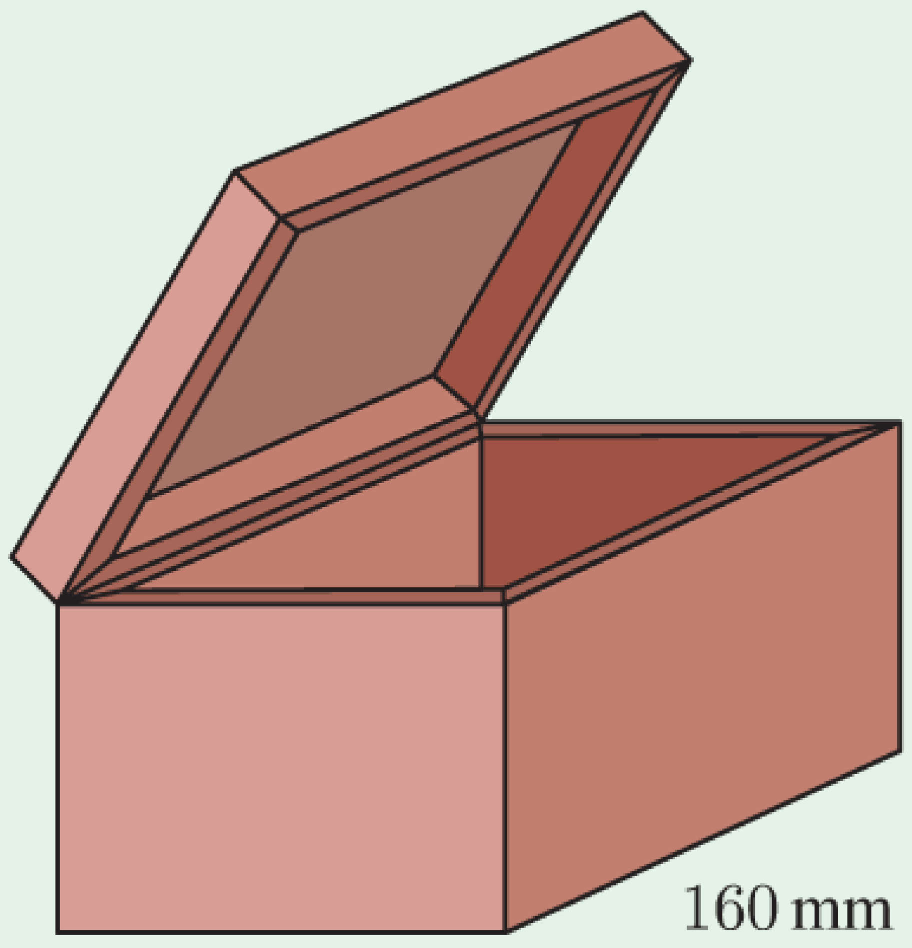
Chapter 6

Measurement

Contents:

OPENING PROBLEM

A jewellery box is made of wood 4 mm thick.
When shut, its height is 88 mm.



Things to think about:

a What is the *external* surface area of the container?

b Why is it useful to specify the "external" surface area when talking about a container?

c Can you find:

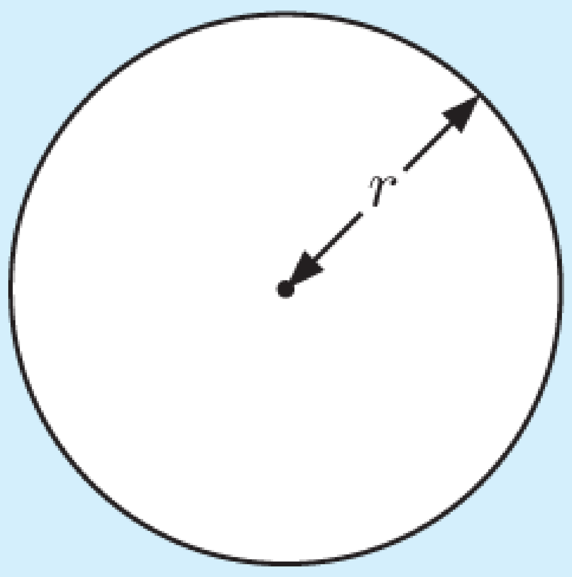
- **i** the *volume* of jewellery the box can hold
- **ii** the *capacity* of the box
- **iii** the *volume* of wood used to make the box?

In previous years you should have studied measurement extensively. In this Chapter we revise measurements associated with parts of a circle, as well as the surface area and volume of 3-dimensional shapes.

A CIRCLES, ARCS, AND SECTORS

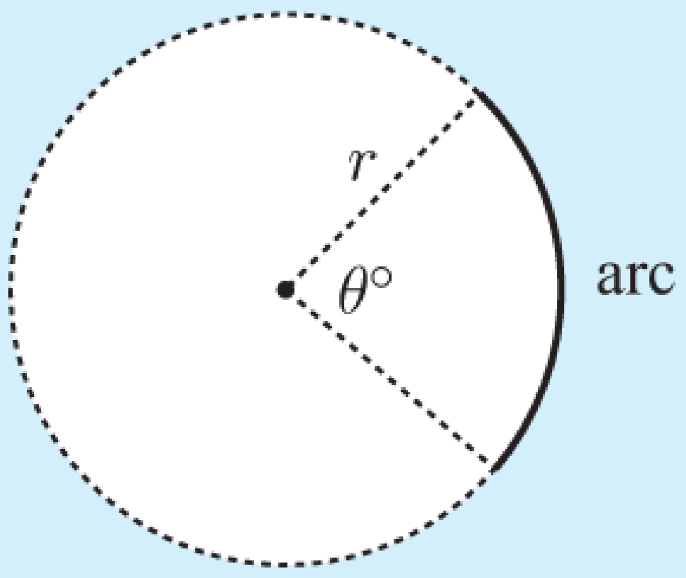
For a **circle** with radius r:

- the **circumference** $C = 2\pi r$
- the **area** $A = \pi r^2$.



An **arc** is a part of a circle which joins any two different points. It can be measured using the angle $\theta°$ subtended by the points at the centre.

Arc length $= \dfrac{\theta}{360} \times 2\pi r$

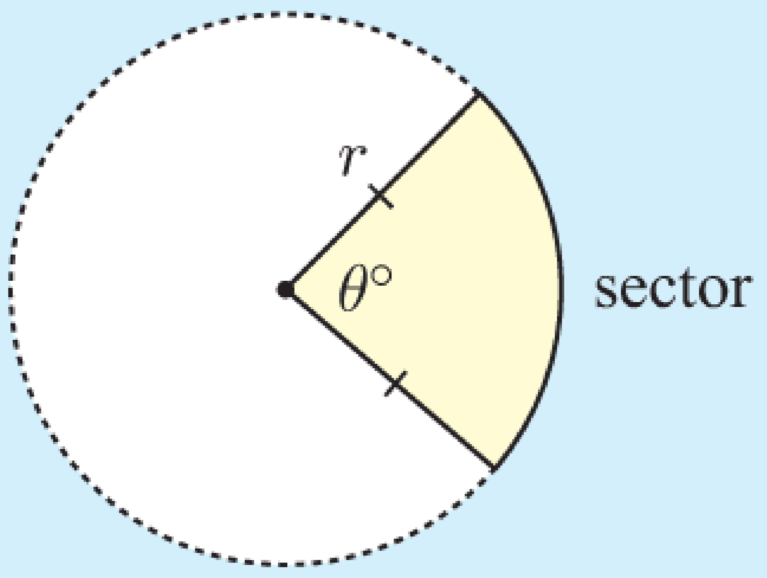


A **sector** is the region between two radii of a circle and the arc between them.

Perimeter = two radii + arc length

$= 2r + \dfrac{\theta}{360} \times 2\pi r$

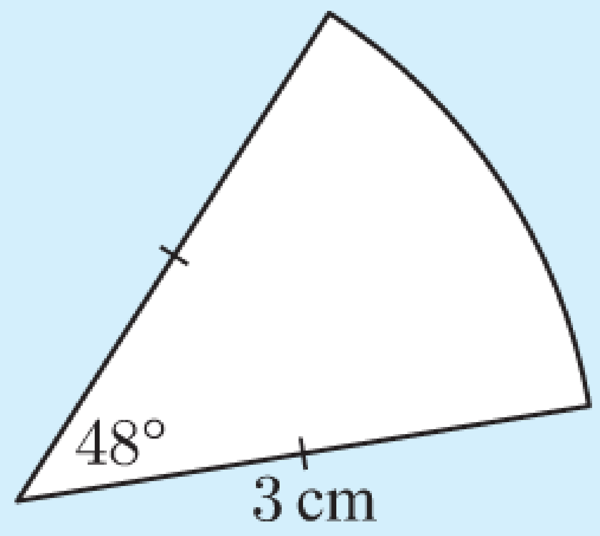
Area $= \dfrac{\theta}{360} \times \pi r^2$



Example 1 **Self Tutor**

For the given figure, find to 3 significant figures:

a the length of the arc
b the perimeter of the sector
c the area of the sector.



