6 Introducing differential calculus

CHAPTER OBJECTIVES:

7.1 Concept of the derivative as a rate of change; tangent to a curve

7.2 The principle that $f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$; the derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where all exponents are integers

7.3 Gradients of curves for given values of x; values of x where $f'(x)$ is given; equation of the tangent at a given point; equation of the line perpendicular to the tangent at a given point (normal)

7.4 Values of x where the gradient of a curve is zero; solution of $f'(x) = 0$; stationary points; local maximum and minimum points

7.5 Optimization problems

Before you start

You should know how to:

1 Use function notation, e.g. If $f(x) = 3x + 7$ what is $f(2)$? $f(2) = 3 \times 2 + 7 = 13$

2 Rearrange formulae, e.g. Make x the subject of the formula:

$$y = 3x + 7$$
$$y - 7 = 3x \Rightarrow \frac{y-7}{3} = x$$

3 Use index notation, e.g. Write without powers

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$$

4 Use the laws of indices, e.g. Simplify:

$5^2 \times 5^4 = 5^{2+4} = 5^6$

$5^4 \div 5^6 = 5^{4-6} = 5^{-2}$

5 Find the equation of a straight line given its gradient and a point, e.g. The line passing through the point (2, 13) with gradient 3

$(y - 13) = 3(x - 2)$

$y - 13 = 3x - 6$

$y = 3x + 7$

Skills check

1 **a** $f(z) = 3 - 2z$, evaluate $f(5)$ and $f(-5)$

b $f(t) = 3t + 5$, evaluate $f(2)$ and $f(-3)$

c $g(y) = y^2$, evaluate $g(5)$ and $g\left(\frac{1}{2}\right)$

d $g(z) = \frac{3}{z}$, evaluate $g(2)$ and $g(15)$

e $f(z) = \frac{z^2}{z+1}$, evaluate $f(4)$ and $f(-3)$

2 Make r the subject of the formula:

a $C = 2\pi r$ **b** $A = \pi r^2$ **c** $A = 4\pi r^2$

d $V = \frac{\pi r^2 h}{3}$ **e** $V = \frac{2\pi r^3}{3}$ **f** $C = \frac{2A}{r}$

3 Write these without powers.

a 4^2 **b** 2^{-3} **c** $\left(\frac{1}{2}\right)^4$

4 Write each expression in the form x^n:

a $\frac{1}{x}$ **b** $\frac{1}{x^4}$ **c** $\frac{x^3}{x}$ **d** $\frac{x^2}{x^5}$ **e** $\frac{\left(x^2\right)^3}{x^5}$

5 Find the equation of the line that passes through

a the point (5, −3) with gradient 2

b the point (4, 2) with gradient −3.

The invention of the differential calculus, in the 17th century, was a milestone in the development of mathematics.

At its simplest it is a method of finding the gradient of a **tangent** to a curve. The gradient of the tangent is a measure of how quickly the function is changing as the x-coordinate changes.

All things move, for example, the hands on a clock, the sprinter in a 100 m race, the molecules in a chemical reaction, the share values on the stock market. Mathematics can be used to model all of these situations. Since each situation is dynamic, the models will involve differential calculus.

For more on the history of calculus, see pages 292–3.

In this chapter, you will investigate certain functions to discover for yourself the method of finding the gradient of a tangent to a curve, and check that this method can be applied to all similar curves. You will apply this technique in a variety of situations, to solve problems about graphs and to use mathematical models in 'real-world' problems.

In the photograph, all the cans have the same basic cylindrical shape. However, they are all different sizes. By the end of this chapter you will be able to determine the optimal design of a cylindrical can – one that uses the smallest amount of metal to hold a given capacity.

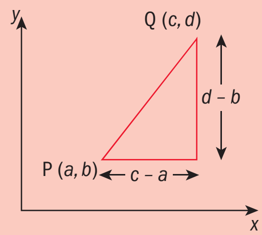
6.1 Introduction to differentiation

You have already met the concept of the gradient of a straight line. **Differentiation** is the branch of mathematics that deals with **gradient functions** of curves.

The gradient measures how fast y is increasing compared to the rate of increase of x.

The gradient of a straight line is constant, which means its direction never changes. The y-values increase at a constant rate.

→ If P is the point (a, b) and Q is (c, d) then the gradient, m, of the straight line PQ is $m = \frac{d - b}{c - a}$.

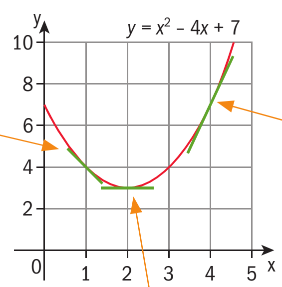


To calculate the gradient of a curve at a particular point you need to draw a tangent at that point. A tangent is a line that just touches the curve.

Here is the curve $y = x^2 - 4x + 7$.
It is a **quadratic** function. Its **vertex** is at the point (2, 3).

The three tangents to the curve are shown in blue.

At the point (1, 4), the curve is decreasing, the gradient of the curve is negative and the tangent to the curve has a negative gradient.



At the point (4, 7), the curve is increasing, the gradient of the curve is positive and the tangent to the curve has a positive gradient.

At the point (2, 3), the gradient of the curve is zero, and the tangent to the curve is horizontal.

The direction of a tangent to the curve changes as the x-coordinate changes. Therefore the gradient of the curve is not constant.

So, for any curve $y = f(x)$ which is not a straight line, its gradient changes for different values of x. The gradient can be expressed as a **gradient function**.

→ Differentiation is a method used to find the equation of the gradient function for a given function, $y = f(x)$.

Extension material on CD:
Worksheet 6 - More about functions

Investigation – tangents and the gradient function

The tangent to a graph at a given point is the straight line with its gradient equal to that of the curve **at that point**. If you find the gradient of the tangent, then you have also found the gradient of the curve at that point. Repeating this for different points, we can use the data obtained to determine the gradient function for the curve.

GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

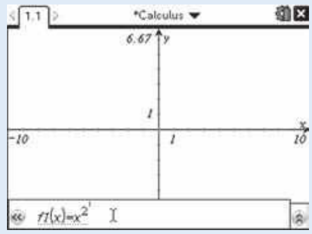
1 Plot the curve $y = x^2$ on your GDC

Open a new document and add a Graphs page.

Save the document as 'Calculus'.

Enter x^2 into the function $f1(x)$.

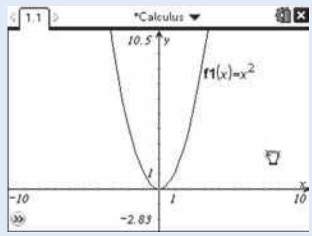
Press enter del.



To get a better view of the curve, you should pan the axes in order to see more of it.

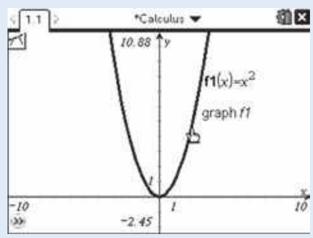
Click down on the touchpad in an area away from the axes, function or any labels.

The ➘ will change to ✋.



Move the grasping hand with the touchpad. The window view will pan with it.

Click the touchpad when the window is in the required position.

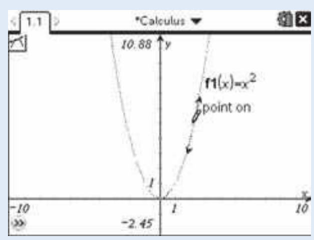


2 Add a tangent to the curve

Press menu 7:Points & Lines | 7:Tangent

Press enter.

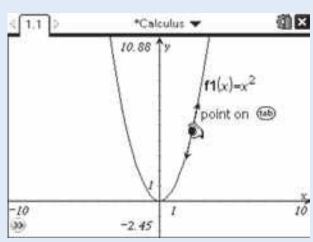
Move the ➘ with the touchpad towards the curve. It will change to a ☝ and the curve will be highlighted.



Click the touchpad.

Choose a point on the curve by clicking the touchpad.

Now you have a tangent drawn at a point on the curve that you can move round to any point on the curve. To get some more information about the tangent, you need the coordinates of the point and the equation of the tangent.



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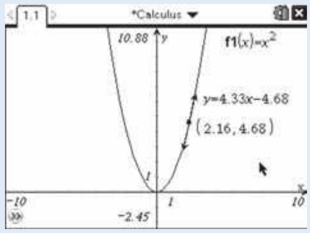
Move the ↖ with the touchpad towards the point.
It will change to a ☜ and you will see 'point on tab'.
Press ctrl menu and select 7: Coordinates and Equations. Press enter.

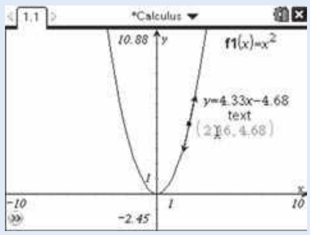
3 Find the equation of the tangent.



Move the ↖ with the touchpad towards the arrow at the end of the tangent. It will change to a ☜ and you will see 'line tab'.

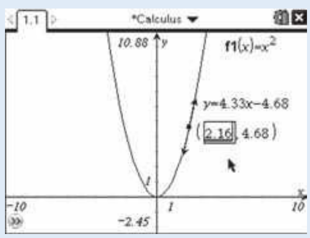
Press ctrl menu and select 7:Coordinates and Equations. Press enter.
You should now have the coordinates of the point and the equation of the tangent labeled.

4 Edit the *x*-coordinate so that the point moves to (1, 1)



Move the ↖ with the touchpad towards the arrow at the *x*-coordinate of the point. It will change to a ☜ and you will see the numbers lighten and the word 'text' appears.

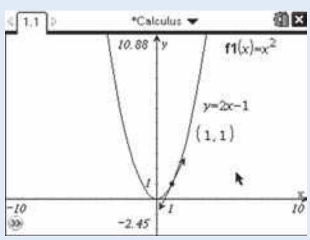
If you move the ↖ very slightly with the touchpad it will change to a I. When it does, click the touchpad.



The *x*-coordinate is now ready for editing.

This is quite tricky and may take a bit of practice. If it does not work, press esc and start again.

Use the del key to delete the current value and type 1. Press enter.



You have drawn the tangent to the curve $y = x^2$ at the point (1, 1)

Its equation is $y = 2x - 1$, so gradient of the tangent is 2.

