Chapter 3

Algebraic expansion and factorisation

Contents:

A Revision of expansion laws
B Further expansion
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F Factorising quadratic trinomials
G Factorisation of $ax^2 + bx + c,\ a \neq 1$
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OPENING PROBLEM

Jody showed her friend Leanne a trick for performing multiplications of 2 digit numbers, such as 42×83:

Step 1: *Multiply the digits in the units column.*
$2 \times 3 = 6$

$$\begin{array}{rrr} & 4 & 2 \\ \times & 8 & 3 \\ \hline & & 6 \end{array}$$

Step 2: *Multiply the digits along the diagonals, then add the results.*
$(4 \times 3) + (8 \times 2) = 28$, so we write 8 and carry the 2.

$$\begin{array}{rrr} & 4 & 2 \\ \times & 8 & 3 \\ {\scriptstyle 2} & & \\ \hline & 8 & 6 \end{array}$$

Step 3: *Multiply the digits in the tens column.*
$4 \times 8 = 32$, adding the 2 gives 34.

$$\begin{array}{rrrr} & & 4 & 2 \\ & \times & 8 & 3 \\ \hline 3 & 4 & 8 & 6 \end{array}$$

So, $42 \times 83 = 3486$.

Things to think about:

Can you use algebra to explain why this trick works?

The study of **algebra** is vital for many areas of mathematics. We need it to manipulate equations, solve problems for unknown variables, and also to develop higher level mathematical theories.

In this chapter we revise the **expansion** of expressions which involve brackets, and the reverse process which is called **factorisation**.

A REVISION OF EXPANSION LAWS

In this section we revise the laws for expanding algebraic expressions.

DISTRIBUTIVE LAW

$$a(b + c) = ab + ac$$

Example 1 **Self Tutor**

Expand the following:

a $2(3x - 1)$ **b** $-3x(x + 2)$

a
$$\begin{aligned} & 2(3x - 1) \\ &= 2 \times 3x + 2 \times (-1) \\ &= 6x - 2 \end{aligned}$$

b
$$\begin{aligned} & -3x(x + 2) \\ &= -3x \times x + -3x \times 2 \\ &= -3x^2 - 6x \end{aligned}$$

THE PRODUCT $(a+b)(c+d)$

$$(a+b)(c+d) = ac + ad + bc + bd$$

Example 2 Self Tutor

Expand and simplify:

a $(x+4)(x-3)$ **b** $(2x-5)(-x+3)$

a $(x+4)(x-3)$
$= x \times x + x \times (-3) + 4 \times x + 4 \times (-3)$
$= x^2 - 3x + 4x - 12$
$= x^2 + x - 12$

b $(2x-5)(-x+3)$
$= 2x \times (-x) + 2x \times 3 - 5 \times (-x) - 5 \times 3$
$= -2x^2 + 6x + 5x - 15$
$= -2x^2 + 11x - 15$

DIFFERENCE OF TWO SQUARES

$$(a+b)(a-b) = a^2 - b^2$$

Example 3 Self Tutor

Expand and simplify:

a $(x+4)(x-4)$ **b** $(3x-2)(3x+2)$

a $(x+4)(x-4)$
$= x^2 - 4^2$
$= x^2 - 16$

b $(3x-2)(3x+2)$
$= (3x)^2 - 2^2$
$= 9x^2 - 4$

PERFECT SQUARES EXPANSION

$$(a+b)^2 = a^2 + 2ab + b^2$$

Example 4 Self Tutor

Expand and simplify:

a $(2x+1)^2$ **b** $(3-4y)^2$

a $(2x+1)^2$
$= (2x)^2 + 2 \times 2x \times 1 + 1^2$
$= 4x^2 + 4x + 1$

b $(3-4y)^2$
$= 3^2 + 2 \times 3 \times (-4y) + (-4y)^2$
$= 9 - 24y + 16y^2$

EXERCISE 3A

1 Expand and simplify:

- **a** $3(2x+5)$
- **b** $4x(x-3)$
- **c** $-2(3+x)$
- **d** $-3x(x+y)$
- **e** $2x(x^2-1)$
- **f** $-x(1-x^2)$
- **g** $-ab(b-a)$
- **h** $x^2(x-3)$
- **i** $3(a^2+3a+1)$
- **j** $5(x^2-3x+2)$
- **k** $-4(2c^2-3c-7)$
- **l** $2a(3a^2-5a+1)$

2 Expand and simplify:

- **a** $2(x+3)+5(x-4)$
- **b** $2(3-x)-3(4+x)$
- **c** $x(x+2)+2x(1-x)$
- **d** $x(x^2+2x)-x^2(2-x)$
- **e** $a(a+b)-b(a-b)$
- **f** $x^2(6-x)+3x(x-4)$