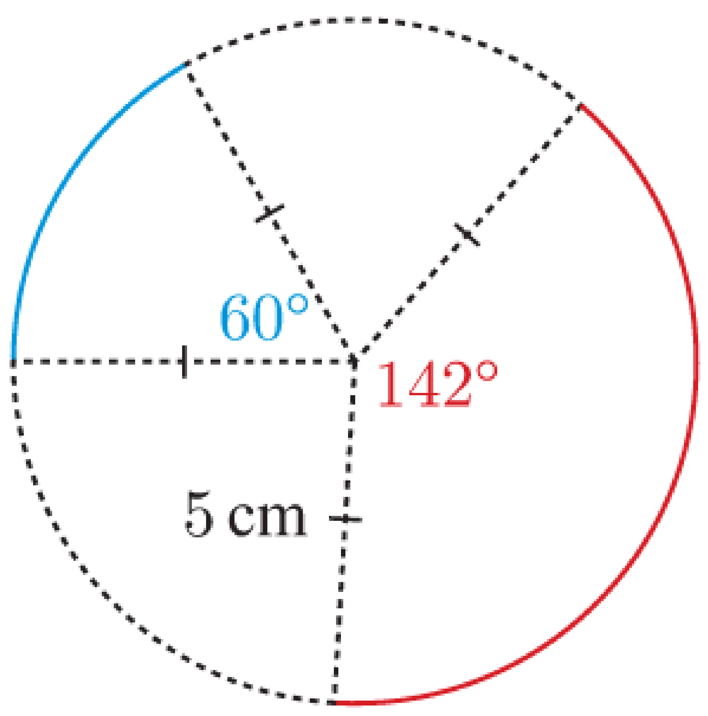
a Arc length $= \frac{\theta}{360} \times 2\pi r$
$= \frac{48}{360} \times 2\pi \times 3$ cm
≈ 2.51 cm

b Perimeter $= 2r +$ arc length
$\approx 2 \times 3 + 2.51$ cm
≈ 8.51 cm

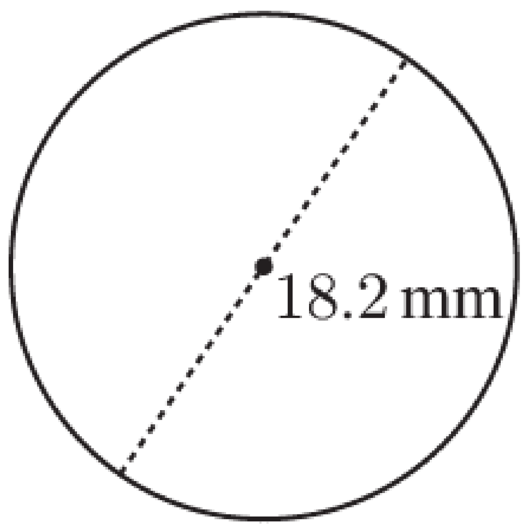
c Area $= \frac{\theta}{360} \times \pi r^2$
$= \frac{48}{360} \times \pi \times 3^2$
$\approx 3.77\text{ cm}^2$

EXERCISE 6A

1 Find the length of:
a the blue arc
b the red arc.



2 Find the perimeter of:

a


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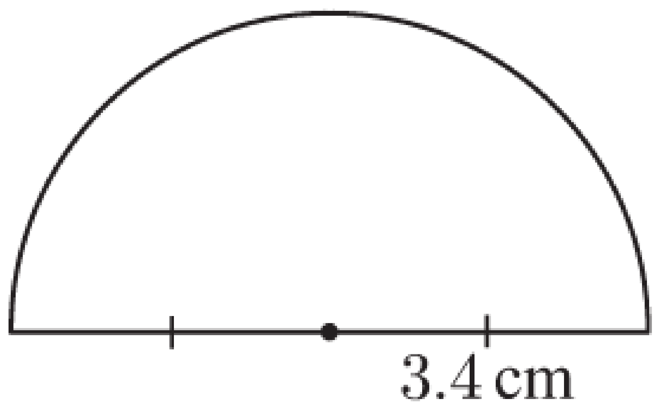
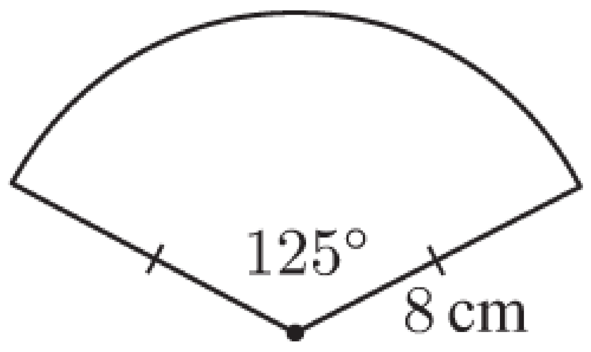


c


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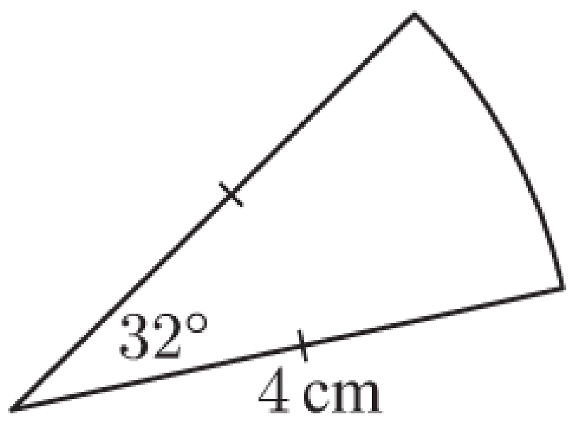

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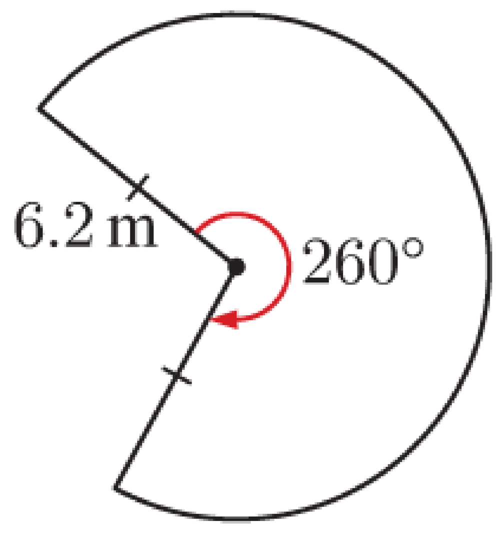

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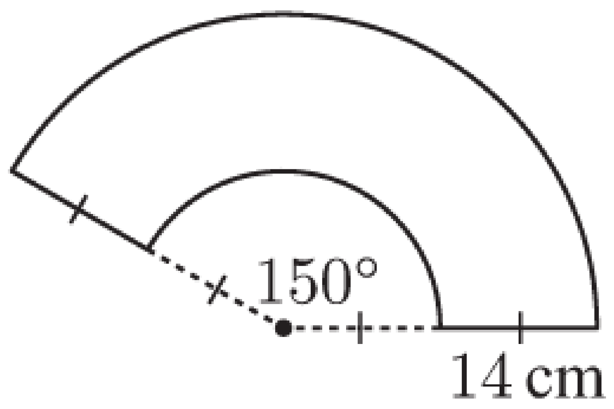
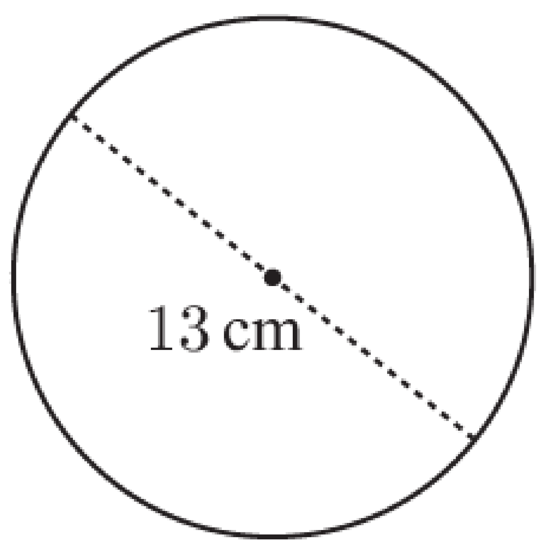


3 An arc of a circle makes a 36° angle at its centre. If the arc has length 26 cm, find the radius of the circle.

4 A sector of a circle makes a 127° angle at its centre. If the arc of the sector has length 36 mm, find the perimeter of the sector.

5 Find the area of:

a


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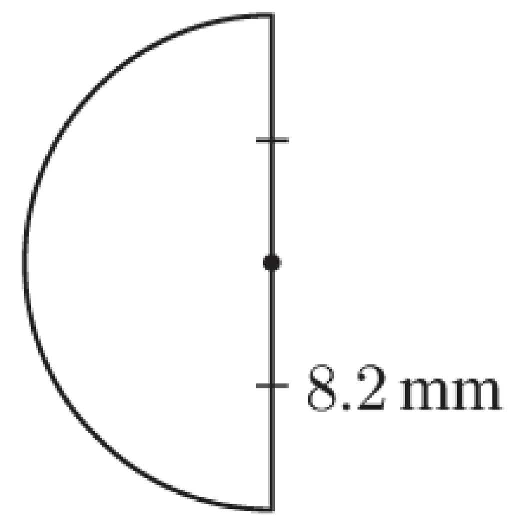
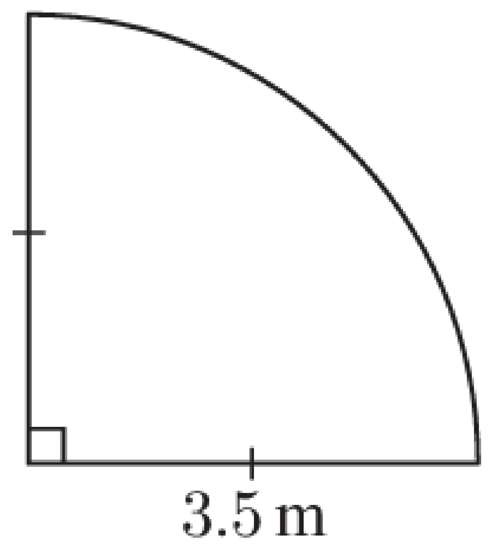


c


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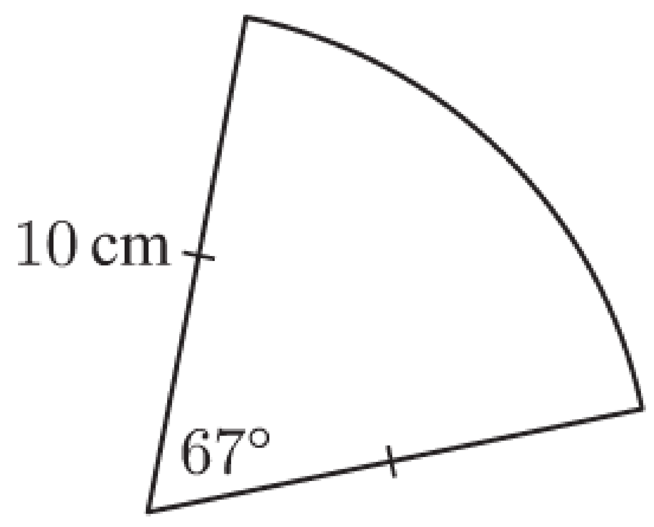