Remember:
In the equation of a straight line $y = mx + c$, m is the gradient

5 Record this information in a table.

$y = x^2$

x-coordinate	−3	−2	−1	0	1	2	3	4	*x*
Gradient of tangent					2				

Worksheet on CD: *This table is Worksheet 6.1 on the CD.*

6 Complete the table

Go back to the graph and edit the *x*-coordinate again. Change it to 2. Write the gradient of the tangent at the point where the *x*-coordinate is 2 in your table. Repeat this until you have completed the table for all values of *x* between −3 and 4.

Continued on next page

7 Look for a simple formula that gives the gradient of the tangent for any value of *x*

Write this formula in the bottom right cell in your copy of the table. Is this formula valid for all values of *x*? Try positive, negative and fractional values.

8 Repeat Steps 1–7 for the curve $y = 2x^2$

Draw the curve, then the tangents and complete this table.

$y = 2x^2$

x-coordinate	−3	−2	−1	0	1	2	3	4	x
Gradient of tangent									

Worksheet on CD: *This table is Worksheet 6.1 on the CD.*

Again, look for a simple formula that gives the gradient of the tangent for any value of *x*. Write it down.

You can repeat this process for other curves, but there is an approach that will save time. The formulae you found in the investigation are called the **gradient functions** of the curves. The gradient function can be written in several ways:

$$\frac{dy}{dx}, \quad \frac{d}{dx}(f(x)), \quad \text{or } f'(x).$$

You can use your GDC to draw a graph of the gradient function for any curve.

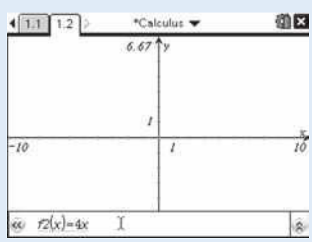
Investigation – GDC and the gradient function

1 Use the GDC to draw the gradient function of $y = 4x$

Add a new Graphs page to your document.

Enter 4*x* into the function *f*2(*x*).

Press enter.



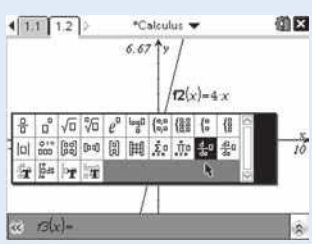
GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

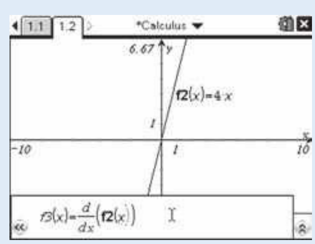
2 Enter the gradient function in *f*3(*x*)

Click the symbol using the touchpad to open the entry line at the bottom of the work area.

Press and use the ◀▶▲▼ keys to select the $\frac{d}{d\square}\square$ template.

Press enter.





Enter *x* and *f*2(*x*) in the template as shown.
Press enter.

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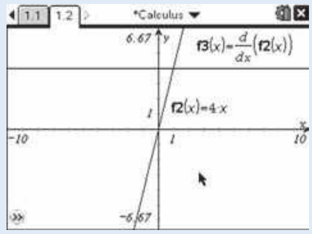
You should have this diagram, with a horizontal line across the graph.



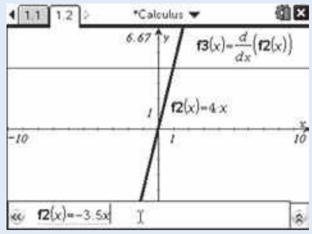
The graph plotter gives you a picture of the gradient function – you have to find the equation of this function.

The GDC drew the line $y = 4$.

The gradient of the line $y = 4x$ is '4'.

3 Repeat for other functions

Click the ›› symbol using the touchpad to open the entry line at the bottom of the work area.



Use the ▲ key to select $f2(x)$.

Enter a new function to replace $4x$.

In this way find the gradient functions for these straight lines.

a $y = -3.5x$

b $y = 2x + 4$

c $y = 5$

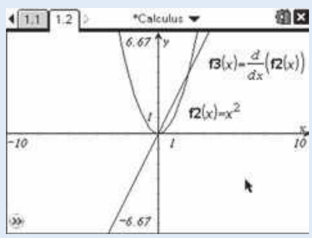
d $y = 3 - x$

e $y = -3.5$

f $y = 2 - \frac{1}{2}x$

> Take care to use the (-) key to enter the – in $-3.5x$ and the (–) key to enter the – in $y = 3 - x$.

4 Change the function to $y = x^2$



A straight line will appear on your screen as in the diagram on the right.

Write down the equation of this new straight line.

The GDC drew the line $y = 2x$.

The gradient function of the curve $y = x^2$ is '$2x$'.

This is the same result that you found by observation in the previous investigation.

Repeat for the curves $y = 2x^2$ and $y = 3x^2$ and write down the gradient functions for these curves.

5 Tabulate your results

You are now building up a set of results that you can use to generalize. To help with this, summarize your findings in a table. You should be able to see patterns in the results.

Continued on next page

Curve	$y = 4x$	$y = -3.5x$	$y = 2x + 4$	$y = 5$	$y = 3 - x$	$y = -3.5$	$y = 2 - \frac{1}{2}x$
Gradient function	4						

Curve	$y = x^2$	$y = 2x^2$	$y = 3x^2$	$y = 4x^2$	$y = -x^2$	$y = -2x^2$	$y = \frac{1}{2}x^2$
Gradient function	$2x$						

6 Extend your results

Complete this table for the curve $y = x^2 + 3x$ using the method from the first Investigation, on page 257.

Worksheet on CD: *This table is Worksheet 6.2 on the CD.*

$\boldsymbol{y = x^2 + 3x}$

x-coordinate	–3	–2	–1	0	1	2	3	4
Gradient of tangent								

What is the algebraic rule that connects the answers for the gradient to the x-coordinates?

Check that your answer is correct by entering $x^2 + 3x$ in $f2(x)$ in the graphs page (Step 2 of this investigation) so that the GDC draws the gradient function.

What is the equation of this straight line?
Is its equation the same as the rule you found?

These should be the same! If they are not, check with your teacher.

Use your GDC to find the gradient functions for the curves below. Look for a pattern developing.

a $y = x^2 + 3x$ **b** $y = x^2 - 5x$ **c** $y = 2x^2 - 3x$ **d** $y = 3x^2 - x$

e $y = 5x - 2x^2$ **f** $y = 2x - x^2$ **g** $y = x^2 + 4$ **h** $y = x^2 - 2$

i $y = 3 - x^2$ **j** $y = x^2 + x - 2$ **k** $y = 2x^2 - x + 3$ **l** $y = 3x - x^2 + 1$

Compare each curve to its gradient function and so determine the formula for the gradient function for the general quadratic curve

$$y = ax^2 + bx + c$$

Write down the gradient functions of the following curves **without using the GDC**.

1 $y = 5x^2 + 7x + 3$

2 $y = 5x + 7x^2 - 4$

3 $y = 3 + 0.5x^2 - 6x$

4 $y = 4 - 1.5x^2 + 8x$

Do not proceed until you have answered these questions correctly.

Investigation – the gradient function of a cubic curve

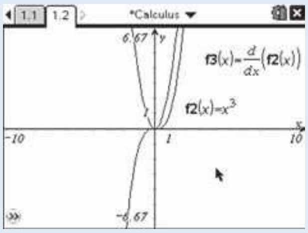
GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

Now consider the simplest cubic curve $y = x^3$.

Change the function to $y = x^3$ using the GDC.

To enter x^3, press [x] [^] [3] ▶.

(You will need to press the ▶ key to get back to the base line from the exponent.)



This time a curve appears, instead of a straight line.

Find the equation of the curve.

This is the gradient function of $y = x^3$.

Once you have the equation of the curve, find the gradient function of $y = 2x^3$, $y = 3x^3$, …
Write down your answers in the worksheet copy of the table.

Have a guess at the equation of the curve. Enter your guess to the gradient function. Adjust your equation until it fits. Then delete it.

Curve	$y = x^3$	$y = 2x^3$	$y = 3x^3$	$y = 4x^3$	$y = -x^3$	$y = -2x^3$	$y = \frac{1}{2}x^3$
Gradient function							

Extend your investigation so that you can find the gradient function of **any** cubic.
Be systematic, so try simple cubic curves first…

Worksheet on CD: *This table is Worksheet 6.3 on the CD.*

Curve	$y = x^3 + 4$	$y = 2x^3 - 3$	$y = x^3 + 5x$	$y = x^3 - 2x$	$y = x^3 + 2x^2$	$y = 2x^3 + \frac{1}{2}x^2$
Gradient function						

Then move on to more complicated cubic curves…

Curve	$y = x^3 + 3x^2 + 2$	$y = x^3 + 4x^2 + 3x$	$y = x^3 + 5x^2 - 4x + 1$	$y = x^3 - x^2 - 5x - 4$
Gradient function				

Generalize your results to determine the formula for the gradient function for the general cubic curve $y = ax^3 + bx^2 + cx + d$

You now have results for the gradient functions of linear functions, quadratic functions and cubic functions. Complete the worksheet copy of the table with these.

Function	Formula	Gradient function
Constant	$y = a$	
Linear	$y = ax + b$	
Quadratic	$y = ax^2 + bx + c$	
Cubic	$y = ax^3 + bx^2 + cx + d$	

Continued on next page

Investigation – the gradient function of any curve

In this investigation you find the gradient function of **any** curve.

Again, take a systematic approach.

1 Find the gradient function of $y = x^4$

2 Find the gradient function of $y = x^5$

3 Generalize these results to find the gradient function of $y = x^n$

GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

Up to this point, all the powers in your curve have been **positive**.

Consider the curves $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, … as well.

To enter $\frac{1}{x}$ on your GDC use the key and select from the template menu.

Remember that $\frac{1}{x} = x^{-1}$

The final result

Function	Gradient function
$y = ax^n$	

Check this result with your teacher. Do not go on until you have done so.

The process of finding the gradient function of a curve is known as **differentiation.** In these investigations, you have learned for yourself how to differentiate.

Finding this result by investigation is not the same as *proving* it to be true. How, without proof, do we know that a result arrived at by pattern building is **always** true?

6.2 The gradient function

Differentiation is the algebraic process used to find the gradient function of a given function.

Two forms of notation are used for differentiation. The notation that you use will depend on the notation used in the question.

Calculus was discovered at almost the same time by both the British mathematician Isaac Newton (1642–1727), and the German mathematician Gottfried Leibniz (1646–1716). The controversy over the rival claims lasted for decades.

➔ To differentiate a function, find the gradient function:

Function	Gradient function
$y = ax^n$	$\frac{dy}{dx} = nax^{n-1}$
$f(x) = ax^n$	$f'(x) = nax^{n-1}$

The process is valid for **all** values of n, both positive and negative.