3 Expand and simplify:

- **a** $(x+2)(x+5)$
- **b** $(x-3)(x+4)$
- **c** $(x+5)(x-3)$
- **d** $(x-2)(x-10)$
- **e** $(2x+1)(x-3)$
- **f** $(3x-4)(2x-5)$
- **g** $(2x+y)(x-y)$
- **h** $(x+3)(-2x-1)$
- **i** $(x+2y)(-x-1)$

4 Expand and simplify:

- **a** $(x+3)(x-1)+3(x-5)$
- **b** $(x+7)(x-5)+(x+1)(x+4)$
- **c** $(2x+3)(x-2)-(x+1)(x+6)$
- **d** $(4t-3)(t+1)-(2t-1)(2t+5)$
- **e** $(4x-1)(3-x)+(2x-3)(3x-2)$
- **f** $5(3x-4)(x+2)-(7-x)(8-5x)$

5 Expand and simplify:

- **a** $(x+7)(x-7)$
- **b** $(3+a)(3-a)$
- **c** $(5-x)(5+x)$
- **d** $(2x+1)(2x-1)$
- **e** $(4-3y)(4+3y)$
- **f** $(3x-4z)(4z+3x)$

6 Expand and simplify:

- **a** $(x+3)(x-3)-(x+6)(x-6)$
- **b** $(5p-2)(5p+2)-p(3p-1)$
- **c** $(3y-z)(3y+z)-(2y+3z)(2y-3z)$
- **d** $(10-x^2)(10+x^2)-(10-3x^2)(10+3x^2)$

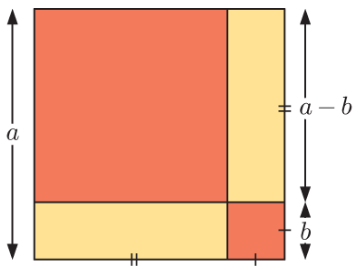
7 Expand and simplify:

- **a** $(x+5)^2$
- **b** $(2x+3)^2$
- **c** $(7+x)^2$
- **d** $(3x+4)^2$
- **e** $(5+x^2)^2$
- **f** $(3x^2+2)^2$
- **g** $(5x+3y)^2$
- **h** $(2x^2+7y)^2$
- **i** $(x^3+8x)^2$

8 Expand and simplify:

- **a** $(x-3)^2$
- **b** $(2-x)^2$
- **c** $(3x-1)^2$
- **d** $(6-5p)^2$
- **e** $(2x-5y)^2$
- **f** $(ab-2)^2$
- **g** $(x^2-5)^2$
- **h** $(4x^2-3y)^2$
- **i** $(-x^2-y^2)^2$

9 Use the diagram alongside to show that $(a-b)^2 = a^2-2ab+b^2$.



10 Expand and simplify:

a $(x+9)^2+(x-2)^2$
b $(3x+1)^2-(2x-3)^2$
c $(x+8)^2-(x+2)(x-5)$
d $(5-p)^2+(p^2-4)^2$
e $(3x^2-1)^2-4(1-x)^2$
f $(5x+y^2)^2-x(x^2-y)^2$

B FURTHER EXPANSION

When expressions containing more than two terms are multiplied together, we can still use the distributive law to expand the brackets. Each term in the first set of brackets is multiplied by each term in the second set of brackets.

If there are 2 terms in the first brackets and 3 terms in the second brackets, there will be $2\times 3=6$ terms in the expansion. However, when we simplify by collecting like terms, the final answer may contain fewer terms.

Example 5 Self Tutor

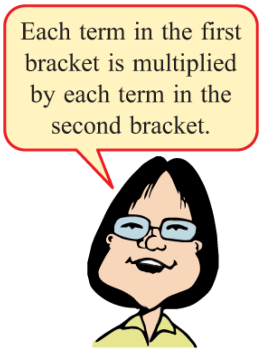
Expand and simplify: $(x+3)(x^2+2x+4)$

$(x+3)(x^2+2x+4)$

$= x^3+2x^2+4x$ {$x\times$ each term in 2nd bracket}

$\quad + 3x^2+6x+12$ {$3\times$ each term in 2nd bracket}

$= x^3+5x^2+10x+12$ {collecting like terms}



EXERCISE 3B

1 Expand and simplify:

a $(x+2)(x^2+x+4)$
b $(x+3)(x^2+2x-3)$
c $(x+3)(x^2+2x+1)$
d $(x+1)(2x^2-x-5)$
e $(2x+3)(x^2+2x+1)$
f $(2x-5)(x^2-2x-3)$
g $(x+5)(3x^2-x+4)$
h $(4x-1)(2x^2-3x+1)$