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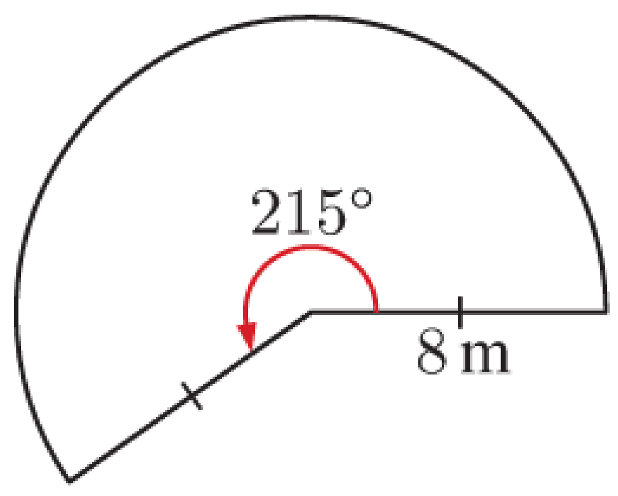

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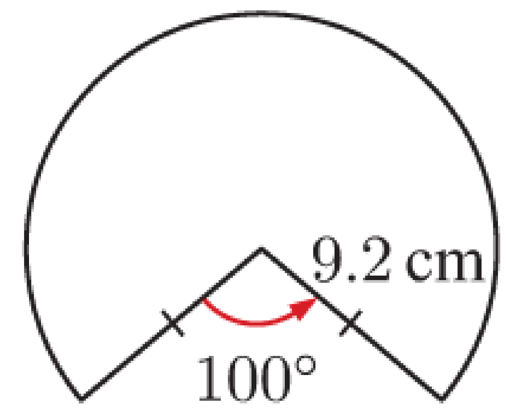
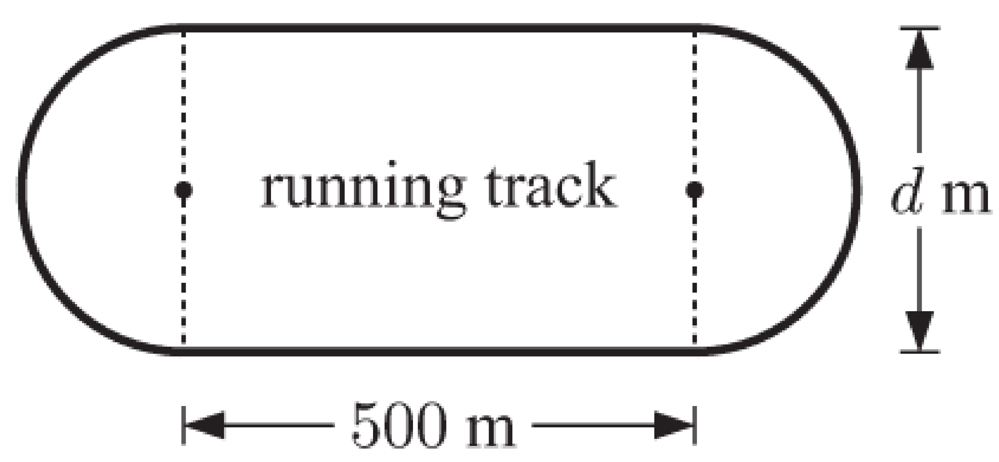


6 Find the radius of a sector with angle 67° and area 16.2 cm^2.

7 Find the perimeter of a sector with angle 136° and area 28.8 cm^2.

8 A running track consists of two straight segments joined by semi-circular ends, as shown. The total perimeter of the track is 1600 metres.



a Determine the diameter of the semi-circular ends.

b Jason takes 4 minutes and 25 seconds to complete a single lap of the track. Calculate Jason's average speed in $\text{m}\,\text{s}^{-1}$.

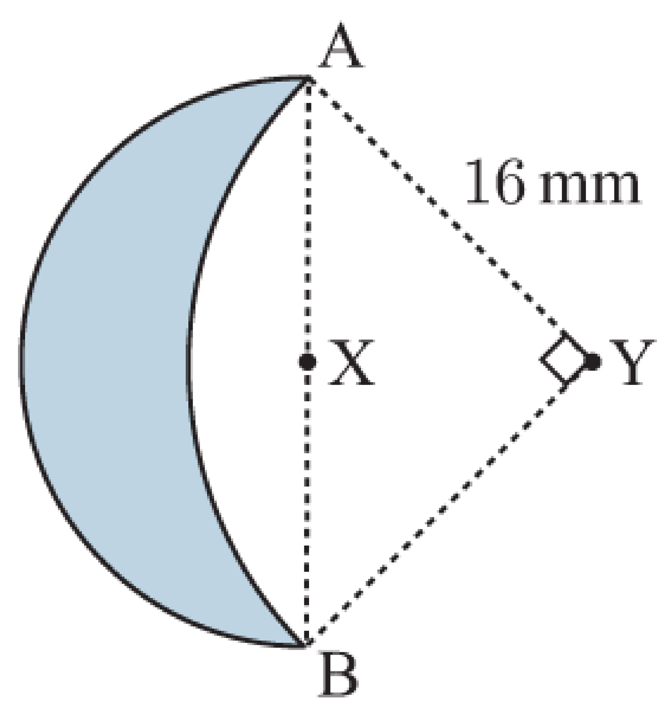
9 X and Y are the centres of the circles containing the two arcs AB shown.

Find:

a the length AX

b the perimeter of the shaded crescent

c the area of the crescent.



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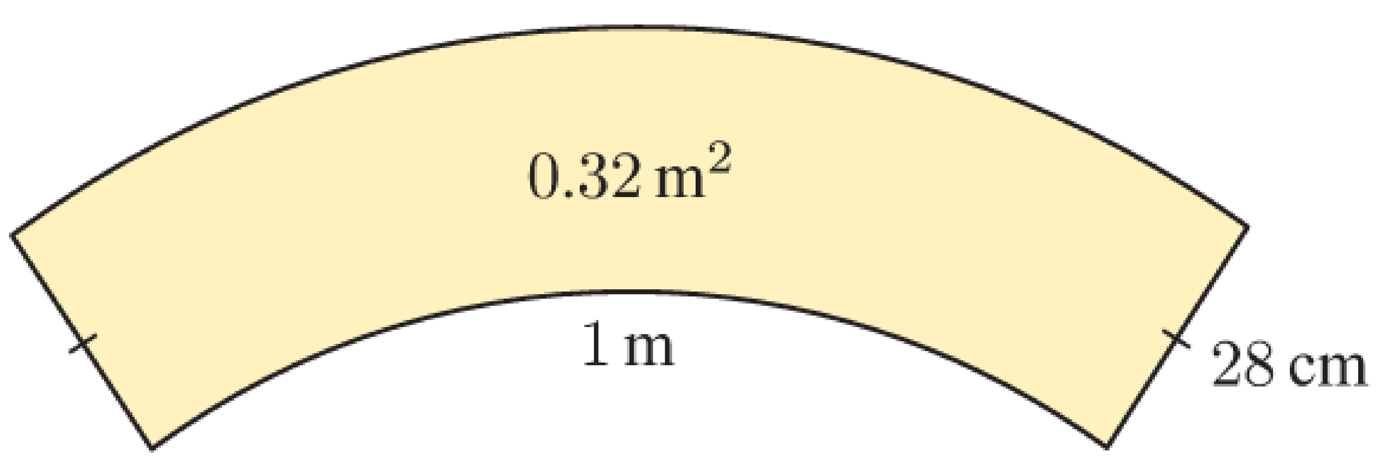
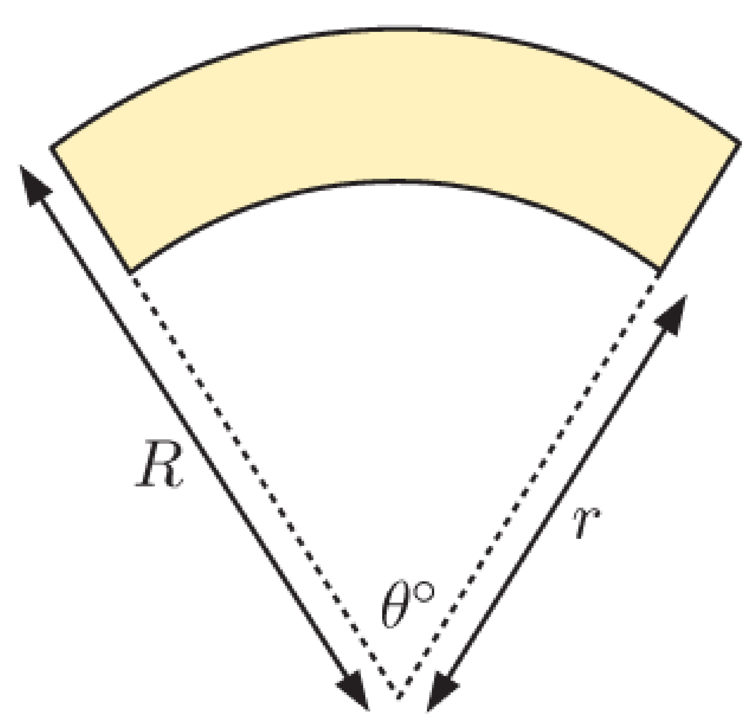


Belinda has made a lampshade with area 0.32 m^2. Its shorter arc has length 1 m, and its slant height is 28 cm.

Suppose the material is cut as the difference between two sectors with common angle $\theta°$, and radii r and R.



a Show that the area of the lampshade is given by $A = \dfrac{0.28\theta}{360}\pi(2r + 0.28)\ \ \text{m}^2$.

b Use the smaller sector to show that $\theta = \dfrac{180}{\pi r}$.

c Find r and θ.

d Hence find the length of the longer arc.

B SURFACE AREA

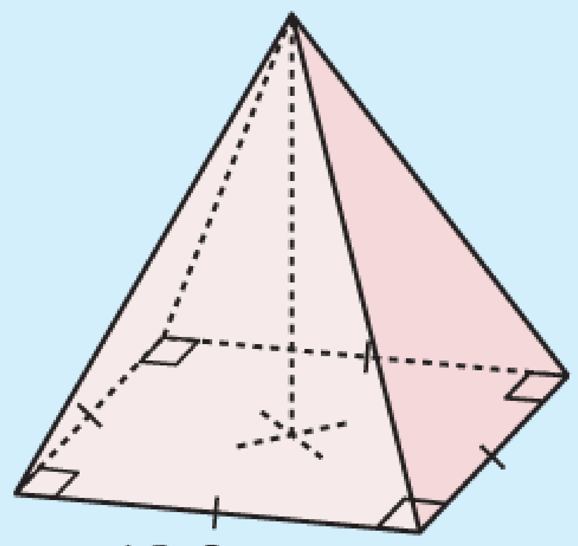
SOLIDS WITH PLANE FACES

The **surface area** of a three-dimensional figure with plane faces is the sum of the areas of the faces.