The $\frac{dy}{dx}$ notation was developed by Leibniz. Newton's notation is now only used in physics. How important is mathematical notation in enhancing your understanding of a subject?

Example 1

Given $y = 4x^7$, find $\frac{dy}{dx}$.

Answer

$\frac{dy}{dx} = 7 \times 4x^{7-1}$

$\frac{dy}{dx} = 28x^6$

$y = ax^n$

$\frac{dy}{dx} = nax^{n-1}$

$a = 4, n = 7$

Example 2

Given $f(x) = 3x^5$, find $f'(x)$.

Answer

$f'(x) = 5 \times 3x^{5-1}$

$f'(x) = 15x^4$

$f(x) = ax^n$

$f'(x) = nax^{n-1}$

$a = 3, n = 5$

The $f'(x)$ notation is from Euler (1707–83), who was perhaps the greatest mathematician of all.

Example 3

Given $f(x) = 3x - 4x^2 + x^3$, find $f'(x)$.

Answer

$f'(x) = 3x^{1-1} - 2 \times 4x^{2-1} + 3 \times x^{3-1}$

$f'(x) = 3 - 8x + 3x^2$

Differentiate each term separately.

Remember that $x^1 = x$ and that $x^0 = 1$.

Exercise 6A

1 Find $\frac{dy}{dx}$.

a $y = 4x^2$ **b** $y = 6x^3$ **c** $y = 7x^4$ **d** $y = 5x^3$

e $y = x^4$ **f** $y = 5x$ **g** $y = x$ **h** $y = 12x$

i $y = 9x^2$ **j** $y = \frac{1}{2}x^3$ **k** $y = \frac{1}{2}x^2$ **l** $y = \frac{3}{4}x^4$

2 Differentiate

a $y = 7$ **b** $y = -3x^3$ **c** $y = -\frac{1}{4}x^4$ **d** $y = -\frac{2}{3}x^3$

e $y = -x$ **f** $y = -3$ **g** $y = 5x^6$ **h** $y = -7x^9$

i $y = \frac{1}{2}x^8$ **j** $y = \frac{3}{4}x^{12}$ **k** $y = -\frac{2}{3}x^9$ **l** $y = \frac{3}{4}$

3 Find $f'(x)$.

a $f(x) = 3x^2 + 5x^3$ **b** $f(x) = 5x^4 - 4x$

c $f(x) = 9x - 11x^3$ **d** $f(x) = x^4 + 3x + 2$

4 Find y'

a $y = 8 - 5x + 4x^6$ **b** $y = 9x^2 - 5x + \frac{1}{2}$

c $y = 7x + 4x^5 - 101$ **d** $y = x(2x + 3)$

y' is another way of writing $\frac{dy}{dx}$.

You can use letters other than x and y for the variables. This changes the notation but not the process.

Example 4

Given $v = 3.5t^8$, find $\frac{dv}{dt}$.

Answer	
$\frac{dv}{dt} = 8 \times 3.5t^{8-1}$	*$v = at^n$*
	$\frac{dv}{dt} = nat^{n-1}$
$\frac{dv}{dt} = 28t^7$	*$a = 3.5,\ n = 8$*

Example 5

Given $f(z) = \frac{3z^4}{2}$, find $f'(z)$.

Answer	
$f(z) = \frac{3z^4}{2} = \frac{3}{2} \times z^4$	*$f(z) = az^n$*
$f'(z) = 4 \times \frac{3}{2} z^{4-1}$	*$f'(z) = naz^{n-1}$*
$f'(z) = 6z^3$	*$a = \frac{3}{2},\ n = 4$*

Example 6

Given $f(t) = (3t - 1)(t + 4)$, find $f'(t)$.

Answer	
$f(t) = 3t^2 + 12t - t - 4$	*Multiply out the brackets.*
$f(t) = 3t^2 + 11t - 4$	
$f'(t) = 6t + 11$	*Differentiate each term separately.*

Exercise 6B

1 Find $\frac{dA}{dt}$.

a $A = 4t(9 - t^2)$ **b** $A = 6(2t + 5)$

c $A = t^2(t - 5)$ **d** $A = (t + 2)(2t - 3)$

e $A = (5 - t)(3 + 2t)$ **f** $A = (6t + 7)(3t - 5)$

g $A = (t^2 + 3)(t - 1)$ **h** $A = 3(t + 3)(t - 4)$

2 Find $f'(r)$.

a $f(r) = \frac{1}{2}(r + 3)(2r - 6)$ **b** $f(r) = (r + 3)^2$

c $f(r) = (2r - 3)^2$ **d** $f(r) = (5 - 2r)^2$

e $f(r) = 3(r + 5)^2$ **f** $f(r) = 5(7 - r)^2$

You can also differentiate functions which have powers of x in the denominator of a fraction. First you must write these terms using negative indices.

Example 7

Given $y = \frac{4}{x^2}$, find $\frac{dy}{dx}$.

Answer	
$y = 4 \times \frac{1}{x^2} = 4x^{-2}$	*Write the function in index form:* $\frac{1}{x^2} = x^{-2}$.
$\frac{dy}{dx} = -2 \times 4x^{-2-1}$	*$a = 4$ and $n = -2$*
$\frac{dy}{dx} = -8x^{-3}$	*Remember the rules for multiplying negative numbers.*
$\frac{dy}{dx} = \frac{-8}{x^3}$	*Rewrite in the original form.*

Example 8

Given $f(x) = \frac{12}{5x^3}$, find $f'(x)$.

Answer	
$f(x) = \frac{12}{5} \times \frac{1}{x^3} = \frac{12}{5} x^{-3}$	*Write the function in index form.* *$a = \frac{12}{5}$ and $n = -3$*
$f'(x) = -3 \times \frac{12}{5} \times x^{-3-1}$	*Be* **very** *careful with minus signs.*
$f'(x) = \frac{-36}{5} \times x^{-4}$	*Simplify.*
$f(x) = \frac{-36}{5x^4}$	*Rewrite in the original form.*

Exercise 6C

Differentiate the following with respect to x.

Remember to use the same notation as the question.

1 $y = \frac{3}{x^2}$ **2** $f(x) = \frac{2}{x^4}$ **3** $y = \frac{7}{x}$

4 $f(x) = \frac{2}{x^8}$ **5** $y = \frac{5}{x^7}$ **6** $y = 9 + \frac{2}{x}$

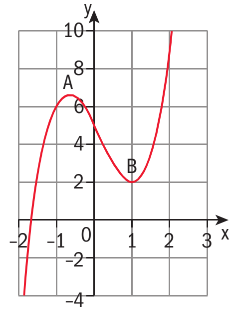
7 $f(x) = 7x^2 + \frac{4}{x^5}$ **8** $y = 7 - 4x + \frac{5}{2x^2}$ **9** $g(x) = x^3 + \frac{3}{x^2}$

10 $y = 4x - \frac{3}{x}$ **11** $g(x) = 5x^3 - \frac{1}{x^4}$ **12** $y = \frac{x^4}{2} - \frac{3}{4x^8}$

13 $y = \frac{x^4}{8} + 3x^2 + \frac{5}{6x^4}$ **14** $g(x) = 2x^3 - x^2 + 2 - \frac{3}{2x^2}$ **15** $A(x) = x^2 - \frac{5}{2x} + \frac{3}{4x^2}$

6.3 Calculating the gradient of a curve at a given point

→ You can use the gradient function to determine the exact value of the gradient at any specific point on the curve.



Here is the curve $y = 2x^3 - x^2 - 4x + 5$ with **domain** $-2 \leq x \leq 2$. The curve intersects the y-axis at (0, 5).

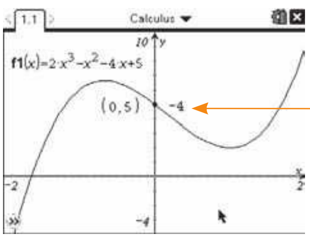
At $x = -2$ the function has a negative value.
It increases to a point A, then decreases to a point B and after $x = 1$ it increases again.

The gradient function of the curve will be negative between points A and B and positive elsewhere.

Will the gradient function be positive or negative at point A and at point B?

Differentiating, the gradient function is $\frac{dy}{dx} = 6x^2 - 2x - 4$.

At the y-intercept (0, 5) the x-coordinate is 0. Substituting this value into $\frac{dy}{dx}$: at $x = 0$, $\frac{dy}{dx} = 6(0)^2 - 2(0) - 4 = -4$



–4 is the gradient at the point (0, 5). Move the point along the curve to find the gradient at other points.

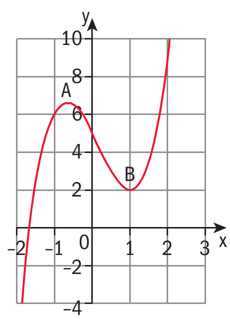
You can check this on your GDC. See Chapter 12, Section 6.1, Example 33.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can use this algebraic method to find the gradient of the curve at other points. For example,

at $x = -1$, $\frac{dy}{dx} = 6(-1)^2 - 2(-1) - 4$

$\frac{dy}{dx} = 4$



This result agrees with what can be seen from the graph.

The gradient of the curve at $x = -1$ is 4 and at $x = 0$ it is −4

Exercise 6D

These questions can be answered using the algebraic method or using a GDC. Make sure you can do both.

1 If $y = x^2 - 3x$, find $\frac{dy}{dx}$ when $x = 4$.

2 If $y = 6x - x^3 + 4$, find $\frac{dy}{dx}$ when $x = 0$.

3 If $y = 11 - 2x^4 - 3x^3$, find $\frac{dy}{dx}$ when $x = -3$.

4 If $y = 2x(5x + 4)$, find the value of $\frac{dy}{dx}$ when $x = -1$.

5 Find the gradient of the curve $y = x^3 - 5x$ at the point where $x = 6$.

6 Find the gradient of the curve $y = 10 - \frac{1}{2}x^4$ at the point where $x = -2$.

7 Find the gradient of the curve $y = 3x(7 - 4x^2)$ at the point (1, 9).

8 Find the gradient of the curve $y = 3x^2 - 5x + 6$ at the point (–2, 28).

9 $s = 40t - 5t^2$

Find $\frac{ds}{dt}$ when $t = 0$.

10 $s = t(35 + 6t)$

Find $\frac{ds}{dt}$ when $t = 3$.

11 $v = 80t + 7$

Find $\frac{dv}{dt}$ when $t = -4$.

12 $v = 0.7t - 11.9$

Find $\frac{dv}{dt}$ when $t = 0.7$.