Example 6 Self Tutor

Expand and simplify: $(x+1)(x-3)(x+2)$

$(x+1)(x-3)(x+2)$

$= (x^2-3x+x-3)(x+2)$ {expanding first two factors}

$= (x^2-2x-3)(x+2)$ {collecting like terms}

$= x^3+2x^2-2x^2-4x-3x-6$ {expanding remaining factors}

$= x^3-7x-6$ {collecting like terms}

2 Expand and simplify:

a $(x+4)(x+3)(x+2)$
b $(x-3)(x-2)(x+4)$
c $(x-3)(x-2)(x-5)$
d $(2x-3)(x+3)(x-1)$
e $(4x+1)(3x-1)(x+1)$
f $(2-x)(3x+1)(x-7)$
g $(x-2)(4-x)(3x+2)$
h $(x+3)^3$
i $(x-2)^3$

3 State how many terms you would obtain by expanding:

a $(a+b)(c+d)$ **b** $(a+b+c)(d+e)$ **c** $(a+b)(c+d+e)$

d $(a+b+c)(d+e+f)$ **e** $(a+b)(c+d)(e+f)$ **f** $(a+b+c)(d+e)(f+g)$

4 Expand and simplify:

a $(x^2+3x+1)(x^2-x+3)$ **b** $(2x^2+x-1)(x^2+3x-2)$

c $(3x^2+x-4)(2x^2-3x+1)$ **d** $(x^2-3x+2)(x+5)(x-3)$

C THE BINOMIAL EXPANSION

Consider $(a+b)^n$ where n is a positive integer.

$\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.

The **binomial expansion** of $(a+b)^n$ is obtained by writing the expression without brackets.

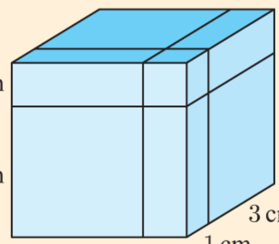
INVESTIGATION 1 THE BINOMIAL EXPANSION OF $(a+b)^3$

In this Investigation we discover the binomial expansion of $(a+b)^3$.

What to do:

1 Find a large potato and cut it to obtain a 4 cm by 4 cm by 4 cm cube.

2 By making 3 cuts parallel to the cube's surfaces, divide the cube into 8 rectangular prisms as shown.



3 How many prisms are:

a 3 by 3 by 3 **b** 3 by 3 by 1

c 3 by 1 by 1 **d** 1 by 1 by 1?

4 Now instead of the 4 cm × 4 cm × 4 cm potato cube, suppose you had a cube with edge length $(a+b)$ cm.

a Explain why the volume of the cube is given by $(a+b)^3$.

b Suppose you made cuts so each edge was divided into a cm and b cm. How many prisms would be:

i a by a by a **ii** a by a by b

iii a by b by b **iv** b by b by b?

DEMO

c By adding the volumes of the 8 rectangular prisms, find an expression for the total volume. Hence write down the binomial expansion of $(a+b)^3$.

Another method of finding the binomial expansion of $(a+b)^3$ is to expand the brackets:

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

$$\boldsymbol{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}.$$

The binomial expansion of $(a+b)^3$ can be used to expand other perfect cubes.

Example 7 Self Tutor

Expand and simplify using the rule $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

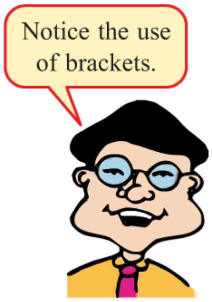
a $(x+4)^3$ **b** $(3x-2)^3$

a We substitute $a = x$ and $b = 4$.

$$\therefore\ (x+4)^3 = x^3 + 3 \times x^2 \times 4 + 3 \times x \times 4^2 + 4^3$$
$$= x^3 + 12x^2 + 48x + 64$$

b We substitute $a = (3x)$ and $b = (-2)$.

$$\therefore\ (3x-2)^3 = (3x)^3 + 3 \times (3x)^2 \times (-2) + 3 \times (3x) \times (-2)^2 + (-2)^3$$
$$= 27x^3 - 54x^2 + 36x - 8$$



EXERCISE 3C

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(x+1)^3$ **b** $(x+3)^3$ **c** $(x+5)^3$ **d** $(x+y)^3$
e $(x-1)^3$ **f** $(x-5)^3$ **g** $(x-4)^3$ **h** $(x-y)^3$
i $(2+y)^3$ **j** $(2x+1)^3$ **k** $(3x+1)^3$ **l** $(2y+3x)^3$
m $(2-y)^3$ **n** $(2x-1)^3$ **o** $(3x-1)^3$ **p** $(2y-3x)^3$

2 By expanding and simplifying $(a+b)^3(a+b)$, show that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

3 Use the binomial expansion $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(x+y)^4$ **b** $(x+1)^4$ **c** $(x+2)^4$ **d** $(x+3)^4$
e $(x-y)^4$ **f** $(x-1)^4$ **g** $(x-2)^4$ **h** $(2x-1)^4$

4 Consider:

$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The expressions on the right hand side of each identity contain the coefficients:

1 1
1 2 1
1 3 3 1
1 4 6 4 1

This triangle of numbers is called **Pascal's triangle**.

a Predict the next two rows of Pascal's triangle, and explain how you found them.

b Hence, write down the binomial expansion for:

i $(a+b)^5$ **ii** $(a-b)^5$ **iii** $(a+b)^6$ **iv** $(a-b)^6$

c **i** Expand and simplify $(x-2)^5$.
ii Check your answer by substituting $x = 1$ into your expansion.

