Chapter 3

Algebraic expansion and factorisation

Contents:

OPENING PROBLEM

Jody showed her friend Leanne a trick for performing multiplications of 2 digit numbers, such as 42×83:

Step 1: *Multiply the digits in the units column.*
$2 \times 3 = 6$

$$\begin{array}{rrr} & 4 & 2 \\ \times & 8 & 3 \\ \hline & & 6 \end{array}$$

Step 2: *Multiply the digits along the diagonals, then add the results.*
$(4 \times 3) + (8 \times 2) = 28,$ so we write 8 and carry the 2.

$$\begin{array}{rrr} & 4 & 2 \\ \times & 8 & 3 \\ {\scriptstyle 2} & & \\ \hline & 8 & 6 \end{array}$$

Step 3: *Multiply the digits in the tens column.*
$4 \times 8 = 32,$ adding the 2 gives 34.

$$\begin{array}{rrrr} & & 4 & 2 \\ & \times & 8 & 3 \\ \hline 3 & 4 & 8 & 6 \end{array}$$

So, $42 \times 83 = 3486$.

Things to think about:

Can you use algebra to explain why this trick works?

The study of **algebra** is vital for many areas of mathematics. We need it to manipulate equations, solve problems for unknown variables, and also to develop higher level mathematical theories.

In this chapter we revise the **expansion** of expressions which involve brackets, and the reverse process which is called **factorisation**.

A REVISION OF EXPANSION LAWS

In this section we revise the laws for expanding algebraic expressions.

DISTRIBUTIVE LAW

$$a(b+c) = ab + ac$$

Example 1 Self Tutor

Expand the following:

a $2(3x-1)$ **b** $-3x(x+2)$

a $2(3x-1)$
$= 2 \times 3x + 2 \times (-1)$
$= 6x - 2$

b $-3x(x+2)$
$= -3x \times x + -3x \times 2$
$= -3x^2 - 6x$

THE PRODUCT $(a+b)(c+d)$

$$(a+b)(c+d) = ac + ad + bc + bd$$

Example 2 Self Tutor

Expand and simplify:

a $(x+4)(x-3)$ **b** $(2x-5)(-x+3)$

a $(x+4)(x-3)$
$= x \times x + x \times (-3) + 4 \times x + 4 \times (-3)$
$= x^2 - 3x + 4x - 12$
$= x^2 + x - 12$

b $(2x-5)(-x+3)$
$= 2x \times (-x) + 2x \times 3 - 5 \times (-x) - 5 \times 3$
$= -2x^2 + 6x + 5x - 15$
$= -2x^2 + 11x - 15$

DIFFERENCE OF TWO SQUARES

$$(a+b)(a-b) = a^2 - b^2$$

Example 3 Self Tutor

Expand and simplify:

a $(x+4)(x-4)$ **b** $(3x-2)(3x+2)$

a $(x+4)(x-4)$
$= x^2 - 4^2$
$= x^2 - 16$

b $(3x-2)(3x+2)$
$= (3x)^2 - 2^2$
$= 9x^2 - 4$

PERFECT SQUARES EXPANSION

$$(a+b)^2 = a^2 + 2ab + b^2$$

Example 4 Self Tutor

Expand and simplify:

a $(2x+1)^2$ **b** $(3-4y)^2$

a $(2x+1)^2$
$= (2x)^2 + 2 \times 2x \times 1 + 1^2$
$= 4x^2 + 4x + 1$

b $(3-4y)^2$
$= 3^2 + 2 \times 3 \times (-4y) + (-4y)^2$
$= 9 - 24y + 16y^2$

EXERCISE 3A

1 Expand and simplify:

a $3(2x+5)$ **b** $4x(x-3)$ **c** $-2(3+x)$

d $-3x(x+y)$ **e** $2x(x^2-1)$ **f** $-x(1-x^2)$

g $-ab(b-a)$ **h** $x^2(x-3)$ **i** $3(a^2+3a+1)$

j $5(x^2-3x+2)$ **k** $-4(2c^2-3c-7)$ **l** $2a(3a^2-5a+1)$

2 Expand and simplify:

a $2(x+3)+5(x-4)$ **b** $2(3-x)-3(4+x)$ **c** $x(x+2)+2x(1-x)$

d $x(x^2+2x)-x^2(2-x)$ **e** $a(a+b)-b(a-b)$ **f** $x^2(6-x)+3x(x-4)$

3 Expand and simplify:

a $(x+2)(x+5)$ **b** $(x-3)(x+4)$ **c** $(x+5)(x-3)$

d $(x-2)(x-10)$ **e** $(2x+1)(x-3)$ **f** $(3x-4)(2x-5)$

g $(2x+y)(x-y)$ **h** $(x+3)(-2x-1)$ **i** $(x+2y)(-x-1)$

4 Expand and simplify:

a $(x+3)(x-1)+3(x-5)$ **b** $(x+7)(x-5)+(x+1)(x+4)$

c $(2x+3)(x-2)-(x+1)(x+6)$ **d** $(4t-3)(t+1)-(2t-1)(2t+5)$

e $(4x-1)(3-x)+(2x-3)(3x-2)$ **f** $5(3x-4)(x+2)-(7-x)(8-5x)$

5 Expand and simplify:

a $(x+7)(x-7)$ **b** $(3+a)(3-a)$ **c** $(5-x)(5+x)$

d $(2x+1)(2x-1)$ **e** $(4-3y)(4+3y)$ **f** $(3x-4z)(4z+3x)$

6 Expand and simplify:

a $(x+3)(x-3)-(x+6)(x-6)$ **b** $(5p-2)(5p+2)-p(3p-1)$

c $(3y-z)(3y+z)-(2y+3z)(2y-3z)$ **d** $(10-x^2)(10+x^2)-(10-3x^2)(10+3x^2)$

7 Expand and simplify:

a $(x+5)^2$ **b** $(2x+3)^2$ **c** $(7+x)^2$

d $(3x+4)^2$ **e** $(5+x^2)^2$ **f** $(3x^2+2)^2$

g $(5x+3y)^2$ **h** $(2x^2+7y)^2$ **i** $(x^3+8x)^2$

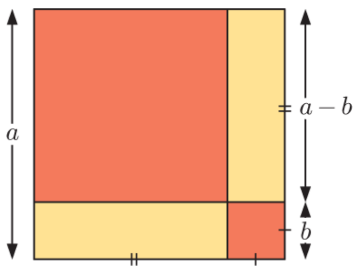
8 Expand and simplify:

a $(x-3)^2$ **b** $(2-x)^2$ **c** $(3x-1)^2$

d $(6-5p)^2$ **e** $(2x-5y)^2$ **f** $(ab-2)^2$

g $(x^2-5)^2$ **h** $(4x^2-3y)^2$ **i** $(-x^2-y^2)^2$

9 Use the diagram alongside to show that $(a-b)^2=a^2-2ab+b^2$.



10 Expand and simplify:

a $(x+9)^2+(x-2)^2$ **b** $(3x+1)^2-(2x-3)^2$ **c** $(x+8)^2-(x+2)(x-5)$

d $(5-p)^2+(p^2-4)^2$ **e** $(3x^2-1)^2-4(1-x)^2$ **f** $(5x+y^2)^2-x(x^2-y)^2$

B FURTHER EXPANSION

When expressions containing more than two terms are multiplied together, we can still use the distributive law to expand the brackets. Each term in the first set of brackets is multiplied by each term in the second set of brackets.

If there are 2 terms in the first brackets and 3 terms in the second brackets, there will be $2 \times 3 = 6$ terms in the expansion. However, when we simplify by collecting like terms, the final answer may contain fewer terms.

Example 5 Self Tutor

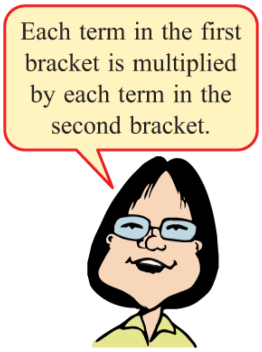
Expand and simplify: $(x+3)(x^2+2x+4)$

$(x+3)(x^2+2x+4)$

$= x^3+2x^2+4x$ {$x \times$ each term in 2nd bracket}

$\quad + 3x^2+6x+12$ {$3 \times$ each term in 2nd bracket}

$= x^3+5x^2+10x+12$ {collecting like terms}



EXERCISE 3B

1 Expand and simplify:

a $(x+2)(x^2+x+4)$ **b** $(x+3)(x^2+2x-3)$ **c** $(x+3)(x^2+2x+1)$

d $(x+1)(2x^2-x-5)$ **e** $(2x+3)(x^2+2x+1)$ **f** $(2x-5)(x^2-2x-3)$

g $(x+5)(3x^2-x+4)$ **h** $(4x-1)(2x^2-3x+1)$

Example 6 Self Tutor

Expand and simplify: $(x+1)(x-3)(x+2)$

$(x+1)(x-3)(x+2)$

$= (x^2-3x+x-3)(x+2)$ {expanding first two factors}

$= (x^2-2x-3)(x+2)$ {collecting like terms}

$= x^3+2x^2-2x^2-4x-3x-6$ {expanding remaining factors}

$= x^3-7x-6$ {collecting like terms}

2 Expand and simplify:

a $(x+4)(x+3)(x+2)$ **b** $(x-3)(x-2)(x+4)$ **c** $(x-3)(x-2)(x-5)$

d $(2x-3)(x+3)(x-1)$ **e** $(4x+1)(3x-1)(x+1)$ **f** $(2-x)(3x+1)(x-7)$

g $(x-2)(4-x)(3x+2)$ **h** $(x+3)^3$ **i** $(x-2)^3$

3 State how many terms you would obtain by expanding:

a $(a+b)(c+d)$ **b** $(a+b+c)(d+e)$ **c** $(a+b)(c+d+e)$

d $(a+b+c)(d+e+f)$ **e** $(a+b)(c+d)(e+f)$ **f** $(a+b+c)(d+e)(f+g)$

4 Expand and simplify:

a $(x^2+3x+1)(x^2-x+3)$ **b** $(2x^2+x-1)(x^2+3x-2)$

c $(3x^2+x-4)(2x^2-3x+1)$ **d** $(x^2-3x+2)(x+5)(x-3)$

C THE BINOMIAL EXPANSION

Consider $(a+b)^n$ where n is a positive integer.