13 $A = 14h^3$

Find $\frac{dA}{dh}$ when $h = \frac{2}{3}$.

14 $W = 7.25p^3$

Find $\frac{dW}{dp}$ at $p = -2$.

15 $V = 4r^2 + \frac{18}{r}$.

Find $\frac{dV}{dr}$ at $r = 3$.

16 $A = 5r + \frac{8}{r^2}$

Find $\frac{dA}{dr}$ at $r = 4$.

17 $V = 7r^3 - \frac{8}{r}$

Find $\frac{dV}{dr}$ at $r = 2$.

18 $A = \pi r^2 - \frac{2\pi}{r}$

Find $\frac{dA}{dr}$ at $r = 1$.

19 $V = 6r + \frac{15}{2r}$

Find $\frac{dV}{dr}$ at $r = 5$.

20 $C = 45r + \frac{12}{r^3}$

Find $\frac{dC}{dr}$ at $r = 1$.

By working backwards you can find the coordinates of a specific point on a curve with a particular gradient.

Example 9

Point A lies on the curve $y = 5x - x^2$ and the gradient of the curve at A is 1. Find the coordinates of A.

Answer	
$\frac{dy}{dx} = 5 - 2x$	*First find* $\frac{dy}{dx}$
at A $\frac{dy}{dx} = 1$ so $5 - 2x = 1$ $x = 2$	*Solve the equation to find x.*
$y = 5(2) - (2)^2 = 6$ A is (2, 6)	*Substitute x = 2 into the equation of the curve to find y.*

Exercise 6E

1 Point P lies on the curve $y = x^2 + 3x - 4$. The gradient of the curve at P is equal to 7.

- **a** Find the gradient function of the curve.
- **b** Find the x-coordinate of P.
- **c** Find the y-coordinate of P.

2 Point Q lies on the curve $y = 2x^2 - x + 1$. The gradient of the curve at Q is equal to –9.

- **a** Find the gradient function of the curve.
- **b** Find the x-coordinate of Q.
- **c** Find the y-coordinate of Q.

3 Point R lies on the curve $y = 4 + 3x - x^2$ and the gradient of the curve at R is equal to –3.

- **a** Find the gradient function of the curve.
- **b** The coordinates of R are (a, b), find the value of a and of b.

EXAM-STYLE QUESTIONS

4 Point R lies on the curve $y = x^2 - 6x$ and the gradient of the curve at R is equal to 6.
Find the gradient function of the curve.
The coordinates of R are (a, b)
Find the value of a and of b.

5 Find the coordinates of the point on the curve $y = 3x^2 + x - 5$ at which the gradient of the curve is 4.

6 Find the coordinates of the point on the curve $y = 5x - 2x^2 - 3$ at which the gradient of the curve is 9.

7 There are **two** points on the curve $y = x^3 + 3x + 4$ at which the gradient of the curve is 6.
Find the coordinates of these two points.

8 There are **two** points on the curve $y = x^3 - 6x + 1$ at which the gradient of the curve is –3.
Find the coordinates of these two points.
Find the equation of the straight line that passes through these two points.

EXAM-STYLE QUESTION

9 There are **two** points on the curve $y = x^3 - 12x + 5$ at which the gradient of the curve is zero.
Find the coordinates of these two points.
Find the equation of the straight line that passes through these two points.

EXAM-STYLE QUESTIONS

10 Point P (1, b) lies on the curve $y = x^2 - 4x + 1$.

a Find the value of b.

b Find the gradient function of the curve.

c Show that at P the gradient of the curve is also equal to b.

d Q (c, d) is the point on the curve at which the gradient of the curve is equal to -2. Show that $d = -2$.

11 Point P (5, b) lies on the curve $y = x^2 - 3x - 3$.

a Find the value of b.

b Find the gradient function of the curve.

c Show that at P the gradient of the curve is also equal to b.

d Q (c, d) is the point on the curve at which the gradient of the curve is equal to -3.
Show that d is also equal to -3.

12 Consider the function $f(x) = 4x - x^2 - 1$.

a Write down $f'(x)$.

b Show that at $x = 5$, $f(x) = f'(x)$.

c Find the coordinates of a second point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

13 Consider the function $f(x) = 2x^2 - x + 1$.

a Write down $f'(x)$.

b Show that at $x = 2$, $f(x) = f'(x)$.

c Find the coordinates of a second point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

14 Consider the function $f(x) = 3x - x^2 - 1$.

a Write down $f'(x)$.

b Show that at $x = 1$, $f(x) = f'(x)$.

c Find the coordinates of a second point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

15 Consider the function $f(x) = 2x^2 - x - 1$.

a Write down $f'(x)$.

b Find the coordinates of the points on the curve $y = f(x)$ for which $f(x) = f'(x)$.

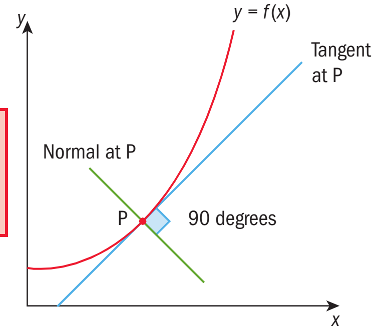
16 Consider the function $f(x) = x^2 + 5x - 5$.

a Write down $f'(x)$.

b Find the coordinates of the points on the curve $y = f(x)$ for which $f(x) = f'(x)$.

17 Consider the function $f(x) = x^2 + 4x + 5$.
Find the coordinates of the point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

6.4 The tangent and the normal to a curve



Here is a curve $y = f(x)$ with a point, P, on the curve.

→ The tangent to the curve at any point P is the straight line which passes through P with gradient equal to the gradient of the curve at P.

The **normal** to the curve at P is the straight line which passes through P that is **perpendicular** to the tangent.

The tangent and the curve are closely related because, at P:

- the x-coordinate of the tangent is equal to the x-coordinate of the curve
- the y-coordinate of the tangent is equal to the y-coordinate of the curve
- the gradient of the tangent is equal to the gradient of the curve.

You can use differentiation to find the equation of the tangent to any curve at a point, P(a, b), provided that you know both the equation of the curve and the x-coordinate, a, of the point P.

→ To find the equation of the tangent to the curve at P(a, b):

1. Calculate b, the y-coordinate of P, using the equation of the curve.
2. Find the gradient function $\frac{dy}{dx}$.
3. Substitute a, the x-coordinate of P, into $\frac{dy}{dx}$ to calculate, m, the value of the gradient at P.
4. Use the equation of a straight line $(y - b) = m(x - a)$.

For more on the equation of a straight line, see Chapter 3.

Example 10

Point P has an x-coordinate 2. Find the equation of the tangent to the curve $y = x^3 - 3$ at P.
Give your answer in the form $y = mx + c$.

Answer

At $x = 2$, $y = (2)^3 - 3 = 5$	*Use $y = x^3 - 3$ to calculate the y-coordinate of P.*
$\frac{dy}{dx} = 3x^2$	*Find the gradient function $\frac{dy}{dx}$.*
At $x = 2$, $\frac{dy}{dx} = 3(2)^2 = 12$ $m = 12$	*Substitute 2, the x-coordinate at P, into $\frac{dy}{dx}$ to calculate m, the value of the gradient at P.*
At P (2, 5) $(y - 5) = 12(x - 2)$ $y - 5 = 12x - 24$ $y = 12x - 19$	*Use the equation $(y - b) = m(x - a)$ with $a = 2$, $b = 5$, $m = 12$. Simplify.*

You can check the equation of the tangent using your GDC.

Exercise 6F

1 Find the equation of the tangent to the given curve at the stated point, P. Give your answers in the form $y = mx + c$.

a $y = x^2$; P(3, 9)
b $y = 2x^3$; P(1, 2)
c $y = 6x - x^2$; P(2, 8)
d $y = 3x^2 - 10$; P(1, −7)
e $y = 2x^2 - 5x + 4$; P(3, 7)
f $y = 10x - x^3 + 5$; P(2, 17)
g $y = 11 - 2x^2$; P(3, −7)
h $y = 5 - x^2 + 6x$; P(2, 13)
i $y = 4x^2 - x^3$; P(4, 0)
j $y = 5x - 3x^2$; P(−1, −8)
k $y = 6x^2 - 2x^3$; P(2, 8)
l $y = 60x - 5x^2 + 7$; P(2, 107)
m $y = \frac{1}{2}x^4 - 7$; P(4, 121)
n $y = 17 - 3x + 5x^2$; P(0, 17)
o $y = 2x(5 - x)$; P(0, 0)
p $y = \frac{1}{4}x^3 - 4x$; P(2, −6)
q $y = \frac{3}{4}x^2 + 3$; P(−2, 6)
r $y = \frac{2}{3}x^3 + \frac{1}{3}$; $P\left(-1, -\frac{1}{3}\right)$
s $y = \frac{1}{4}x^3 - 7x^2 + 5$; P(−2, −25)

2 Find the equation of the tangent to the given curve at the stated point. Give your answers in the form $ax + by + c = 0$

a $y = \frac{12}{x^2}$; (2, 3)
b $y = 5 + \frac{6}{x^3}$; (1, 11)
c $y = 6x - \frac{8}{x^2}$; (−2, −14)
d $y = x^3 + \frac{6}{x^2}$; (−1, 5)
e $y = 5x - \frac{8}{x}$; (4, 18)

To find the equation of the normal to a curve at a given point you need to do one extra step.

→ The normal is perpendicular to the tangent so its gradient, m', is found using the formula $m' = \frac{-1}{m}$, where m is the gradient of the tangent.

Example 11

Point P has x-coordinate −4.
Find the equation of the normal to the curve $y = \frac{12}{x}$ at P.
Give your answer in the form
$ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$.

Answer

At $x = -4$, $y = \frac{12}{(-4)} = -3$

Use $y = \frac{12}{x}$ to calculate the y-coordinate of P.

$\frac{dy}{dx} = -\frac{12}{x^2}$

Find the gradient function $\frac{dy}{dx}$. (Remember, $y = 12x^{-1}$.)

▶ Continued on next page

You learned about gradient of a perpendicular line in Chapter 3.