D REVISION OF FACTORISATION

Factorisation is the process of writing an expression as a **product** of its **factors**.

Factorisation is the reverse process of expansion, so we use the expansion laws in reverse.

FACTORISING WITH COMMON FACTORS

If every term in an expression has the same common factor, then we can place this factor in front of a set of brackets. We use the reverse of the distributive law for expansion.

Example 8 Self Tutor

Fully factorise:

a $6x^2 + 4x$ **b** $-4(a+1) + (a+2)(a+1)$

a $6x^2 + 4x$
$= 2 \times 3 \times x \times x + 2 \times 2 \times x$
$= 2x(3x+2)$

b $-4(a+1) + (a+2)(a+1)$
$= (a+1)[-4 + (a+2)]$
$= (a+1)(a-2)$

DIFFERENCE OF TWO SQUARES FACTORISATION

$$a^2 - b^2 = (a+b)(a-b)$$

Example 9 Self Tutor

Fully factorise:

a $4 - 9y^2$ **b** $9a - 16a^3$

a $4 - 9y^2$
$= 2^2 - (3y)^2$
$= (2+3y)(2-3y)$

b $9a - 16a^3$
$= a(9 - 16a^2)$
$= a(3^2 - (4a)^2)$
$= a(3+4a)(3-4a)$

PERFECT SQUARES FACTORISATION

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

Example 10 Self Tutor

Factorise:

a $4x^2 + 4x + 1$ **b** $8x^2 - 24x + 18$

a $4x^2 + 4x + 1$
$= (2x)^2 + 2 \times 2x \times 1 + 1^2$
$= (2x+1)^2$

b $8x^2 - 24x + 18$
$= 2(4x^2 - 12x + 9)$
$= 2((2x)^2 - 2 \times 2x \times 3 + 3^2)$
$= 2(2x-3)^2$

EXERCISE 3D

1 Fully factorise:

a $x^2 - 5x$ **b** $2x^2 + 6x$ **c** $4x - 2xy$ **d** $3ab - 6b$

e $2x^2 + 8x^3$ **f** $-6x^2 + 12x$ **g** $x^3 + x^2$ **h** $3ab^2 - 9a^2b$

2 Fully factorise:

a $3(x+5) + x(x+5)$ **b** $a(b+3) - 5(b+3)$ **c** $x(x+4) + x + 4$

d $x(x+2) + (x+2)(x+5)$ **e** $a(c-d) + b(c-d)$ **f** $y(2+y) - y - 2$

g $ab\,(x-1) + c\,(x-1)$ **h** $a\,(x+2) - x - 2$ **i** $(x-3)^2 + x - 3$

j $(x+5)^2 + 3x + 15$ **k** $2(x-2)^2 + 4x - 8$ **l** $(x+y)^3 - x - y$

3 Fully factorise:

a $x^2 - 16$ **b** $64 - x^2$ **c** $9x^2 - 1$

d $49 - 4x^2$ **e** $y^2 - 4x^2$ **f** $4a^2 - 25b^2$

g $81x^2 - 16y^2$ **h** $4x^4 - y^2$ **i** $9a^2b^2 - 16$

j $(x+3)^2 - 4$ **k** $(3x-2)^2 - 16$ **l** $(2x-5)^2 - (x-4)^2$

4 Fully factorise:

a $2x^2 - 8$ **b** $3y^2 - 27$ **c** $2 - 18x^2$ **d** $4x - 9x^3$

e $a^3b - ab^3$ **f** $50 - 2x^2y^2$ **g** $9b^3 - 4b$ **h** $x^5 - xy^4$

5 Factorise:

a $x^2 + 4x + 4$ **b** $x^2 - 10x + 25$ **c** $9x^2 + 30x + 25$

d $x^2 - 8x + 16$ **e** $4x^2 + 28x + 49$ **f** $x^2 - 20x + 100$

6 Factorise:

a $-9x^2 + 6x - 1$ **b** $3x^2 + 18x + 27$ **c** $-18x^2 + 12x - 2$

d $2x^2 - 50$ **e** $2x^2 - 16x + 32$ **f** $-3x^2 - 18x - 27$

E FACTORISING EXPRESSIONS WITH FOUR TERMS

Some expressions with four terms do not have an overall common factor, but can be factorised by pairing the four terms.