The surface area is therefore the same as the area of the **net** required to make the figure.

Example 2 Self Tutor

The pyramid shown is 10.8 cm high.
Find its surface area.



The net of the pyramid includes one square with side length 13.2 cm, and four isosceles triangles with base 13.2 cm.

Let the height of the triangles be h cm.

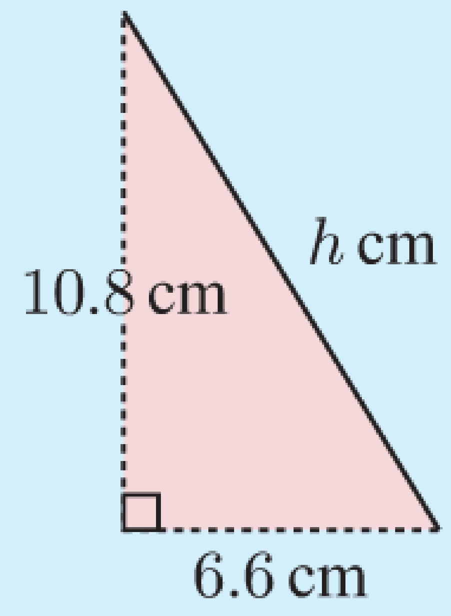
Now $h^2 = 10.8^2 + 6.6^2$ {Pythagoras}

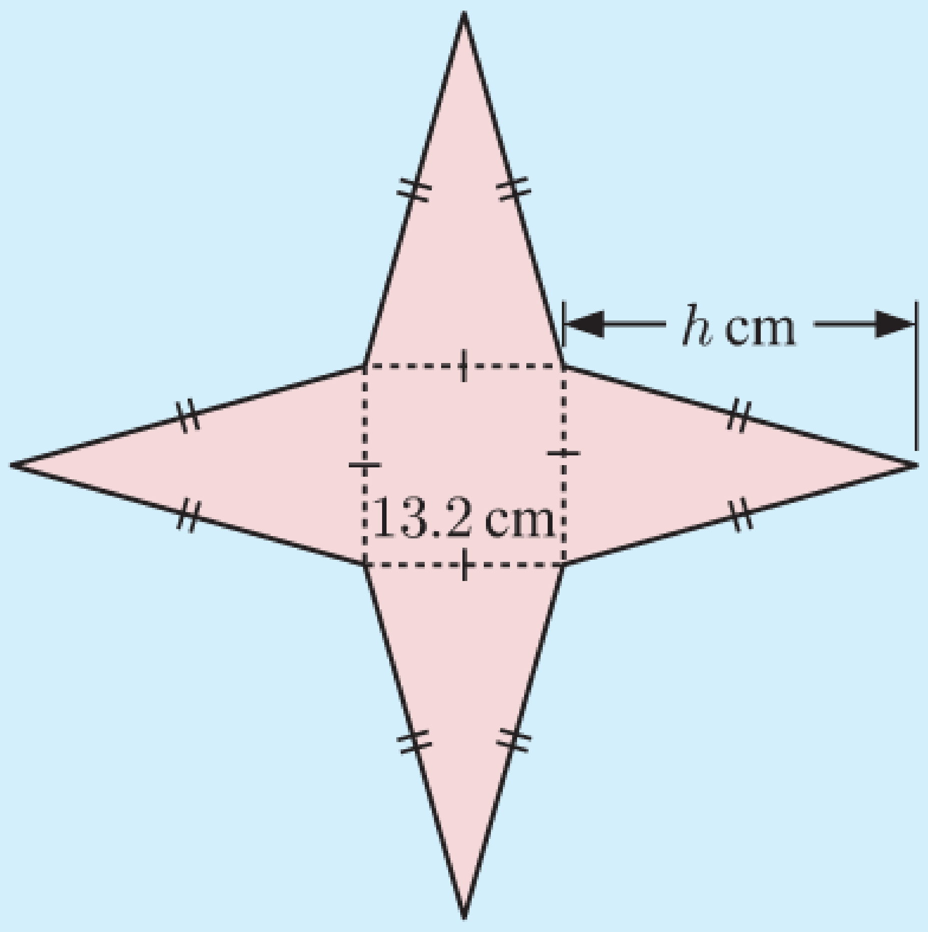
$\therefore\ h = \sqrt{10.8^2 + 6.6^2} \approx 12.66$

$\therefore$ the surface area

$\approx 13.2^2 + 4 \times (\frac{1}{2} \times 13.2 \times 12.66)\ \text{cm}^2$

$\approx 508\ \text{cm}^2$

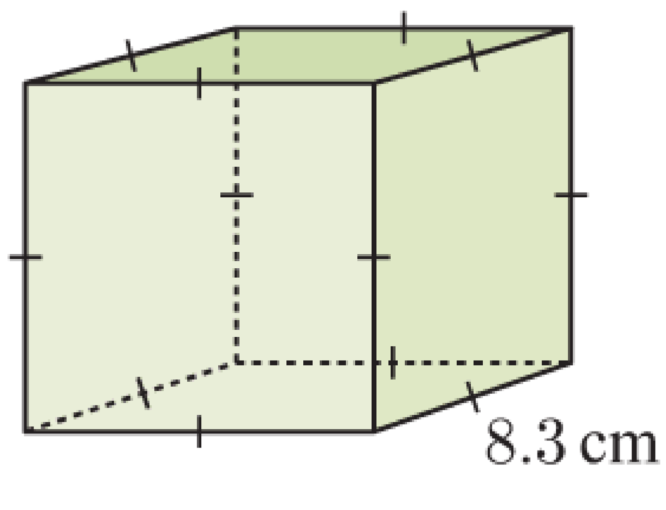




EXERCISE 6B.1

1 Find the surface area of each rectangular prism:

a



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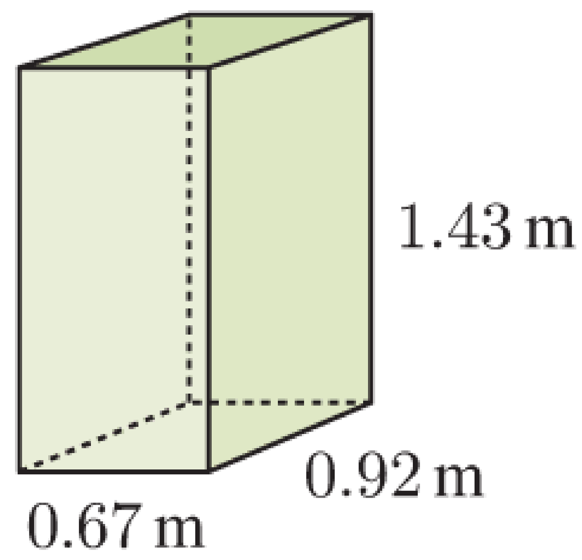

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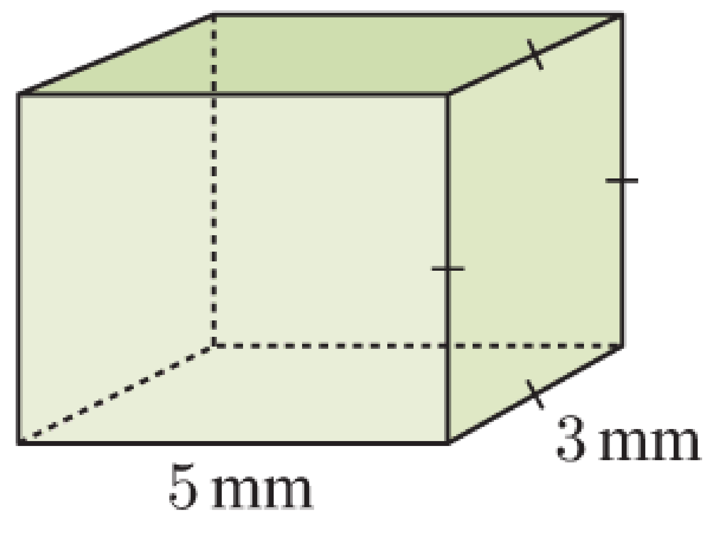
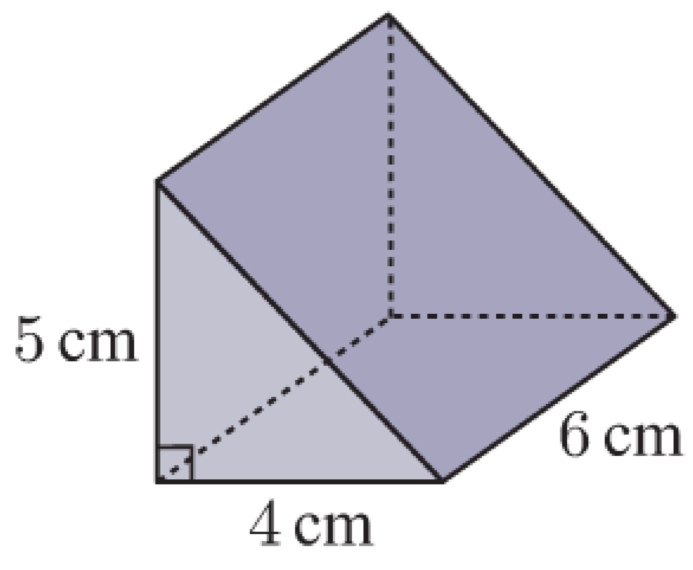
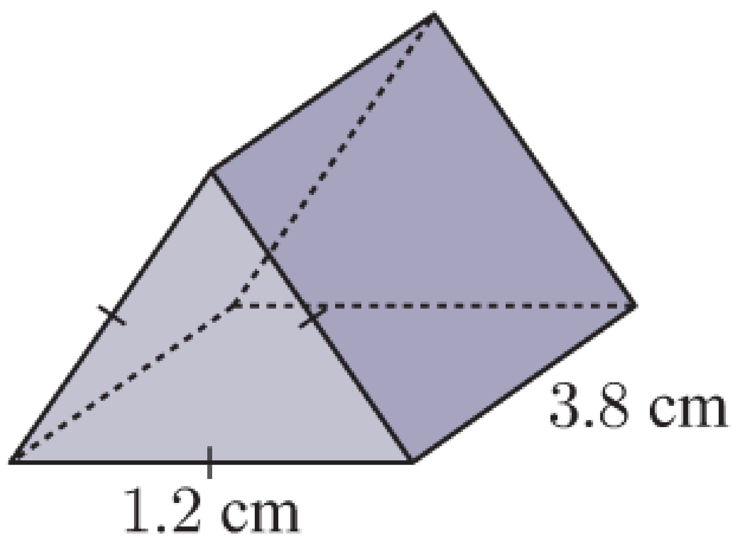