At $x = -4$, $\frac{dy}{dx} = -\frac{12}{(-4)^2} = -\frac{3}{4}$ The gradient of the **tangent**, $m = -\frac{3}{4}$	*Substitute the value of x into $\frac{dy}{dx}$ to calculate, m, the value of the gradient at P.*
Hence, the gradient of the normal, $m' = \frac{4}{3}$	*The normal is perpendicular to the tangent.*
The equation of the normal to $y = \frac{12}{x}$ at P(–4, –3) is $(y-(-3)) = \frac{4}{3}(x-(-4))$	*Use the equation of a straight line $(y - b) = m(x - a)$ with $a = -4$, $b = -3$, $m = \frac{4}{3}$*
$3(y + 3) = 4(x + 4)$ $3y + 9 = 4x + 16$ $4x - 3y + 7 = 0$	*Simplify.* *Rearrange to the form $ax + bx + c = 0$, where $a, b, c \in \mathbb{Z}$*

The gradient of a line perpendicular to a line whose gradient is m is $-\frac{1}{m}$.

You cannot find the equation of a normal directly from the GDC.

Exercise 6G

Find the equation of the normal to the given curve at the stated point P. Give your answers in the form $ax + by + c = 0$

1 $y = 2x^2$; P(1, 2)

2 $y = 3 + 4x^3$; P(0.5, 3.5)

3 $y = \frac{x}{2} - x^2$; P(2,–3)

4 $y = \frac{3x^2}{2} + x$; P(–2,4)

5 $y = (x + 2)(5 - x)$; P(0, 10)

6 $y = (x + 2)^2$; P(0, 4)

7 $y = \frac{4}{x}$; P(2, 2)

8 $y = \frac{6}{x^2}$; P(–1,6)

9 $y = 6x + \frac{8}{x}$; P(1, 14)

10 $y = x^4 - \frac{3}{x^3}$; P(–1, 4)

11 $y = 4 - 2x - \frac{1}{x}$; P(0.5, 1)

12 $y = 5x - \frac{9}{2x}$; P(3, 13.5)

Example 12

The gradient of the tangent to the curve $y = ax^2$ at the point P (3, b) is 30. Find the values of a and b.

Answer $\frac{dy}{dx} = 2ax$	*As the gradient of the tangent is given, find $\frac{dy}{dx}$.*
$2a(3) = 30$ $\Rightarrow a = 5$	*When $x = 3$, $\frac{dy}{dx} = 30$*
The equation of the curve is $y = 5x^2$. $b = 5(3)^2 \Rightarrow b = 45$	*Substitute $x = 3$ to find b.*

Exercise 6H

1 Find the equation of the tangent to the curve $y = (x - 4)^2$ at the point where $x = 5$.

EXAM-STYLE QUESTIONS

2 Find the equation of the tangent to the curve $y = x(x^2 - 3)$ at the point where $x = -2$.

3 Find the equation of the normal to the curve $y = x + \frac{6}{x}$ at the point where $x = 4$.

4 Find the equation of the normal to the curve $y = x^2 - \frac{1}{x^2}$ at the point where $x = -1$.

5 Find the equations of the tangents to the curve $y = 3x^2 - 2x$ at the points where $y = 8$.

6 Find the equations of the tangents to the curve $y = 2x(3 - x)$ at the points where $y = -20$.

7 Find the equation of the normal to the curve $y = 7 - 5x - 2x^3$ at the point where it intersects the x-axis.

8 Find the equation of the normal to the curve $y = x^3 + 3x - 2$ at the point where $y = -6$.

9 **a** Find the value of x for which the gradient of the tangent to the curve $y = (4x - 3)^2$ is zero.

b Find the equation of the tangent at this point.

EXAM-STYLE QUESTION

10 **a** Find the value of x for which the gradient of the tangent to the curve $y = x^2 + \frac{16}{x}$ is zero.

b Find the equation of the tangent at this point.

11 **a** Find the value of x for which the gradient of the tangent to the curve $y = \frac{x^2}{2} + x - 3$ is 5.

b Find the equation of the tangent at this point.

12 **a** Find the value of x for which the gradient of the tangent to the curve $y = x^4 + 3x - 3$ is 3.

b Find the equation of the tangent at this point.

c Find the equation of the normal at this point.

13 **a** Find the value of x for which the gradient of the tangent to the curve $y = 4x + \frac{3}{x^4}$ is 16.

b Find the equation of the tangent at this point.

c Find the equation of the normal at this point.

14 There are two points on the curve $y = 2x^3 + 9x^2 - 24x + 5$ at which the gradient of the curve is equal to 36. Find the equations of the tangents to the curve at these points.

EXAM-STYLE QUESTION

15 The gradient of the tangent to the curve $y = x^2 + kx$ at the point P $(3, b)$ is 7.
Find the value of k and the value of b.

16 The gradient of the tangent to the curve $y = x^2 + kx$ at the point P $(-2, b)$ is 1.
Find the value of k and that of b.

17 The gradient of the tangent to the curve $y = kx^2 - 2x + 3$ at the point P $(4, b)$ is 2.
Find the value of k and that of b.

18 The gradient of the tangent to the curve $y = 4 + kx - x^3$ at the point P $(-2, b)$ is -5.
Find the value of k and that of b.

19 The gradient of the tangent to the curve $y = px^2 + qx$ at the point P $(2, 5)$ is 7.
Find the value of p and that of q.

20 The gradient of the tangent to the curve $y = px^2 + qx - 5$ at the point P $(-3, 13)$ is 6.
Find the value of p and that of q.

6.5 Rates of change

The gradient function, $f'(x)$, of a function $f(x)$ is a measure of how $f(x)$ changes as x increases. We say that $f'(x)$ measures the **rate of change of *f* with respect to *x*.**

→ For the graph $y = f(x)$, the gradient function $\frac{dy}{dx} = f'(x)$ gives the rate of change of y with respect to x.

In general, the **rate of change** of one variable with respect to another is the gradient function.

Other variables can also be used, for example:

if $A = f(t)$, then $\frac{dA}{dt} = f'(t)$ measures the **rate of change of *A* with respect to *t*.**

If the variable t represents time, then the gradient function measures the rate of change with respect to the *time* that passes.

This is an important concept. If you measure how a variable changes as time is passing then you are applying mathematics to situations that are **dynamic** – to situations that are moving.

For example, if C represents the value of a car (measured on a day-to-day basis) we can say that C is a function of time: $C = f(t)$.

Then, $\frac{dC}{dt} = f'(t)$ represents the rate at which the value of the car is changing – it measures the rate of change of C with respect to t, the rate of inflation or deflation of the price of the car.

Similarly, if s represents the distance measured from a fixed point to a moving object then s is a function of time: $s = g(t)$ and $\frac{ds}{dt} = g'(t)$ measures the rate of change of this distance, s, with respect to t.

$\frac{ds}{dt}$ measures the **velocity** of the object at time t.

If v is the velocity of an object, what does $\frac{dv}{dt}$ represent?

Example 13

The volume of water in a container, V cm^3, is given by the formula $V = 300 + 2t - t^2$, where t is the time measured in seconds.

a What does $\frac{dV}{dt}$ represent?

b What units are used for $\frac{dV}{dt}$?

c Find the value of $\frac{dV}{dt}$ when $t = 3$.

d What does the answer to **c** tell you?

Answers

a $\frac{dV}{dt}$ represents the rate of change of the volume of water in the container.

The rate at which the water is entering (or leaving) the container.

b $\frac{dV}{dt}$ is measured in cm^3 per second (cm^{3}s^{-1}).

The volume is measured in cm^3 and time is measured in seconds.

c $\frac{dV}{dt} = 2 - 2t$

At $t = 3$,

$\frac{dV}{dt} = 2 - 2(3) = -4$

$\frac{dV}{dt}$ is negative, so

d Since this value is **negative**, the water is **leaving** the container at 4 cm^3 per second.

the volume is decreasing.

How would you decide by considering $\frac{dv}{dt}$ whether the water was **entering** or **leaving** the container?

Example 14

A company mines copper, where the mass of copper, x, is measured in thousands of tonnes. The company's profit, P, measured in millions of dollars, depends on the amount of copper mined. The profit is given by the function $P(x) = 2.3x - 0.05x^2 - 12$

a Find $P(0)$ and $P(6)$ and interpret these results.

b Find $\frac{dP}{dx}$. What does $\frac{dP}{dx}$ represent?

c Find the value of P and $\frac{dP}{dx}$ when $x = 20$ and when $x = 25$.

d Interpret the answers to **c**.

e Find the value of x for which $\frac{dP}{dx} = 0$.

f Determine P for this value of x, and interpret this value.

You can graph any function on the GDC. This could give you further insight into the problem.

Answers

a $P(0) = -12$; a loss of 12 million dollars.
$P(6) = 0$; there is no profit and no loss, this is the break-even point.

Substitute $x = 0$ in to $P(x)$.

b $\frac{dP}{dx} = -0.1x + 2.3$

$\frac{dP}{dx}$ represents the rate of change of the profit as the amount of copper mined increases.

$\frac{dP}{dx}$ measures the rate of change of P with respect to x.

c At $x = 20$, $P = 14$ and $\frac{dP}{dx} = 0.3$

At $x = 25$, $P = 14.25$ and $\frac{dP}{dx} = -0.2$

Substitute $x = 20$ and $x = 25$ into $P(x)$ and $\frac{dP}{dx}$.

d At both points the company is profitable.

At $x = 20$, $\frac{dP}{dx} > 0$ so a further increase in production will make the company **more profitable**.

At $x = 20$, $P(x)$ is increasing.

At $x = 25$, $\frac{dP}{dx} < 0$ so a further increase in production will make the company **less profitable**.

At $x = 25$, $P(x)$ is decreasing.

e $\frac{dP}{dx} = -0.1x + 2.3 = 0$

$0.1x = 2.3$

$x = \frac{2.3}{0.1} = 23$

Set $\frac{dP}{dx}$ equal to 0.

Solve for x.

23 000 tonnes of copper needs to be mined to maximize the company's profit.

x is measured in thousands of tonnes.

f $P(23) = 14.45$

14.45 million dollars is the **maximum** profit that the company can make.

Substitute $x = 23$ into $P(x)$.

Exercise 6I

EXAM-STYLE QUESTION

1 The volume of water in a container, V cm^3, is given by the formula $V = 100 + 2t + t^3$, where t is the time measured in seconds.

  a How much water is there in the container initially?

  b How much water is there in the container when $t = 3$?

  c What does $\frac{dV}{dt}$ represent?

  d Find the value of $\frac{dV}{dt}$ when $t = 3$.

  e Use your answers to **b** and **d** to explain what is happening to the volume of water in the tank.

Initially $t = 0$

2 The area, A, of a pool of water forming under a leaking pipe is $A = 4t + t^2$ cm^2 after t seconds.

  a What is the area of the pool initially?

  b What is the area of the pool when $t = 5$?

  c What does $\frac{dA}{dt}$ represent?

  d Find the value of $\frac{dA}{dt}$ when $t = 5$.

  e Use your answers to **b** and **d** to explain what is happening to the area of the pool.