For example,

$$\underbrace{ab + ac} + \underbrace{bd + cd}$$
$$= a(b+c) + d(b+c) \qquad \{\text{factorising each pair separately}\}$$
$$= (b+c)(a+d) \qquad \{\text{removing common factor } (b+c)\}$$

Example 11 **Self Tutor**

Factorise: $3ab + d + 3ad + b$.

$$3ab + d + 3ad + b$$
$$= \underbrace{3ab + b} + \underbrace{3ad + d} \qquad \{\text{putting terms containing } b \text{ together}\}$$
$$= b(3a+1) + d(3a+1) \qquad \{\text{factorising each pair}\}$$
$$= (3a+1)(b+d) \qquad \{(3a+1) \text{ is a common factor}\}$$

EXERCISE 3E

1 Factorise:

a $2a + 2 + ab + b$
b $4d + ac + ad + 4c$
c $ab + 6 + 2b + 3a$
d $mn + 3p + np + 3m$
e $2xy - 5 + 10y - x$
f $6a - bc - 2ac + 3b$

Example 12 Self Tutor

Factorise:

a $x^2 + 2x + 5x + 10$
b $x^2 + 3x - 4x - 12$

a $\underbrace{x^2 + 2x} + \underbrace{5x + 10}$

$= x(x+2) + 5(x+2)$ {factorising each pair}

$= (x+2)(x+5)$ {$(x+2)$ is a common factor}

b $\underbrace{x^2 + 3x}\ \underbrace{- 4x - 12}$

$= x(x+3) - 4(x+3)$ {factorising each pair}

$= (x+3)(x-4)$ {$(x+3)$ is a common factor}

2 Factorise:

a $x^2 + 2x + 4x + 8$
b $x^2 + 3x + 7x + 21$
c $x^2 + 5x + 4x + 20$
d $2x^2 + x + 6x + 3$
e $3x^2 + 2x + 12x + 8$
f $20x^2 + 12x + 5x + 3$

3 Factorise:

a $x^2 - 4x + 5x - 20$
b $x^2 - 7x + 2x - 14$
c $x^2 - 3x - 2x + 6$
d $x^2 - 5x - 3x + 15$
e $x^2 + 7x - 8x - 56$
f $2x^2 + x - 6x - 3$
g $3x^2 + 2x - 12x - 8$
h $4x^2 - 3x - 8x + 6$
i $9x^2 + 2x - 9x - 2$

F FACTORISING QUADRATIC TRINOMIALS

A **quadratic trinomial** is an algebraic expression of the form $ax^2 + bx + c$ where x is a variable and a, b, c are constants, $a \neq 0$.

Consider the expansion of the product $(x+2)(x+5)$:

$$\begin{aligned}(x+2)(x+5) &= x^2 + 5x + 2x + 2 \times 5 \quad \text{\{using FOIL\}}\\ &= x^2 + [5+2]x + [2 \times 5]\\ &= x^2 + [\textbf{sum} \text{ of 2 and 5}]x + [\textbf{product} \text{ of 2 and 5}]\\ &= x^2 + 7x + 10\end{aligned}$$

$$\boldsymbol{x^2 + px + q = (x+a)(x+b)}$$

where a and b are two numbers whose sum is p, and whose product is q.

So, if we want to factorise the quadratic trinomial $x^2 + 7x + 10$ into $(x + ...)(x + ...)$ we must find two numbers to fill the vacant places which have a *sum* of 7 and a *product* of 10. The numbers are 2 and 5, so $x^2 + 7x + 10 = (x+2)(x+5)$.

Example 13 **Self Tutor**

Factorise:

a $x^2 - 7x + 12$ **b** $x^2 - 2x - 15$

a We need two numbers with sum -7 and product 12.
The numbers are -3 and -4.
$\therefore \quad x^2 - 7x + 12 = (x - 3)(x - 4)$

b We need two numbers with sum -2 and product -15.
The numbers are -5 and 3.
$\therefore \quad x^2 - 2x - 15 = (x - 5)(x + 3)$

EXERCISE 3F

1 Fully factorise:

a $x^2 + 3x + 2$ **b** $x^2 + 5x + 6$ **c** $x^2 - x - 6$ **d** $x^2 + 3x - 10$
e $x^2 + 4x - 21$ **f** $x^2 + 8x + 16$ **g** $x^2 - 14x + 49$ **h** $x^2 + 3x - 28$
i $x^2 - 11x + 24$ **j** $x^2 + 15x + 44$ **k** $x^2 - x - 56$ **l** $x^2 - 18x + 81$
m $x^2 - 4x - 32$ **n** $x^2 + 4x - 45$ **o** $x^2 - 4x - 96$ **p** $x^2 + 4x - 96$

Example 14 **Self Tutor**

Fully factorise by first removing a common factor:

a $3x^2 + 6x - 72$ **b** $77 + 4x - x^2$

a $3x^2 + 6x - 72$
$= 3(x^2 + 2x - 24)$ {3 is a common factor}
$= 3(x + 6)(x - 4)$ {sum $= 2$, product $= -24$
$\therefore$ the numbers are 6 and -4}

b $77 + 4x - x^2$
$= -x^2 + 4x + 77$ {writing in descending powers of x}
$= -1(x^2 - 4x - 77)$ {-1 is a common factor}
$= -(x - 11)(x + 7)$ {sum $= -4$, product $= -77$
$\therefore$ the numbers are -11 and 7}