$\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.

The **binomial expansion** of $(a+b)^n$ is obtained by writing the expression without brackets.

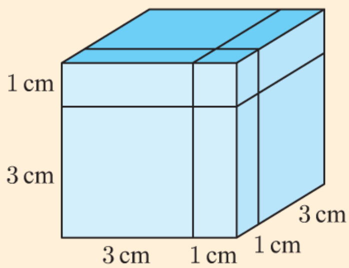
INVESTIGATION 1 THE BINOMIAL EXPANSION OF $(a+b)^3$

In this Investigation we discover the binomial expansion of $(a+b)^3$.

What to do:

1 Find a large potato and cut it to obtain a 4 cm by 4 cm by 4 cm cube.

2 By making 3 cuts parallel to the cube's surfaces, divide the cube into 8 rectangular prisms as shown.



3 How many prisms are:

a 3 by 3 by 3 **b** 3 by 3 by 1

c 3 by 1 by 1 **d** 1 by 1 by 1?

4 Now instead of the 4 cm $\times$ 4 cm $\times$ 4 cm potato cube, suppose you had a cube with edge length $(a+b)$ cm.

a Explain why the volume of the cube is given by $(a+b)^3$.

b Suppose you made cuts so each edge was divided into a cm and b cm. How many prisms would be:

i a by a by a **ii** a by a by b

iii a by b by b **iv** b by b by b?

DEMO

c By adding the volumes of the 8 rectangular prisms, find an expression for the total volume. Hence write down the binomial expansion of $(a+b)^3$.

Another method of finding the binomial expansion of $(a+b)^3$ is to expand the brackets:

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

$$\boldsymbol{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}.$$

The binomial expansion of $(a+b)^3$ can be used to expand other perfect cubes.

Example 7 — Self Tutor

Expand and simplify using the rule $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

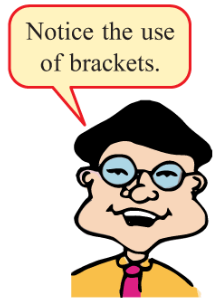
a $(x+4)^3$ **b** $(3x-2)^3$

a We substitute $a = x$ and $b = 4$.

$$\therefore (x+4)^3 = x^3 + 3 \times x^2 \times 4 + 3 \times x \times 4^2 + 4^3$$
$$= x^3 + 12x^2 + 48x + 64$$

b We substitute $a = (3x)$ and $b = (-2)$.

$$\therefore (3x-2)^3 = (3x)^3 + 3 \times (3x)^2 \times (-2) + 3 \times (3x) \times (-2)^2 + (-2)^3$$
$$= 27x^3 - 54x^2 + 36x - 8$$



EXERCISE 3C

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(x+1)^3$ **b** $(x+3)^3$ **c** $(x+5)^3$ **d** $(x+y)^3$

e $(x-1)^3$ **f** $(x-5)^3$ **g** $(x-4)^3$ **h** $(x-y)^3$

i $(2+y)^3$ **j** $(2x+1)^3$ **k** $(3x+1)^3$ **l** $(2y+3x)^3$

m $(2-y)^3$ **n** $(2x-1)^3$ **o** $(3x-1)^3$ **p** $(2y-3x)^3$

2 By expanding and simplifying $(a+b)^3(a+b)$, show that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

3 Use the binomial expansion $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(x+y)^4$ **b** $(x+1)^4$ **c** $(x+2)^4$ **d** $(x+3)^4$

e $(x-y)^4$ **f** $(x-1)^4$ **g** $(x-2)^4$ **h** $(2x-1)^4$

4 Consider:

$$\begin{aligned}
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

The expressions on the right hand side of each identity contain the coefficients:

```
        1     1
      1    2    1
   1    3    3    1
1    4    6    4    1
```

This triangle of numbers is called **Pascal's triangle**.

a Predict the next two rows of Pascal's triangle, and explain how you found them.

b Hence, write down the binomial expansion for:

i $(a+b)^5$ **ii** $(a-b)^5$ **iii** $(a+b)^6$ **iv** $(a-b)^6$

c **i** Expand and simplify $(x-2)^5$.

ii Check your answer by substituting $x = 1$ into your expansion.