3 The weight of oil in a storage tank, W, varies according to the formula $W = 5t^2 + \frac{640}{t} + 40$ where W is measured in tonnes and t is the time measured in hours, $1 \le t \le 10$.

  a Find the weight of oil in the tank at $t = 1$.

  b Find $\frac{dW}{dt}$.

  c Find the rate of change of the weight of the oil in the tank when

    i $t = 3$ ii $t = 5$.

  d What does your answer to **c** tell you?

  e Find the value of t for which $\frac{dW}{dt} = 0$.

  f Interpret your answer to **e**.

4 The volume of water, V, measured in m^3, in a swimming pool after t minutes, where $t > 0$, is $V = 10 + 6t + t^2$.

  a Find the rate at which the volume is increasing when $t = 1$.

  b Find the rate at which the volume is increasing when there are 65 m^3 of water in the pool.

5 Water is flowing out of a tank. The depth of the water, y cm, at time t seconds is given by $y = 500 - 4t - t^3$.

  a Find the rate at which the depth is decreasing at 2 seconds and at 3 seconds.

  b Find the time at which the tank is empty.

6 The area, A cm², of a blot of ink is growing so that, after t seconds, $A = \frac{3t^2}{4} + \frac{t}{2}$.

  a Find the rate at which the area is increasing after 2 seconds.

  b Find the rate at which the area is increasing when the area of the blot is 30 cm².

7 The weight of oil in a storage tank, W, varies according to the formula $W = 10t + \frac{135}{t^2} + 4$ where W is measured in tonnes and t is the time measured in hours, $1 \le t \le 10$.

  a Find the rate at which the weight is changing after 2 hours.

  b Find the value of t for which $\frac{dW}{dt} = 0$.

8 The angle turned through by a rotating body, θ degrees, in time t seconds is given by the relation $\theta = 4t^3 - t^2$.

  a Find the rate of increase of θ when $t = 2$.

  b Find the value of t at which the body changes direction.

9 A small company's profit, P, depends on the amount x of 'product' it makes. This profit can be modeled by the function $P(x) = -10x^3 + 40x^2 + 10x - 15$. P is measured in thousands of dollars and x is measured in tonnes.

  a Find $P(0)$ and $P(5)$ and interpret these results.

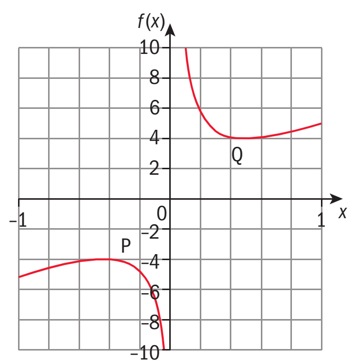
  b Find $\frac{dP}{dx}$.

  c Find the value of P and $\frac{dP}{dx}$ when **i** $x = 2$ **ii** $x = 3$.

  d Interpret your answers to **c**.

  e Find the value of x and of P for which $\frac{dP}{dx} = 0$. What is the importance of this point?

6.6 Local maximum and minimum points (turning points)



Here is the graph of the function

$$f(x) = 4x + \frac{1}{x}, \quad x \neq 0$$

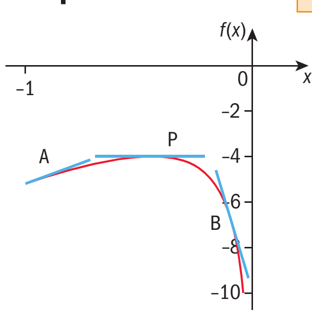
The graph has two branches, because the function is **not defined** at the point $x = 0$.

$\frac{1}{0}$ is undefined; it has no value.

First, look at the left-hand branch of the graph, for the **domain** $x < 0$.

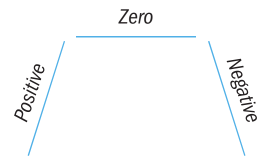
As x increases, the curve increases to the point P. After point P, the curve decreases. P is said to be a local **maximum point**.

You can determine that P is a local maximum point because just before P (for example, at A) the gradient of the curve is positive, and just after P (for example, at B) the gradient of the curve is negative.



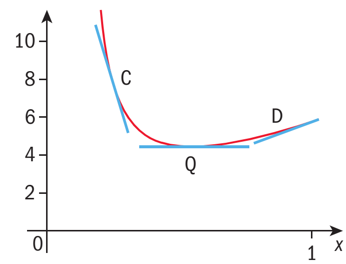
At P itself, most importantly, the gradient of the curve is zero.

→ At a local maximum, the curve stops increasing and changes direction so that it 'turns' and starts decreasing. So, as x increases, the three gradients occur in the order: positive, zero, negative. Where the gradient is zero is the maximum point.



Now look at the right-hand branch of the graph, with the domain $x > 0$.

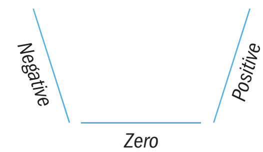
As x increases, the curve decreases to the point Q. After Q, the curve increases. Q is said to be a local **minimum point**.



You can determine that Q is a local minimum point because just before Q (for example, at C) the gradient of the curve is negative and just after Q (for example, at D) the gradient of the curve is positive.

At Q itself, the gradient of the curve is zero.

→ At a local minimum, the curve stops decreasing and changes direction; it 'turns' and starts increasing. So, as x increases, the three gradients occur in the order: negative, zero, positive. Where the gradient is zero is the minimum point.



Local maximum and local minimum points are known as **stationary points** or **turning points**.

→ At any stationary or turning point – either local maximum or local minimum – $f'(x)$ is zero.

At a stationary point, if $y = f(x)$ then $\frac{dy}{dx} = 0$.

To find the coordinates of P (the local maximum) and of Q (the local minimum) for the function $f(x) = 4x + \frac{1}{x}$, use the fact that at each of these points $f'(x)$ is zero.

$f(x) = 4x + \frac{1}{x}$, so $f'(x) = 4 - \frac{1}{x^2}$

Remember that $\frac{1}{x} = x^{-1}$.

Set $f'(x) = 0$ which gives $4 - \frac{1}{x^2} = 0$

Adding $\frac{1}{x^2}$: $\quad 4 = \frac{1}{x^2}$

Multiplying by x^2: $\quad 4x^2 = 1$

Dividing by 4: $\quad x^2 = \frac{1}{4}$

Taking square roots: $x = \frac{1}{2}$ or $x = -\frac{1}{2}$

Substitute each x-value into $f(x)$ to find the y-coordinate of each turning point.

You can find local maximum and local minimum points using a GDC, without using differentiation. See Chapter 12, Section 6.3.

At $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) + \frac{1}{\left(\frac{1}{2}\right)} = 4$

At $x = -\frac{1}{2}$, $f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right) + \frac{1}{\left(-\frac{1}{2}\right)} = -4$

So, the coordinates of the turning points are $\left(\frac{1}{2}, 4\right)$ and $\left(-\frac{1}{2}, -4\right)$

To determine which is the local maximum and which is the local minimum, look at the graph of the function: $\left(\frac{1}{2}, 4\right)$ is the local minimum and $\left(-\frac{1}{2}, -4\right)$ the local maximum.

You cannot decide which is the maximum and which is the minimum simply by looking at the coordinates.

→ To find turning points, first set the gradient function equal to zero and solve this equation. This gives the x-coordinate of the turning point.

Exercise 6J

Find the values of x for which $\frac{dy}{dx} = 0$. Verify your answers by using your GDC.

1 $y = x^2 - 6x$

2 $y = 12x - 2x^2$

3 $y = x^2 + 10x$

4 $y = 3x^2 + 15x$

5 $y = x^3 - 27x$

6 $y = 24x - 2x^3$

7 $y = 4x^3 - 3x$

8 $y = 3x - 16x^3$

9 $y = 2x^3 - 9x^2 + 12x - 7$

10 $y = 5 + 9x + 6x^2 + x^3$

11 $y = x^3 - 3x^2 - 45x + 11$

12 $y = 12x^2 + x^3 + 36x - 8$

13 $y = 2x^3 - 6x^2 + 7$

14 $y = 17 + 30x^2 - 5x^3$

15 $f(x) = x + \frac{1}{x}$

16 $y = x + \frac{4}{x}$

17 $y = 4x + \frac{9}{x}$

18 $y = 8x + \frac{1}{2x}$

19 $y = 27x + \frac{4}{x^2}$

20 $y = x + \frac{1}{2x^2}$

Once you have found the x-coordinate of any turning point, you can then calculate the y-coordinate of the point and decide if it is a maximum or minimum.

Example 15

Find the coordinates of the turning points of the curve $y = 3x^4 - 8x^3 - 30x^2 + 72x + 5$. Determine the nature of these points.

'Determine the nature' means decide whether the point is a local maximum or a local minimum.

Answer

$y = 3x^4 - 8x^3 - 30x^2 + 72x + 5$

$\frac{dy}{dx} = 12x^3 - 24x^2 - 60x + 72$ *Differentiate.*

$\frac{dy}{dx} = 12x^3 - 24x^2 - 60x + 72 = 0$ *At each turning point $\frac{dy}{dx} = 0$.*

Continued on next page

$x = -2$, $x = 1$, $x = 3$

Solve this equation with your GDC.

At $x = -2$,
$y = 3(-2)^4 - 8(-2)^3 - 30(-2)^2 + 72(-2) + 5 = -95$
so $(-2, -95)$ is a turning point.
At $x = 1$, $y = 3(1)^4 - 8(1)^3 - 30(1)^2 + 72(1) + 5 = 42$
so $(1, 42)$ is a turning point.
At $x = 3$, $y = 3(3)^4 - 8(3)^3 - 30(3)^2 + 72(3) + 5 = -22$
so $(3, -22)$ is a turning point.

Substitute the three values of x to find the y-coordinates.

x-coordinate		−2		1		3	
Gradient		0		0		0	

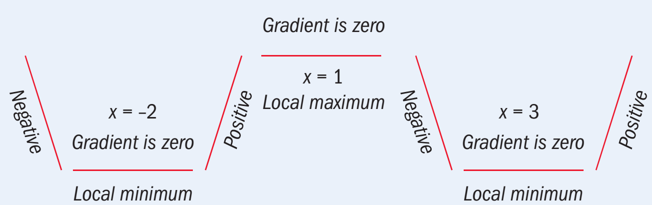
To decide if points are maximum or minimum (without using the GDC) find the gradient at points on each side of the turning points. First, fill in the information on the turning points.