2 Fully factorise by first removing a common factor:

a $2x^2 + 10x + 8$ **b** $3x^2 - 21x + 18$ **c** $2x^2 + 14x + 24$
d $5x^2 - 30x - 80$ **e** $4x^2 - 8x - 12$ **f** $3x^2 - 42x + 99$
g $2x^2 - 2x - 180$ **h** $3x^2 - 6x - 24$ **i** $2x^2 + 18x + 40$
j $x^3 - 7x^2 - 8x$ **k** $4x^2 - 24x + 36$ **l** $3x^2 + 18x - 81$
m $2x^2 - 44x + 240$ **n** $x^3 - 3x^2 - 28x$ **o** $x^4 + 2x^3 + x^2$

3 Fully factorise:

a $-x^2 - 3x + 54$ **b** $-x^2 - 7x - 10$ **c** $-x^2 - 10x - 21$
d $4x - x^2 - 3$ **e** $-4 + 4x - x^2$ **f** $3 - x^2 - 2x$

4 Fully factorise:

a $-x^2 + 2x + 48$ **b** $6x - x^2 - 9$ **c** $30x - 3x^2 - 63$
d $-2x^2 + 4x + 126$ **e** $20x - 2x^2 - 50$ **f** $-x^3 + x^2 + 2x$

5 Given that $x^2 + bx + c = (x + m)(x + n)$, factorise $x^2 - bx + c$.

G FACTORISATION OF $ax^2 + bx + c$, $a \neq 1$

In this section we will learn how to factorise quadratic trinomials where the coefficient of x^2 is not 1, and we cannot remove a common factor.

Consider the quadratic trinomial $4x^2 + 11x + 6$.

Using the FOIL rule, we observe that
$$\begin{aligned} &(4x+3)(x+2) \\ &= 4x^2 + 8x + 3x + 6 \\ &= 4x^2 + 11x + 6 \end{aligned}$$

We will now *reverse* the process to factorise $4x^2 + 11x + 6$:

$$\begin{aligned} &4x^2 + 11x + 6 \\ &= 4x^2 + 8x + 3x + 6 && \{\text{'splitting' the middle term}\} \\ &= (4x^2 + 8x) + (3x + 6) && \{\text{grouping in pairs}\} \\ &= 4x(x+2) + 3(x+2) && \{\text{factorising each pair separately}\} \\ &= (4x+3)(x+2) && \{\text{completing the factorisation}\} \end{aligned}$$

But how do we know how to correctly 'split' the middle term? How do we know that $11x$ should be written as $8x + 3x$ rather than $6x + 5x$ or $10x + x$?

INVESTIGATION 2 'SPLITTING' THE MIDDLE TERM

Consider the general quadratic trinomial $ax^2 + bx + c$.

Suppose we 'split' the middle term into $px + qx$, so $ax^2 + bx + c = ax^2 + px + qx + c$.

What to do:

1 Explain why $p + q = b$.

2 Show that $ax^2 + bx + c = x(ax + p) + (qx + c)$.

3 We can only factorise this expression further if the two terms have a common factor. This means that $ax + p = k(qx + c)$ for some k.

a By equating coefficients, show that $kq = a$ and $kc = p$.

b Hence, show that $pq = ac$.

This tells us that factorisation by 'splitting' the middle term only works if we can choose p and q such that $p + q = b$ and $pq = ac$.

4 Since $pq = ac$, we let $q = \frac{ac}{p}$. When we 'split' the middle term, we therefore either write $ax^2 + px + \frac{ac}{p}x + c$ or $ax^2 + \frac{ac}{p}x + px + c$.

Show that factorising gives the result $\left(x + \frac{c}{p}\right)(ax + p)$ in either case.

The following procedure is used to factorise $ax^2 + bx + c$ by 'splitting' the middle term:

Step 1: Find two numbers p and q whose sum is b and whose product is ac.

Step 2: Replace bx by $px + qx$.

Step 3: Complete the factorisation.

Example 15 Self Tutor

Factorise:

a $3x^2 + 17x + 10$ **b** $6x^2 - 11x - 10$

a For $3x^2 + 17x + 10$, $ac = 3 \times 10 = 30$ and $b = 17$.
We need two numbers with sum 17 and product 30. These are 2 and 15.

$$\therefore\ 3x^2 + 17x + 10 = 3x^2 + 2x + 15x + 10$$
$$= x(3x+2) + 5(3x+2)$$
$$= (3x+2)(x+5)$$

b For $6x^2 - 11x - 10$, $ac = 6 \times -10 = -60$ and $b = -11$.
We need two numbers with sum -11 and product -60. These are -15 and 4.

$$\therefore\ 6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$$
$$= 3x(2x-5) + 2(2x-5)$$
$$= (2x-5)(3x+2)$$

EXERCISE 3G

1 Consider the quadratic trinomial $3x^2 + 7x + 2$.

a Factorise the expression by 'splitting' the middle term into:

i $+6x + x$ **ii** $+x + 6x$

b Are your factorisations in **a** equivalent?

