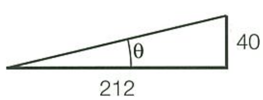
(c) The resultant speed of the components in the easterly and northerly directions is:

$\sqrt{150^2 + 150^2} \approx 212$ km h^{-1}

So the angle of elevation θ satisfies

$\tan\theta = \frac{40}{212}$

$\therefore \theta \approx 10.7°$



The velocity is made up of a component 'along the ground' (the resultant of the $\boldsymbol{i}$ and $\boldsymbol{j}$ components) and a vertical ($\boldsymbol{k}$) component.

Even though 212 km h^{-1} is written in the working, the calculation should be done using the full accuracy stored in the calculator.

Practice questions 7.8

19. The path of a flying bird is modelled by $\boldsymbol{r} = 4t\boldsymbol{i} + 6t\boldsymbol{j} + (12 - t)\boldsymbol{k}$ metres where t is in seconds.

(a) Find the speed of the bird.

(b) Find the angle of depression of its flight.

(c) How far has the bird travelled when it lands on the ground?

20. A ship starts at $0\boldsymbol{i} + 0\boldsymbol{j}$ km and moves with velocity $20\boldsymbol{i} + 5\boldsymbol{j}$ km h^{-1}. A second ship starts at $7\boldsymbol{i} + 12\boldsymbol{j}$ km and moves with velocity $-19\boldsymbol{i} - 8\boldsymbol{j}$ km h^{-1}. Show that the ships have the same speed but do not collide.

Mixed practice 7

1. Let $\boldsymbol{a}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}1\\1\\p\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}2\\22\\8\end{pmatrix}$.

   (a) Find $\boldsymbol{a}\times\boldsymbol{b}$.

   (b) Find the value of p such that $\boldsymbol{a}\times\boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

2. Point A has position vector $3\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$. Point D lies on the line with equation $\boldsymbol{r}=(\boldsymbol{i}-\boldsymbol{j}+5\boldsymbol{k})+\lambda(-\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})$. Find the value of λ such that (AD) is parallel to the x-axis.

3. Two vectors are given by $\boldsymbol{a}=\begin{pmatrix}\sin\theta\\\cos\theta\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}\cos\theta\\\sin 2\theta\end{pmatrix}$, where $\theta\in[0, 2\pi]$. Find all possible values of θ for which $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

4. Three points have coordinates A(3, 1, 1), B(4, 1, 3) and C(3, $q+1$, $q+1$), where $q>0$.

   (a) Find a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   (b) Given that the area of the triangle ABC is $6\sqrt{2}$, find the value of q.

   (c) Find the Cartesian equation of the plane containing the points A, B and C.

5. (a) Find a vector equation of the line l passing through the points P(3, –1, 2) and Q(–1, 1, 7).

   (b) The point M has position vector $3\boldsymbol{i}-4\boldsymbol{j}+\boldsymbol{k}$. Find the acute angle between (PM) and l.

   (c) Hence find the shortest distance from M to the line l.

6. Three points have coordinates A(3, 0, 2), B(–1, 4, 1) and C(–4, 1, 3). Find the coordinates of point D such that ABCD is a parallelogram.

7. The tetrahedron MABC has vertices with coordinates A(1, –1, 3), B(2, 1, 2), C(0, 1, 5) and M(9, 2, 5).

   (a) (i) Calculate $\overrightarrow{AB}\times\overrightarrow{AC}$.

   (ii) Find the Cartesian equation of the plane ABC.

   (b) (i) Write down the equation of the line through M which is perpendicular to the plane ABC.

   (ii) Find the coordinates of the foot of the perpendicular from M to the plane ABC.

   (iii) Hence find the volume of the tetrahedron.

   (c) Calculate the angle between the edge [MB] and the base ABC.

8. The angle between vectors $\boldsymbol{p}=\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{q}=x\boldsymbol{i}+x\boldsymbol{j}+2\boldsymbol{k}$ is 60°.

   (a) Find constants a, b and c such that $ax^2+bx+c=0$.

   (b) Hence find the angle between the vector $\boldsymbol{q}$ and the z-axis.

9. PQRS is a rhombus with $\overrightarrow{PQ} = \boldsymbol{a}$ and $\overrightarrow{QR} = \boldsymbol{b}$. The midpoints of the sides [PQ], [QR], [RS] and [SP] are A, B, C and D, respectively.

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) (i) Explain why $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{b}$.

(ii) Show that [AB] and [BC] are perpendicular.

(c) What type of quadrilateral is ABCD?

10. Three points have coordinates A(4, 1, 2), B(1, 5, 1) and C(λ, λ, 3).

(a) Find the value of λ for which the triangle ABC has a right angle at B.

(b) For this value of λ, find the coordinates of point D on the side [AC] such that AD = 2DC.

11. Find the angle between the two planes with equations $2x + 3y - 4z = 5$ and $6x - 2y - 3z = 4$.

12. (a) Show that the point A with coordinates (–1, 5, 3) does not lie on the line l with equation

$$\boldsymbol{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

(b) Find the coordinates of the point B on l such that [AB] is perpendicular to l.

13. (a) Find the coordinates of the point Q where the line l_1, with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, meets the plane with equation $x - 3y + z = 17$.

(b) Write down the equation of the line l_2 through Q which is parallel to the line with equation $\frac{x+1}{3} = \frac{2-y}{7} = \frac{z}{3}$.

(c) Find the angle between l_1 and l_2.

14. The line l_1 has vector equation $\boldsymbol{r} = \lambda\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ and the line l_2 has vector equation $\boldsymbol{r} = \begin{pmatrix} 9 \\ 15 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

(a) Show that the two lines meet and find the point of intersection.

(b) Find the angle between the two lines.

Two flies are flying in an empty room. One fly starts in a corner at position (0, 0, 0) and every second flies 3 cm in the x direction, 5 cm in the y direction and 1 cm up in the vertical z direction.

(c) Find the speed of this fly.

The second fly starts at the point (9, 15, 3) cm and each second travels 2 cm in the x direction and 1 cm down in the vertical z direction.

(d) Show that the two flies do not meet.

(e) Find the distance between the flies when they are at the same height.

(f) Find the minimum distance between the two flies.

Going for the top 7

1. Two unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are given such that $|\boldsymbol{a}+2\boldsymbol{b}|=|\boldsymbol{b}-2\boldsymbol{a}|$.

   (a) Find the value of $\boldsymbol{a}\cdot\boldsymbol{b}$.

   (b) Hence find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

2. Two lines have vector equations $l_1 : \boldsymbol{r}=\begin{pmatrix}1\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $l_2 : \boldsymbol{r}=\begin{pmatrix}5\\-1\\-6\end{pmatrix}+\mu\begin{pmatrix}1\\1\\3\end{pmatrix}$.

   Points A on l_1 and B on l_2 are such that (AB) is perpendicular to both lines.

   (a) Show that $\mu_A-\lambda_B=1$.

   (b) Find another linear equation connecting λ_B and μ_A.

   (c) Hence find the shortest distance between the two lines.

3. Three planes have equations: Π_1: $x-2y+z=0$, Π_2: $3x-z=4$ and Π_3: $x+y-z=k$.

   (a) Show that for all values of k, the planes do not intersect at a unique point.

   (b) Find the value of k for which the intersection of the three planes is a line, and find the vector equation of this line.

4. Two planes have equations Π_1: $x-2y+3z=7$ and Π_2: $x-2y+3z=-21$. By finding the shortest distance between each plane and the origin, determine the distance between Π_1 and Π_2.