2 Find the surface area of each triangular prism:

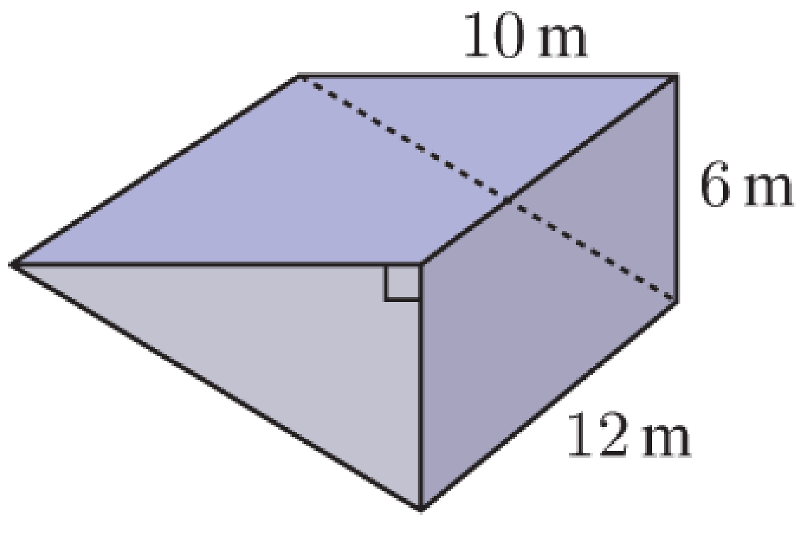
a



b



c



3 Draw each solid and hence find its surface area:

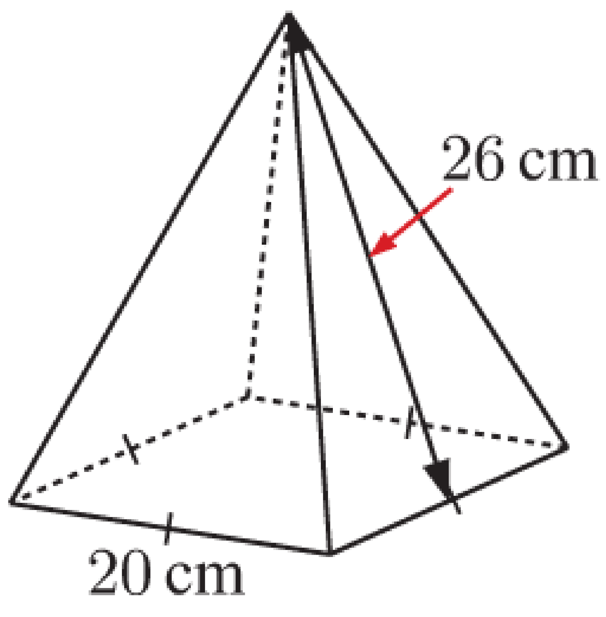
a an ice cube with side length 2.5 cm

b a block of cheese measuring 14 cm by 8 cm by 3 cm

c a wooden wedge with length 6.2 cm and a cross-section which is a right angled triangle with legs 10.6 cm and 2.8 cm.

4 Find the surface area of each pyramid:

a



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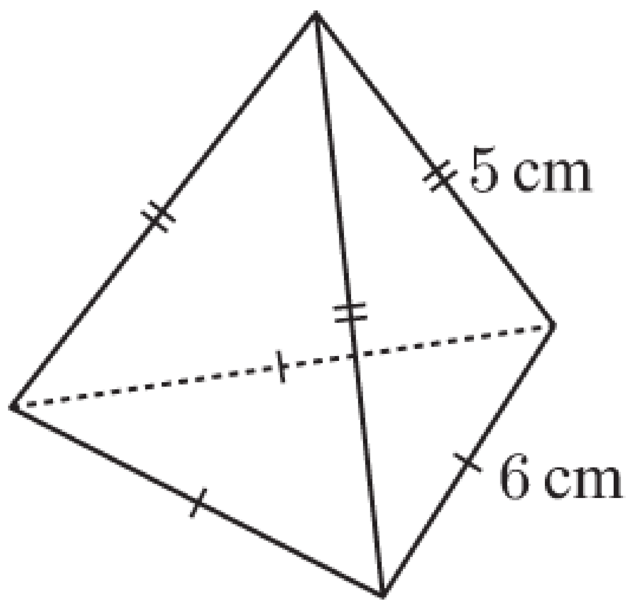

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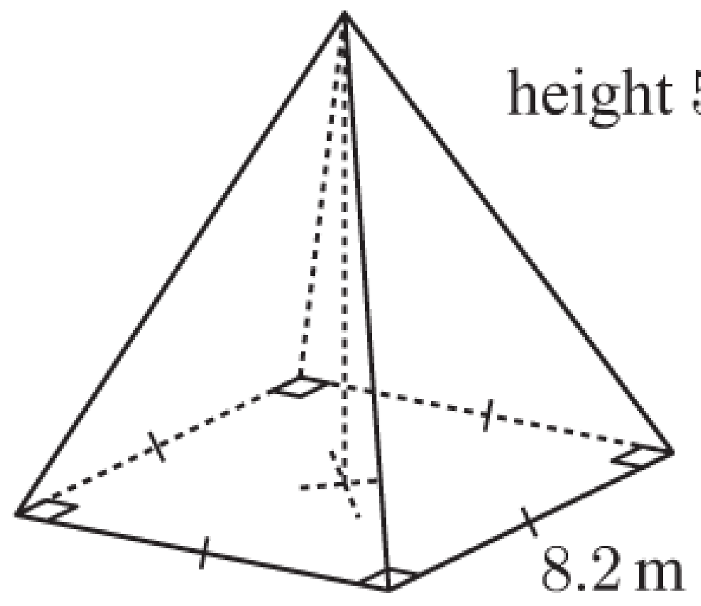

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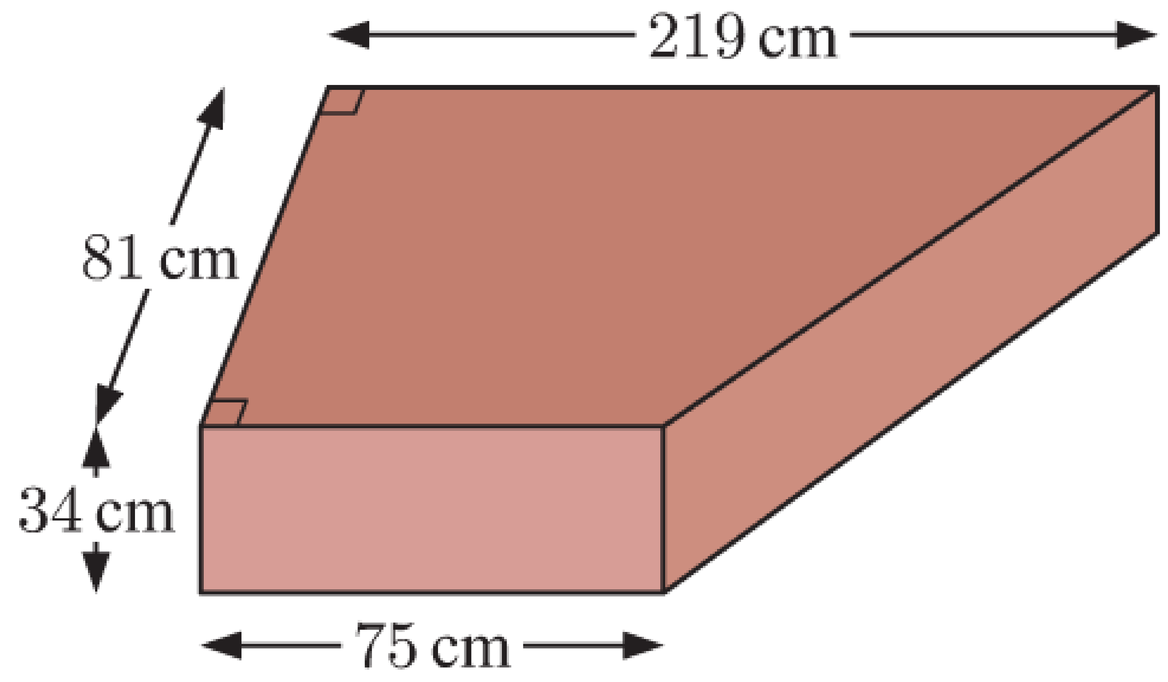


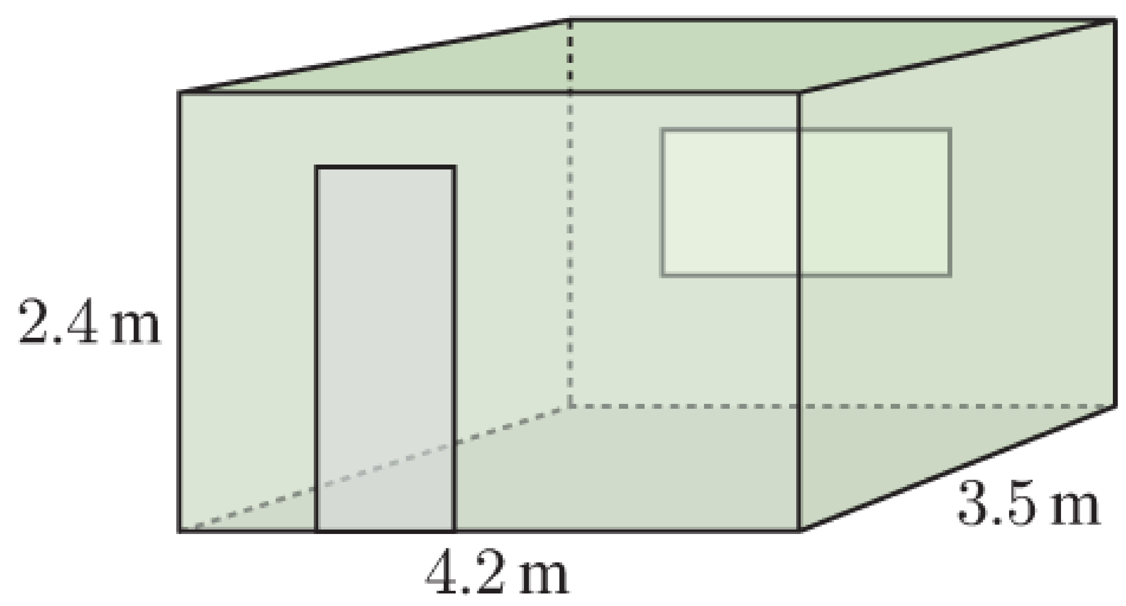
A harpsichord case has the dimensions shown.

a Find the total area of the top and bottom surfaces.

b Find the area of each side of the case.

c If the timber costs €128 per square metre, find the value of the timber used to construct this case.