2 Fully factorise:

a $2x^2 + 5x + 3$ **b** $2x^2 + 13x + 18$ **c** $7x^2 + 9x + 2$ **d** $3x^2 + 13x + 4$

e $3x^2 + 8x + 4$ **f** $3x^2 + 16x + 21$ **g** $8x^2 + 14x + 3$ **h** $21x^2 + 17x + 2$

i $6x^2 + 5x + 1$ **j** $6x^2 + 19x + 3$ **k** $10x^2 + 17x + 3$ **l** $14x^2 + 37x + 5$

3 Consider the quadratic trinomial $4x^2 + 4x - 3$.

a Factorise the expression by 'splitting' the middle term into:

i $+6x - 2x$ **ii** $-2x + 6x$

b Are your factorisations in **a** equivalent?

4 Fully factorise:

a $2x^2 - 9x - 5$ **b** $3x^2 + 5x - 2$ **c** $3x^2 - 5x - 2$ **d** $2x^2 + 3x - 2$

e $2x^2 + 3x - 5$ **f** $5x^2 - 8x + 3$ **g** $11x^2 - 9x - 2$ **h** $2x^2 - 3x - 9$

i $3x^2 - 17x + 10$ **j** $5x^2 - 13x - 6$ **k** $3x^2 + 10x - 8$ **l** $2x^2 + 17x - 9$

m $2x^2 + 9x - 18$ **n** $15x^2 + x - 2$ **o** $21x^2 - 62x - 3$

Example 16 **Self Tutor**

Fully factorise: $-5x^2 - 7x + 6$

We remove -1 as a common factor first.

$$\begin{aligned}
&-5x^2 - 7x + 6 \\
&= -1[5x^2 + 7x - 6] \\
&= -[5x^2 + 10x - 3x - 6] \\
&= -[5x(x+2) - 3(x+2)] \\
&= -[(x+2)(5x-3)] \\
&= -(x+2)(5x-3)
\end{aligned}$$

For $5x^2 + 7x - 6$, $ac = -30$ and $b = 7$. The two numbers with product -30 and sum 7 are 10 and -3.

5 Fully factorise by first removing -1 as a common factor:

- **a** $-3x^2 - x + 14$
- **b** $-5x^2 + 11x - 2$
- **c** $-4x^2 - 9x + 9$
- **d** $-9x^2 + 12x - 4$
- **e** $-8x^2 - 14x - 3$
- **f** $-12x^2 + 16x + 3$

6
- **a** Show that $(3x+5)^2 - (2x-3)^2 = 5x^2 + 42x + 16$ by expanding the LHS.
- **b** Factorise $5x^2 + 42x + 16$ by 'splitting' the middle term.
- **c** Factorise $(3x+5)^2 - (2x-3)^2$ using the difference of two squares.

H MISCELLANEOUS FACTORISATION

In the following **Exercise** you will need to determine which factorisation method to use.

This flowchart may prove useful:

Expression to be factorised.

↓

Remove any **common factors**.

↓

- Look for the **difference of two squares**.
- Look for **perfect squares**.
- For four terms, look for **grouping in pairs**.
- Look for the **sum and product** type.
- Look for **'splitting' the middle term**.

EXERCISE 3H

1 Fully factorise:

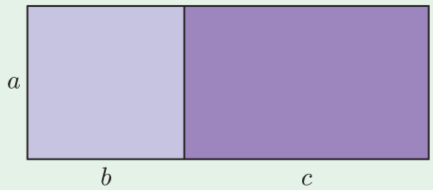
- **a** $3x^2 + 2x$
- **b** $x^2 - 81$
- **c** $2p^2 + 8$
- **d** $3b^2 - 75$
- **e** $2x^2 - 32$
- **f** $n^4 - 4n^2$
- **g** $x^2 - 8x - 9$
- **h** $d^2 + 6d - 7$
- **i** $x^2 + 8x - 9$
- **j** $4t + 8t^2$
- **k** $4x^2 + 12x + 5$
- **l** $2g^2 - 12g - 110$
- **m** $4a^2 - 9d^2$
- **n** $5a^2 - 5a - 10$
- **o** $2c^2 - 8c + 6$
- **p** $2x^2 + 17x + 21$
- **q** $d^4 + 2d^3 - 3d^2$
- **r** $x^3 + 4x^2 + 4x$