$x = -10$ for $x < -2$ $f'(-10) = -12\,268$
$x = 0$ for $-2 < x < 1$ $f'(0) = 72$
$x = 2$ for $1 < x < 3$ $f'(2) = -48$
$x = 5$ for $x > 3$ $f'(5) = 672$

Now choose x-coordinates of points on each side of the turning points. Calculate the gradient at each point and enter them in the table.

x-coordinate	−10	−2	0	1	2	3	5
Gradient	−12 268	0	72	0	−48	0	672

Choose points close to the stationary point.



Sketch the pattern of the gradients from the table.

$(-2, -95)$ is a local **minimum**.

As the curve moves through (−2, −95), the gradient changes negative → zero → positive.

$(1, 42)$ is a local **maximum**.

As the curve moves through (1, 42), the gradient changes positive → zero → negative.

$(3, -22)$ is also a local **minimum**.

As the curve moves through (3, −22), the gradient changes negative → zero → positive.

Exercise 6K

Determine the coordinates of any turning points on the given curves.
For each, decide if it is a maximum or minimum.
Check your answers by using your GDC.

1 $y = x^3 - 9x^2 + 24x - 20$ **2** $y = x^3 + 6x^2 + 9x + 5$

3 $y = x(9 + 3x - x^2)$ **4** $y = x^3 - 3x^2 + 5$

5 $y = x(27 - x^2)$

6 $y = x^2(9 - x)$

7 $f(x) = x + \dfrac{1}{x}$

8 $f(x) = x + \dfrac{9}{x}$

9 $f(x) = \dfrac{x}{2} + \dfrac{8}{x}$

10 $f(x) = \dfrac{9}{x} + \dfrac{x}{4}$

11 $f : x \to = x^2 - \dfrac{16}{x}$

12 $f : x \to = 9x + \dfrac{1}{6x^2}$

'$f : x \to$' is read as 'f such that x maps to' and means the same as '$f(x) =$'.

You can sometimes determine the nature of a turning point without checking points on either side.

Example 16

Find the coordinates of any turning points of the curve $y = 9x - 3x^2 + 8$ and determine their nature.

Answer

At turning points:

$$\frac{dy}{dx} = 9 - 6x = 0$$

$$x = 1.5$$

Solve for x.

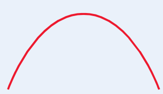
$y = 9(1.5) - 3(1.5)^2 + 8$
$= 14.75$

Substitute $x = 1.5$ into $y = 9x - 3x^2 + 8$.

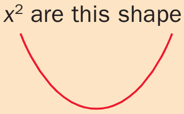
The turning point is (1.5, 14.75).

The turning point is a local maximum.

Quadratic graphs with a negative coefficient of x^2 are this shape:



Quadratic graphs with a positive coefficient of x^2 are this shape:



Exercise 6L

Find the coordinates of the local maximum or local minimum point for each quadratic curve.
State the nature of this point.

1 $y = x^2 - 4x + 10$ **2** $y = 18x - 3x^2 + 2$ **3** $y = x^2 + x - 3$

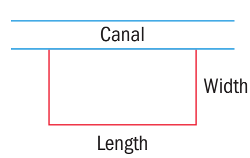
4 $y = 8 - 5x + x^2$ **5** $y = 3x + 11 - x^2$ **6** $y = 20 - 6x^2 - 15x$

7 $y = (x - 3)(x - 7)$ **8** $y = x(x - 18)$ **9** $y = x(x + 4)$

6.7 Using differentiation in modeling: optimization

An introductory problem

In Chapter 4, you used quadratic functions to model various situations. One of the optimization problems was to maximize the area of a rectangular field that bordered a straight canal and was enclosed on three sides by 120 m of fencing.



A model is a mathematical function that describes the situation. In this case, we need a model for the area of the field (the rectangle) for different widths.

First, identify the **variables** in the problem.
These are:

- the width of the field
- the length of the field
- the area of the field.

Second, identify any **constraints** in the problem. The constraint in this problem is that 120 m of fencing is used for three sides.

It often helps to try a few numerical examples in order to put the problem in context and to indicate the method. For example

1 If the width were 20 m, then the length would be $120 - 2(20) = 80$ m
the area would be $20 \times 80 = 1600\,\text{m}^2$

2 If the width were 50 m, then the length would be $120 - 2(50) = 20$ m
the area would be $50 \times 20 = 1000\,\text{m}^2$

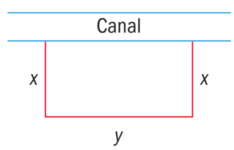
Note that, although the length of the fencing is constant, the size of the enclosed area varies.

Setting up the model

The model is for the area of the field and is a function of *both* its width and its length.

1 Define the variables.
Let A be the area of the field, x be the width of the field and y be the length of the field.
Then $A = xy$

2 Write the constraint algebraically.
$120 = 2x + y$



$x + x + y = 2x + y = 120$

If you define the variables in a different way, you obtain a different function. Here you could have defined the **length** to be x and the width as y. The area $A(x)$ would then have been a different – but correct – function.

3 Use the formula for the constraint to write the area function using just one variable.
Rearrange the constraint: $y = 120 - 2x$
Substitute in the area function: $A = xy = x(120 - 2x)$

So a model for the area of the field is $A(x) = x(120 - 2x)$, where x is the width of the field.

To determine the maximum area (the optimum solution) set the gradient function to zero.

The formula for the area is: $A(x) = x(120 - 2x)$

Expand the brackets: $A(x) = 120x - 2x^2$

Differentiate: $\frac{\mathrm{d}A}{\mathrm{d}x} = A'(x) = 120 - 4x$

Equate $\frac{\mathrm{d}A}{\mathrm{d}x}$ to zero: $120 - 4x = 0$

Solve: $4x = 120 \Rightarrow x = 30$

The quadratic function $A(x)$ has a negative coefficient of x^2 so the turning point is a maximum.

The width of the optimum rectangle is 30 m. To find the length substitute $x = 30$ into $y = 120 - 2x$.

$120 - 2(30) = 60$ m

The dimensions of the rectangle are width 30 m and length 60 m.

To find the maximum area substitute $x = 30$ into $A(x) = x(120 - 2x)$.

The maximum area is $A(30) = (30)(120 - 2(30)) = 1800\text{ m}^2$

→ In optimization problems, use differentiation to find an optimal value (either the maximum or the minimum) of a function as two variables interact.

You need to find an equation for this function in terms of these two variables and a constraint formula which links the variables. The constraint formula is used to remove one of the variables.

You can only use differentiation in functions with one variable.

Example 17

Optimize the function $A = 3xy$ subject to the constraint $x + y = 20$.

Answer	
$y = 20 - x$	*Rearrange the constraint so y is the subject.*
$A = 3xy = 3x(20 - x)$	*Substitute y into the function.*
$A(x) = 60x - 3x^2$	*Simplify.*
$\frac{dA}{dx} = 60 - 6x$	*Differentiate.*
$60 - 6x = 0 \Rightarrow x = 10$	*Set $\frac{dA}{dx}$ to zero and solve for x.*
$A(10) = 60(10) - 3(10)^2 = 300$ The optimal value of A is 300.	*Substitute the value of x into A(x) to find the optimal value of A.*

$A(x)$ is a quadratic function. Is the value 300 a maximum or a minimum?

Exercise 6M

Choosing which variable to eliminate is an important skill. A bad choice will make the function more complicated.

1 $A = bh$, subject to the constraint $b - h = 7$.

a Use the constraint to express b in terms of h.

b Express A in terms of h.

2 $V = 3xt$ subject to the constraint $x + t = 10$.

a Use the constraint to express x in terms of t.

b Express V in terms of t.

3 $p = x^2y$ subject to the constraint $2x + y = 5$.

a Use the constraint to express y in terms of x.

b Express p in terms of x.

4 $R = \frac{1}{2}nr^2$ subject to the constraint $n - r = 25$.

a Express R in terms of r. **b** Express R in terms of n.

5 $L = 2m(m + x)$ subject to the constraint $\frac{1}{2}(x + 5m) = 50$.

a Express L in terms of m. **b** Express L in terms of x.

6 $V = \pi r^2 h$ and $2r + h = 17$

a Express V in terms of r. **b** Express V in terms of h.

7 $y = 5x^2 + c$ and $12x - 2c = 3$

a Express y in terms of x.

b Use differentiation to find $\frac{dy}{dx}$.

c Hence find the minimum value of y.

d Find the value of c that corresponds to this minimum value.

How do you know, without testing the gradient, that it is a minimum?

8 $N = 2n(5 - x)$ and $12n + 10x = 15$

a Express N in terms of n.

b Use differentiation to find $\frac{dN}{dn}$.

c Hence find the minimum value of N.

d Find the value of x that corresponds to this minimum value.

9 Given $A = \frac{1}{2}LB$ and $3L - 5B = 18$, express A in terms of L. Hence find the minimum value of A and the value of B that corresponds to this minimum value.

10 Given $C = \pi fr$ and $r = 30 - 3f$, express C in terms of either f or r. Hence find the maximum value of C and the values of f and of r that correspond to this maximum value.

11 Given $a - b = 10$ and $X = 2ab$, find the minimum value of X.

12 Given $x + 2t = 12$, find the maximum/minimum value of tx and determine the nature of this optimum value.

Let $A = tx$

13 Given $3y + x = 30$, find the maximum/minimum value of $2xy$ and determine its nature.

14 Given $2M - L = 28$, find the values of L and M which give $3LM$ a maximum/minimum value. Find this optimum value, and determine its nature.

15 Given $c + g = 8$, express $c^2 + g^2$ in terms of g only. Hence find the minimum value of $c^2 + g^2$ subject to the constraint $c + g = 8$.

Let $A = c^2 + g^2$

16 The sum of two numbers is 6. Find the values of these numbers such that the sum of their squares is a minimum.

17 Given that $r + h = 6$, express r^2h in terms of r only. Hence find the maximum value of r^2h subject to the constraint $r + h = 6$.

18 Given that $m + n = 9$, find the maximum/minimum values of m^2n and distinguish between them.

At the beginning of this chapter we defined the optimal design of a can as the one that uses the smallest amount of metal to hold a given capacity. Example 18 calculates the minimum surface area for a can holding 330 cm^3.

Example 18

Find the minimum surface area of a cylinder which has a volume of 330 cm^3.

Answer

Let

A be the total surface area of the cylinder. — *Define the variables.*

r be the radius of the base of the cylinder.

h be the height of the cylinder.