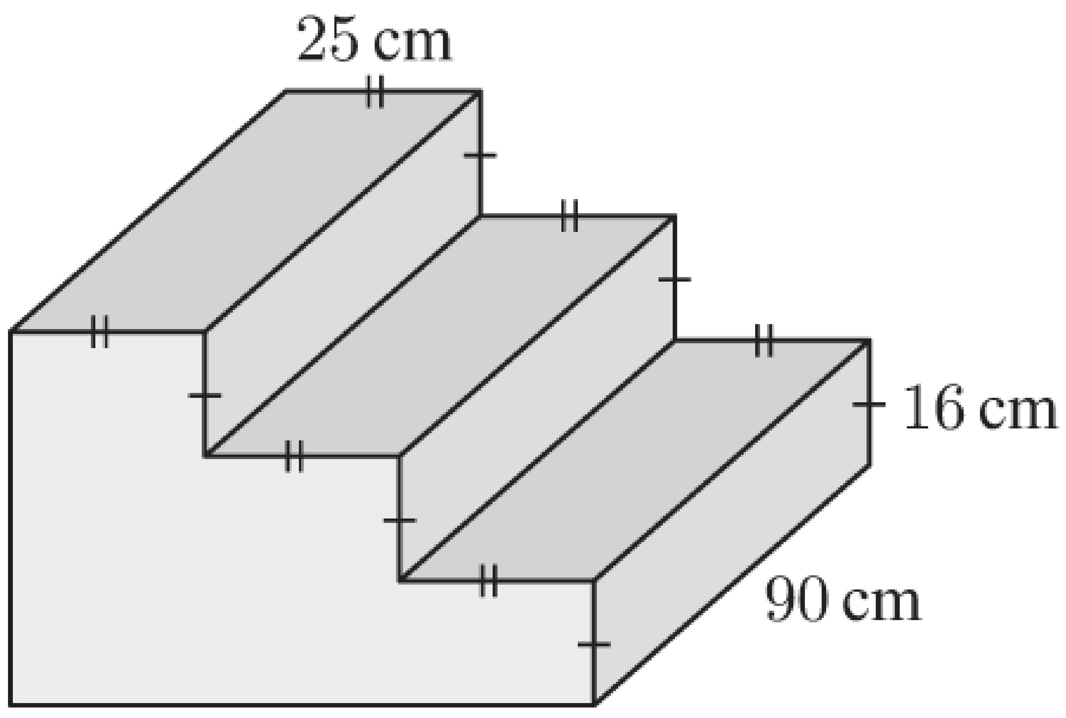
6 The walls and ceiling of this room need to be painted with two coats of paint. The door is 0.8 m by 2.2 m and the window is 183 cm by 91 cm. The door also has to be stained on *both* sides with two coats of stain. Use the following table to calculate the total cost of the stain and paint:



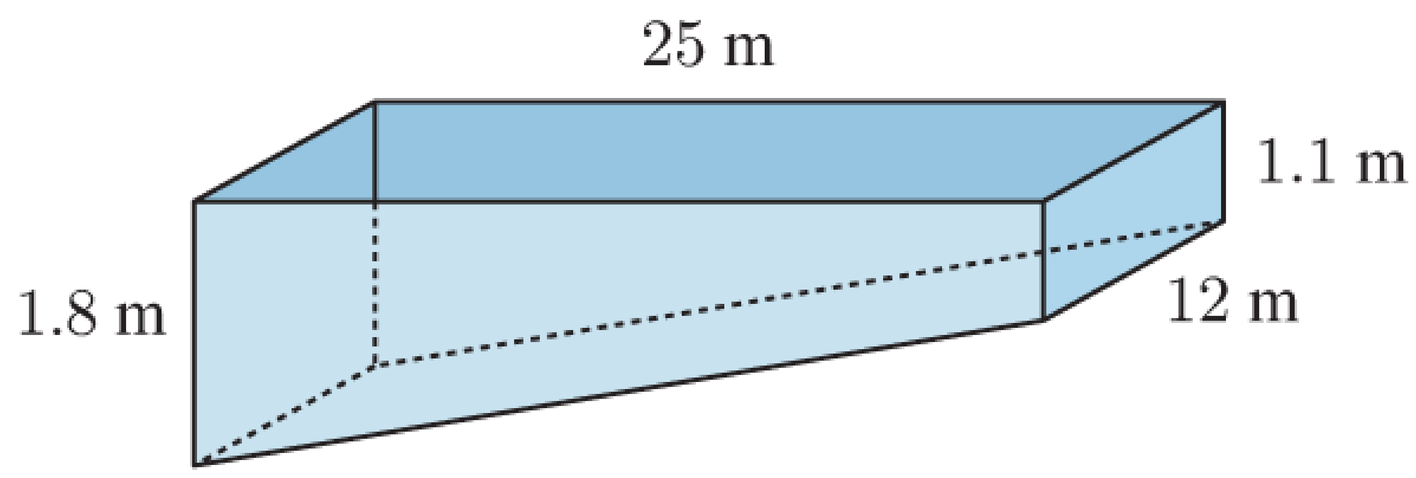
Type of paint	*Size*	*Area covered*	*Cost per tin*
wall paint	4 litres	16 m^2	$32.45
	2 litres	8 m^2	$20.80
wood stain	2 litres	10 m^2	$23.60
(for doors)	1 litre	5 m^2	$15.40

7 Find the surface area of:

a this set of steps

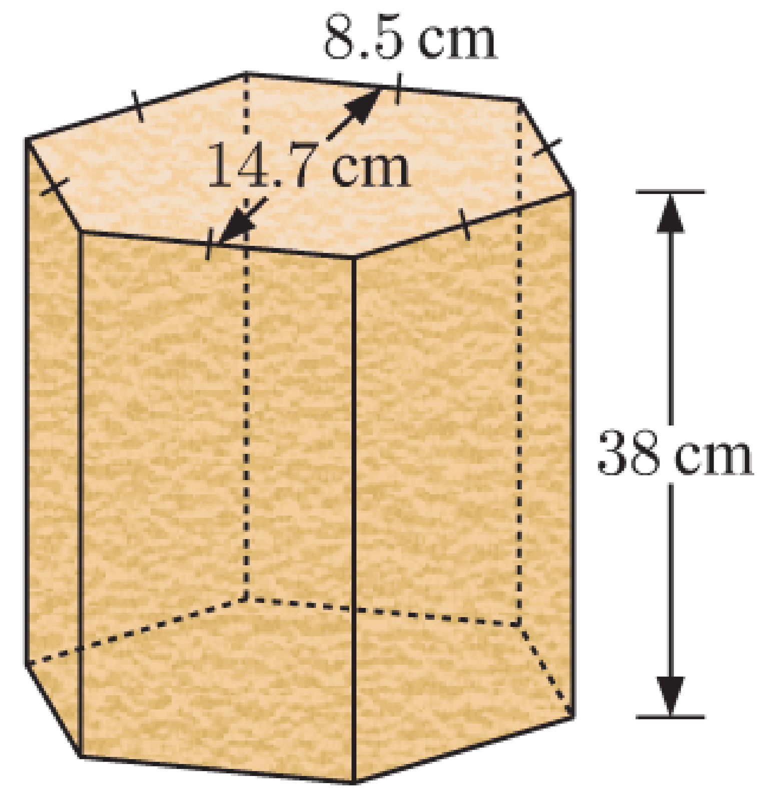


b the sides and base of this swimming pool.



8 The **Taylor Prism** is a regular hexagonal prism made of clay with a historical record written on its sides. It was found by archaeologist **Colonel Taylor** in 1830. If the ancient Assyrians had written on all the surfaces, what total surface area would the writing have covered?

Hint: A regular hexagon can be divided into six equilateral triangles.