2 Fully factorise:

a $7x - 35y$ **b** $2g^2 - 8$ **c** $-5x^2 - 10x$
d $m^2 + 3mp$ **e** $a^2 + 8a + 15$ **f** $m^2 - 6m + 9$
g $5x^2 + 5xy - 5x^2y$ **h** $xy + 2x + 2y + 4$ **i** $y^2 + 5y - 9y - 45$
j $2x^2 + 10x + x + 5$ **k** $3y^2 - 147$ **l** $6x^2 - 29x - 5$
m $4c^2 - 1$ **n** $3x^2 + 3x - 36$ **o** $2bx - 6b + 10x - 30$
p $12x^2 + 13x + 3$ **q** $-2x^2 - 6 + 8x$ **r** $16x^2 + 8x + 1$
s $4x^2 - 2x^3 - 2x$ **t** $(a + b)^2 - 9$ **u** $12x^2 - 38x + 6$

REVIEW SET 3A

1 Use the diagram alongside to show that $a(b + c) = ab + ac$.



2 Expand and simplify:

a $(x + 5)(x - 6)$ **b** $(2x + 5)(3x - 1)$
c $(x + 3)(x + 2) - (2x - 1)(x - 6)$

3 Fully factorise:

a $7x^2 - 4x$ **b** $x^3 + 5x^2 - 6x$ **c** $x(x - 8) + 5(x - 8)$

4 Expand and simplify:

a $(x + 5)(x - 2)(x + 1)$ **b** $(2x - 3)(x^2 + 4x + 2)$

5 Fully factorise:

a $16 - 9m^2$ **b** $x^3 - 81x$ **c** $(x + 7)^2 - 25$

6 Expand and simplify:

a $(t + 7)(t - 7)$ **b** $(2y + 5)(2y - 5)$ **c** $(2m - 5n)^2$

7 Fully factorise:

a $2x^2 + 20x + 50$ **b** $2b - dc + 2d - bc$

8 Use the binomial expansion of $(a + b)^3$ to expand and simplify:

a $(2k + 3)^3$ **b** $(r - 4t)^3$

9 Fully factorise:

a $x^2 + 7x - 18$ **b** $3x^2 - 9x - 30$ **c** $64 - 2x^2 + 8x$

10 Fully factorise:

a $8x^2 + 10x + 3$ **b** $5x^2 - 13x + 6$ **c** $-9x^2 + 3x + 2$

11 **a** Show that $(2x + 9)^2 - (x - 3)^2 = 3x^2 + 42x + 72$ by expanding the LHS.
b Factorise $3x^2 + 42x + 72$ by first taking out a common factor.
c Factorise $(2x + 9)^2 - (x - 3)^2$ using the difference of two squares.

12 **a** Write down the binomial expansion of:

i $(a+b)^2$ **ii** $(a+b)^3$ **iii** $(a+b)^4$ **iv** $(a+b)^5$

b In $(a+b)^2 = a^2+2ab+b^2$, the sum of the coefficients of the expansion is $1+2+1=4$.
Find the sum of the coefficients in the expansion of:

i $(a+b)^3$ **ii** $(a+b)^4$ **iii** $(a+b)^5$

c What do you suspect is the sum of the coefficients in the expansion of $(a+b)^n$?

d Prove your result by letting $a=b=1$.

REVIEW SET 3B

1 Expand and simplify:

a $5(4x-5)$ **b** $-4x(x-3)$ **c** $2(x+6)+x(3x-7)$

2 Expand and simplify:

a $x(x^2-3)+5(x-4)$ **b** $(a+b)(a-b)-(a+2b)(a-2b)$

3 Fully factorise:

a $2x^2-98$ **b** $(3x+1)^2-(x-4)^2$

4 Answer the **Opening Problem** on page **48**.
Hint: The 2 digit number with digit form 'ab' represents the value $10a+b$.

5 Fully factorise:

a $x^2+3x-54$ **b** $3x^2+24x+48$

6 How many terms would you obtain by expanding $(a+b+c+d)(e+f)(g+h)$?

7 Expand and simplify:

a $(3x^2-5)^2$ **b** $(2a-b)^3$

8 Fully factorise:

a $x^2-5x-66$ **b** $2x^2+20x-78$ **c** $4x^2-8x-21$

9 Expand and simplify: $(x^2-x+4)(x^2+2x+3)$

10 Fully factorise:

a $-x^2+x+12$ **b** $-6x^2-5x+50$

11 Consider factorising the expression $6x^2+17x+12$.

a Explain why the middle term $17x$ should be 'split' into $9x$ and $8x$.

b Factorise $6x^2+17x+12$ by writing $17x$ as $9x+8x$.

c Now factorise $6x^2+17x+12$ by writing $17x$ as $8x+9x$. Check that you get the same answer as in **b**.

12 **a** Use your calculator to find:

i 23^2 and 27^2 **ii** 18^2 and 32^2 **iii** 11^2 and 39^2 **iv** 14^2 and 36^2.

b If a and b are two integers whose sum is 50, what can we say about the last 2 digits of the squares a^2 and b^2?

c Prove that your answer to **b** is correct.
Hint: Write b in terms of a, then find the difference between the two squares.