Then $A = 2\pi r^2 + 2\pi rh$

$\pi r^2 h = 330$ — *The constraint is that the volume of the cylinder is 330 cm^3.*

$h = \frac{330}{\pi r^2}$ — *Rearrange to make h the subject.*

$A = 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r\left(\frac{330}{\pi r^2}\right)$ — *Substitute the expression for h into the area function to reduce it to just one variable.*

$= 2\pi r^2 + \frac{660}{r}$ — *Simplify.*

$A = 2\pi r^2 + 660r^{-1}$ — *Write using indices.*

$\frac{dA}{dr} = 4\pi r + (-1)660r^{-2}$ — *Differentiate.*

$\frac{dA}{dr} = 4\pi r - \frac{660}{r^2}$ — *Simplify.*

$4\pi r - \frac{660}{r^2} = 0$ — *Equate $\frac{dA}{dx}$ to zero to find the minimum.*

$4\pi r = \frac{660}{r^2}$ — *Solve.*

$4\pi r^3 = 660$

$r^3 = \frac{660}{4\pi}$

$r^3 = \frac{165}{\pi} \Rightarrow r = \sqrt[3]{\frac{165}{\pi}}$

$r = 3.74$ cm to 3 sf.

You could solve this using a GDC.

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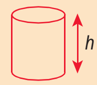
Is a drinks can perfectly cylindrical? What modeling assumptions do you need to make?

The surface area of a cylinder, $A = 2\pi r^2 + 2\pi rh$



The volume of a cylinder, $V = \pi r^2 h$

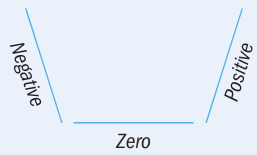
At $r = 1$, $\frac{dA}{dr} = 4\pi(1) - \frac{660}{(1)^2} < 0$

At $r = 10$, $\frac{dA}{dr} = 4\pi(10) - \frac{660}{(10)^2} > 0$

So, the minimum surface area is

$A = 2\pi(3.74)^2 + \frac{660}{(3.74)} = 264 \text{ cm}^2$

Check that the answer for r gives a local minimum point by checking the gradient on each side of r = 3.74.



r = 3.74 is a minimum.

You could find the height of the cylinder with this area by substituting $r = 3.74$ into $h = \frac{330}{\pi r^2}$.

Exercise 6N

Draw a diagram first.

1 A gardener wishes to enclose a rectangular plot of land using a roll of wire-netting that is 40 m long. One side of the plot is to be the wall of the garden.
How should he bend the wire-netting to enclose the maximum area?

2 The sum of two numbers is 20. Let the first number be x. Write down an expression for the second number in terms of x.
Find the value of x given that twice the square of the first number added to three times the square of the second number is a minimum.

EXAM-STYLE QUESTIONS

3 An **open** rectangular box has its length double its width. The total surface area of the box is 150 cm^2.
The width of the box is x cm, and its height is h cm. Express the total surface area of the box in terms of x and h.
Use this expression (constraint) to find the volume of the box in terms of x only.
Hence, find the greatest possible volume of the box, and the width, length and height of the box required to give this volume.

4 A piece of wire 24 cm long is to be bent to form a rectangle with just one side duplicated for extra strength. Find the dimensions of the rectangle that give the maximum area.

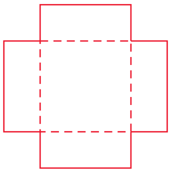
5 A long strip of metal 120 cm wide is bent to form the base and two sides of a chute with a rectangular cross-section.
Find the width of the base that makes the area of the cross-section a maximum.

6 The sum of the height and the radius of the base of a cone is 12 cm. Find the maximum volume of the cone and the values of the height and the radius required to give this volume.

7 A closed box with a square base is to be made out of 600 cm^2 of metal. Find the dimensions of the box so that its volume is a maximum. Find the value of this maximum volume.

8 The total surface area of a closed cylindrical tin is to be 600 cm^2. Find the dimensions of the tin if the volume is to be a maximum.

9 A square sheet of metal of side 24 cm is to be made into an open tray of depth x cm by cutting out of each corner a square of side x cm and folding up along the dotted lines as shown in the diagram. Show that the volume of the tray is $4x(144 - 24x + x^2)$ cm^3. Find the value of x for this volume to be a maximum.



10 A rectangular sheet of metal measures 16 cm by 10 cm. Equal squares of side x cm are cut out of each corner and the remainder is folded up to form a tray of depth x cm. Show that the volume of the tray is $4x(8 - x)(5 - x)$ cm^3, and find the maximum volume.

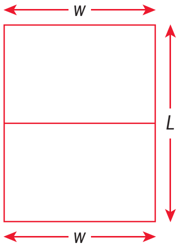
11 A tin of soup is made in the shape of a cylinder so that the amount of metal used in making the tin is a minimum. The volume of the tin is 350 cm^3.

- **a** If the radius of the base of the tin is 5 cm, find the height of the tin.
- **b** If the radius of the base of the tin is 2 cm, find the height of the tin.
- **c** **i** Use the volume of the tin to write down the constraint between the radius of the tin and its height.
  - **ii** Show that the constraint can be written as $h = \frac{350}{\pi r^2}$
  - **iii** Find an expression for A, the total surface area of a cylinder, in terms of r only.
  - **iv** Find the dimensions of the tin that minimize the total surface area of the tin.
  - **v** Find the value of this minimum area.

The metal used in making the tin is the surface area of the cylinder.

12 The diagram shows a rectangular field with an area of 50 000 m^2. It has to be divided in half and also fenced in. The most efficient way to enclose the area is to construct the fencing so that the total length of the fence is minimized.

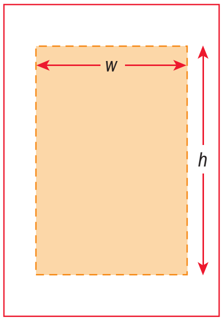
- **a** If the length (L) of the field is 200 m, what is the width?
- **b** Find the total length of the fencing in this case.
- **c** Use the fixed area to write down the problem constraint algebraically.
- **d** Find the dimensions of the field that make the length of fencing a minimum.
  Find the perimeter of the field in this case.



13 A second rectangular field is identical to that in question **12**. The cost of the fencing around the perimeter is \$3 per metre. The cost of the dividing fence is \$5 per metre. The most efficient way to enclose the area minimizes the total **cost** of the fence.

- **a** Find the total cost of the fencing when the length (L) is 200 m.
- **b** Use the fixed area to write down the problem constraint algebraically.
- **c** Find the dimensions of the field that make the **cost** of the fencing a minimum. Find the cost in this case.

14 The page of a mathematics book is designed to have a printable area of 144 cm² plus margins of 2 cm along each side and 3 cm at the top and the bottom. The diagram for this is shown with the printable area shaded.



a If the width of the printable area (w) is 9 cm, find its height (h). Using these values, find the area of the page.

b If the width of the printable area is 14 cm, calculate the area of the page.

c Write down an expression for the printable area in terms of w and h.

d Write down an expression for P, the area **of the page** in terms of w and h.

e Use the results of **c** and **d** to show that $P = 168 + 4h + \frac{864}{h}$.

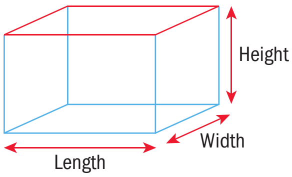
f Find the dimensions of the page that minimize the page area.

15 A fish tank is to be made in the shape of a cuboid with a rectangular base, with a length twice the width. The volume of the tank is fixed at 225 litres. The tank is to be made so that the total length of steel used to make the frame is minimized.

a **i** If the length of the base is 100 cm, what is its width?

ii Show that in this case, the height of the tank is 45 cm.

iii Find the total length of the steel frame.



b If the width of the tank is x, find an expression for the volume of the tank in terms of x and h, the height of the tank.

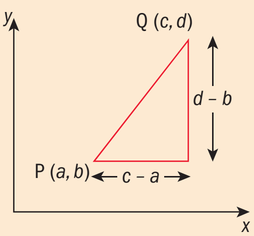
c Show that L, the total length of the steel frame, can be written as $L = 6x + \frac{450\,000}{x^2}$.

d Find the dimensions of the tank that minimize the length of the steel frame. Find also the length of the frame in this case.

CHAPTER 6 SUMMARY

Introduction to differentiation

- If P is the point (a, b) and Q is (c, d) then the gradient, m, of the straight line PQ is $m = \frac{d-b}{c-a}$.



The gradient function

- To differentiate a function, find the gradient function:

Function	Gradient function
$y = ax^n$	$\frac{dy}{dx} = nax^{n-1}$
$f(x) = ax^n$	$f'(x) = nax^{n-1}$

The process is valid for **all** values of n, both positive and negative.

Continued on next page

Calculating the gradient of a curve at a given point

- You can use the gradient function to determine the exact value of the gradient at any specific point on the curve.
- At a local maximum or minimum, $f'(x) = 0$ $\left(\frac{dy}{dx} = 0\right)$

The tangent and the normal to a curve

- The tangent to the curve at any point P is the straight line which passes through P with gradient equal to the gradient of the curve at P.
- To find the equation of the tangent to the curve at P(a, b):
  1. Calculate b, the y-coordinate of P, using the equation of the curve.
  2. Find the gradient function $\frac{dy}{dx}$.
  3. Substitute a, the x-coordinate of P, into $\frac{dy}{dx}$ to calculate, m, the value of the gradient at P.
  4. Use the equation of a straight line $(y - b) = m(x - a)$.
- The normal is perpendicular to the tangent so its gradient, m', is found using the formula $m' = \frac{-1}{m}$, where m is the gradient of the tangent.

Rates of change

- For the graph $y = f(x)$, the gradient function $\frac{dy}{dx} = f'(x)$ gives the rate of change of y with respect to x.

Local maximum and minimum points (turning points)

- At a local maximum, the curve stops increasing and changes direction so that it 'turns' and starts decreasing. So, as x increases, the three gradients occur in the order: positive, zero, negative. Where the gradient is zero is the maximum point.
- At a local minimum, the curve stops decreasing and changes direction; it 'turns' and starts increasing. So, as x increases, the three gradients occur in the order: negative, zero, positive. Where the gradient is zero is the minimum point.
- At any stationary or turning point – either local maximum or local minimum – $f'(x)$ is zero.

Using differentiation in modeling: optimization

- In optimization problems, use differentiation to find an optimal value (either the maximum or the minimum) of a function as two variables interact.