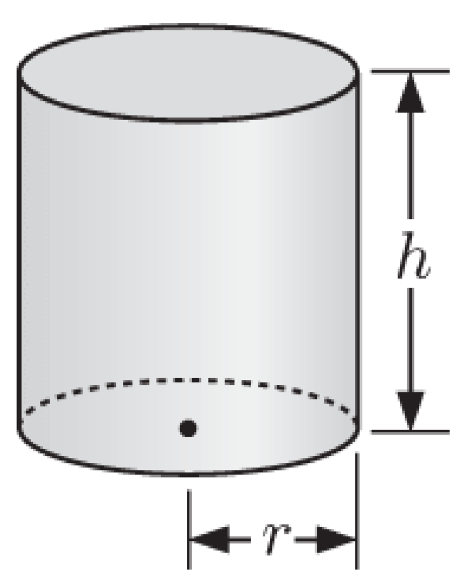
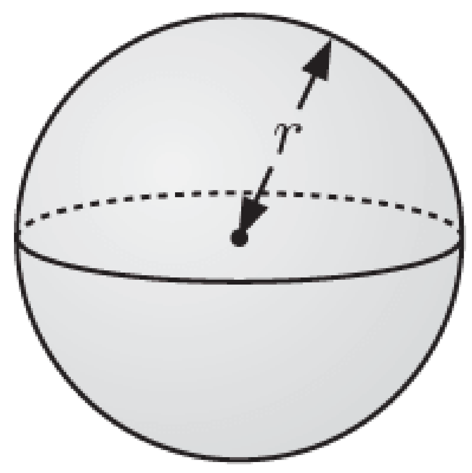
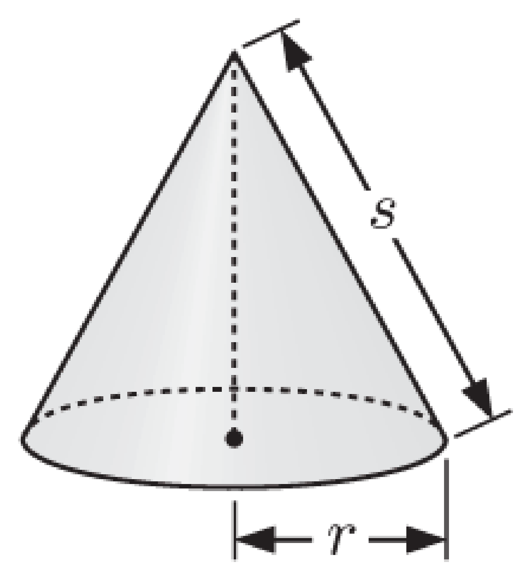
9 Write a formula for the surface area of:

a a rectangular prism with side lengths x cm, $(x+2)$ cm, and $2x$ cm

b a square-based pyramid for which every edge has length x cm.

SOLIDS WITH CURVED SURFACES

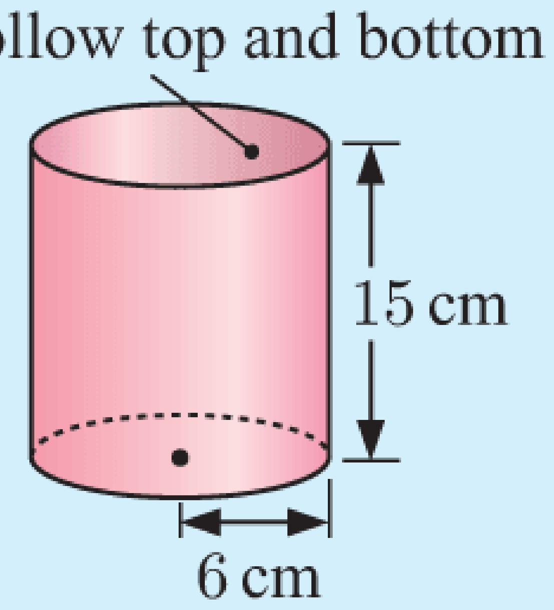
These objects have curved surfaces, but their surface areas can still be calculated using formulae.

Cylinder	**Sphere**	**Cone**
		
$\boldsymbol{A}$ = **curved surface** + **2 circular ends** $= \boldsymbol{2\pi rh + 2\pi r^2}$	$\boldsymbol{A = 4\pi r^2}$	$\boldsymbol{A}$ = **curved surface** + **circular base** $= \boldsymbol{\pi rs + \pi r^2}$

Example 3 **Self Tutor**

Find, to 1 decimal place, the outer surface area of:

a hollow top and bottom



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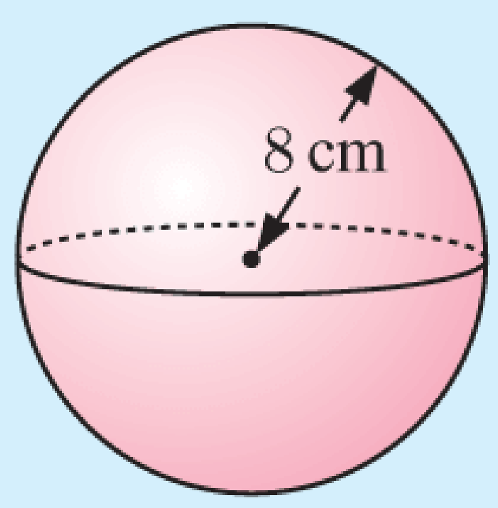

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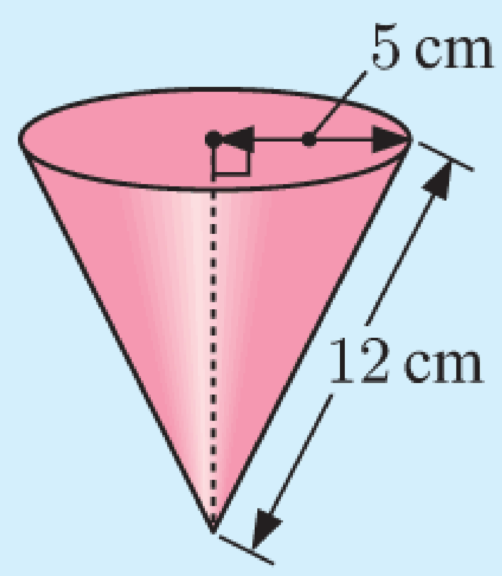


a The cylinder is hollow top and bottom, so we only have the curved surface.

$A = 2\pi rh$

$= 2 \times \pi \times 6 \times 15$

$\approx 565.5 \text{ cm}^2$

b $A = 4\pi r^2$

$= 4 \times \pi \times 8^2$

$\approx 804.2 \text{ cm}^2$

c $A = \pi rs + \pi r^2$

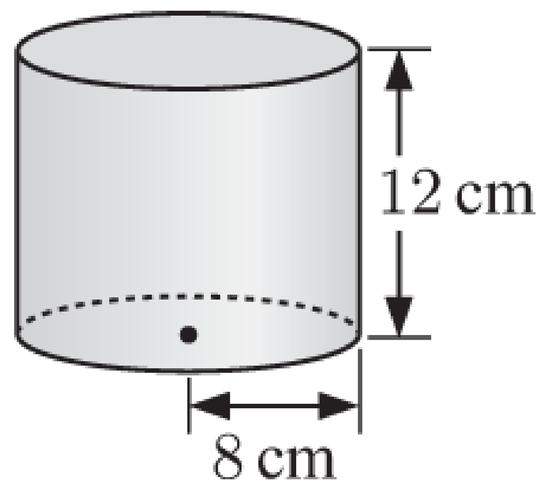
$= \pi \times 5 \times 12 + \pi \times 5^2$

$\approx 267.0 \text{ cm}^2$

EXERCISE 6B.2

1 Find, to 1 decimal place, the outer surface area of:

a



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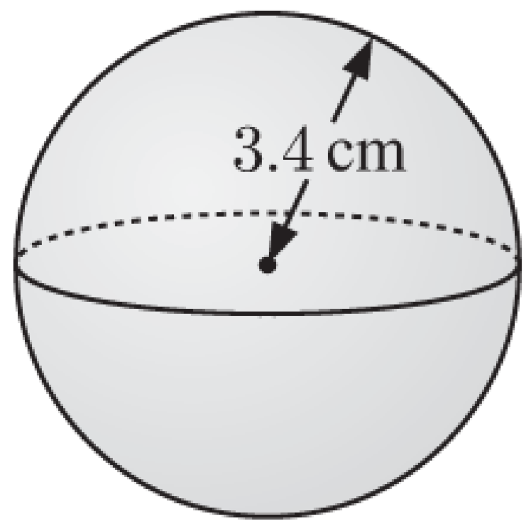

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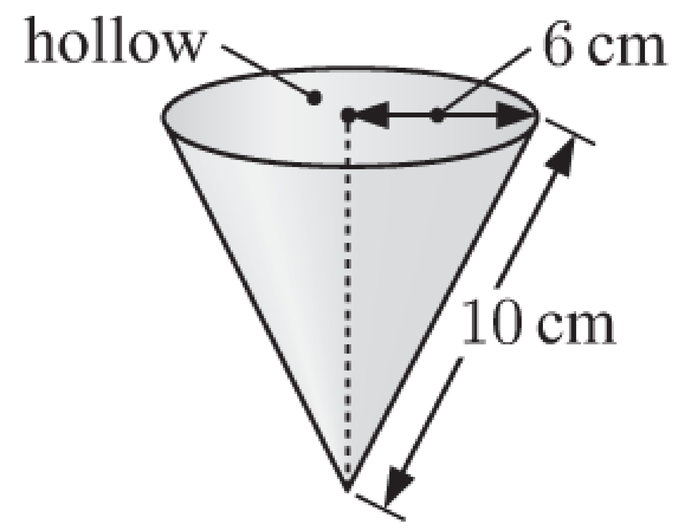

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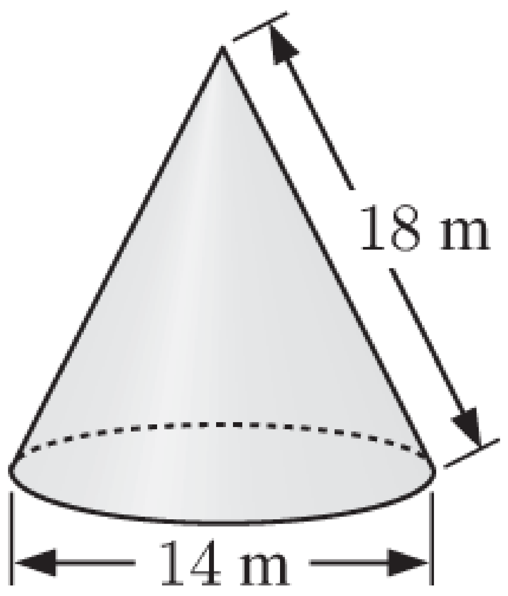

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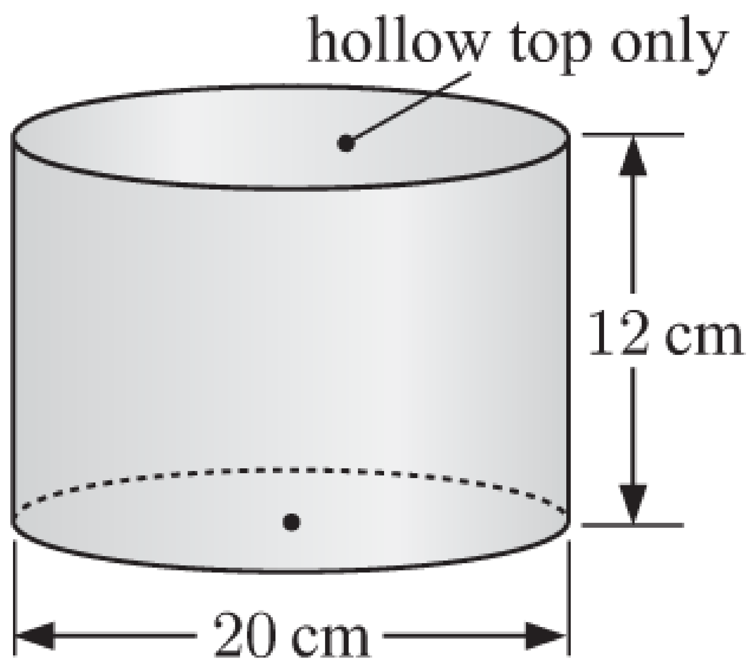

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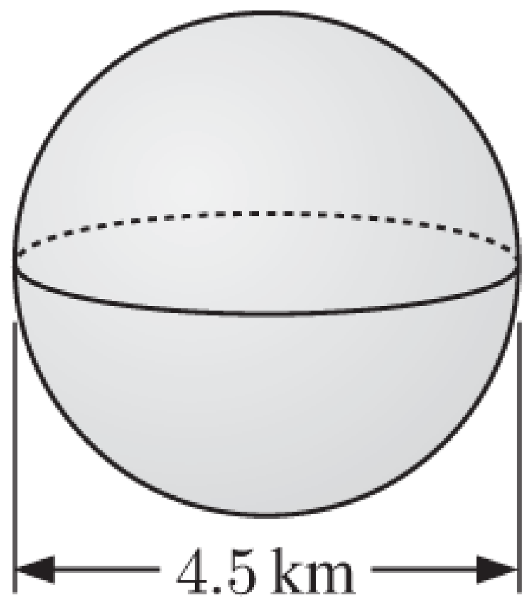
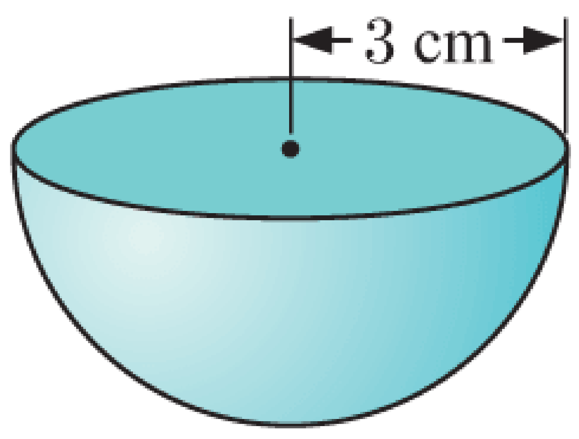


2 Find the total surface area of the solid hemisphere shown.

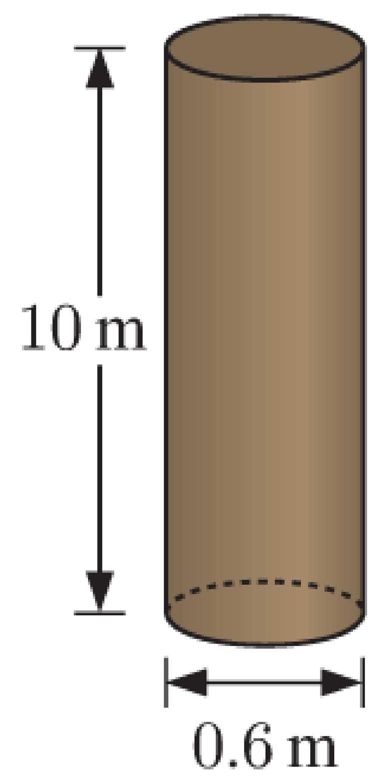


3 Find the surface area of:

- **a** a cylinder with height 36 cm and radius 8 cm
- **b** a sphere with diameter 4.6 m
- **c** a cone with radius 38 mm and slant height 86 mm
- **d** a cone with radius 1.2 cm and height 1.6 cm.

4 A new wharf has 24 cylindrical concrete pylons, each with diameter 0.6 m and length 10 m. The pylons will be coated with a salt resistant material.

- **a** Find the total surface area of one pylon.
- **b** Coating the pylons with the material costs \$45.50 per m^2. Find the cost of coating one pylon.
- **c** Find the total cost of coating the 24 pylons, to the nearest dollar.



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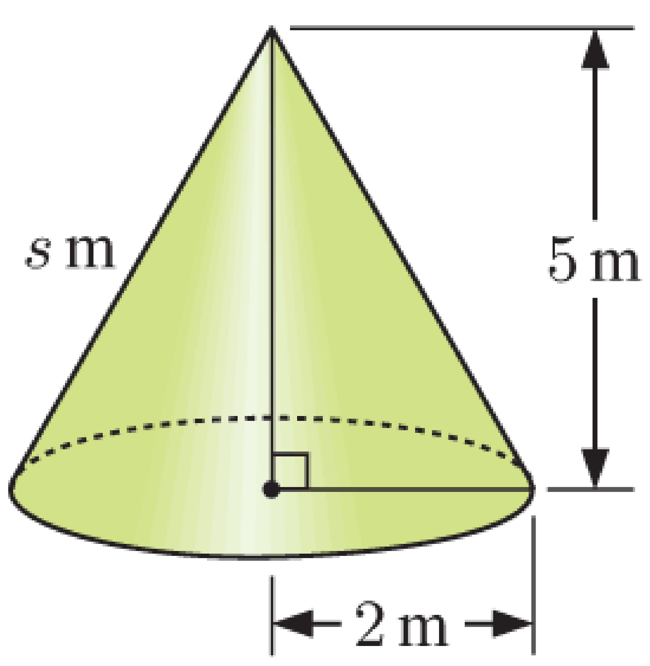


A conical tent has base radius 2 m and height 5 m.

- **a** Find the slant height s, to 2 decimal places.
- **b** Find the area of canvas necessary to make the tent, including the base.
- **c** If canvas costs \$18 per m^2, find the cost of the canvas.

6 A cylindrical tank of base diameter 8 m and height 6 m requires a non-porous lining on its circular base and curved walls. The lining costs \$23.20 per m^2 for the base, and \$18.50 per m^2 for the sides.

- **a** Find the area of the base.
- **b** Find the cost of lining the base.
- **c** Find the area of the curved wall.
- **d** Find the cost of lining the curved wall.
- **e** Find the total cost of the lining, to the nearest \$10.

Example 4 **Self Tutor**

The length of a hollow pipe is three times its radius.

- **a** Write an expression for its outer surface area in terms of its radius r.
- **b** If the outer surface area is 301.6 m^2, find the radius of the pipe.

a Let the radius be r m, so the length is $3r$ m.

Surface area $= 2\pi rh$
$= 2\pi r \times 3r$
$= 6\pi r^2\text{ m}^2$

b The surface area is 301.6 m^2

$\therefore\ 6\pi r^2 = 301.6$

$\therefore\ r^2 = \dfrac{301.6}{6\pi}$

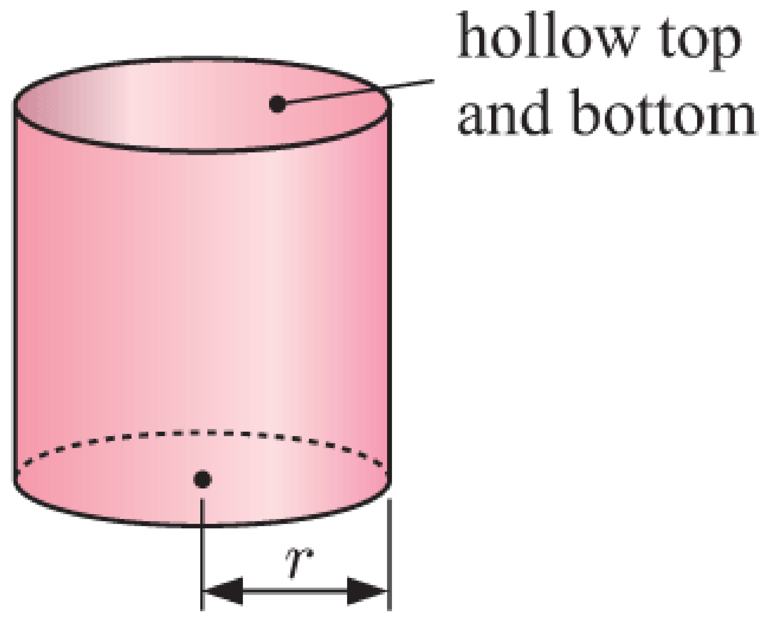
$\therefore\ r = \sqrt{\dfrac{301.6}{6\pi}}$ {as $r > 0$}

$\therefore\ r \approx 4.00$

The radius of the pipe is 4 m.

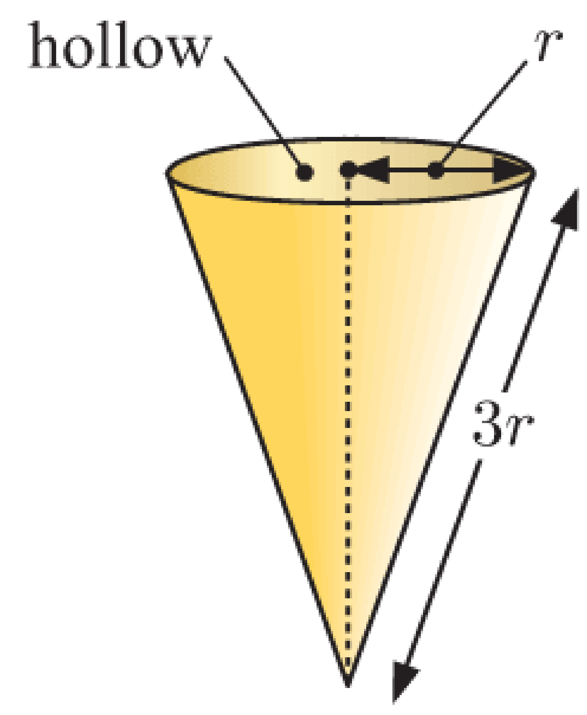
7 The height of a hollow cylinder is the same as its diameter.

- **a** Write an expression for the outer surface area of the cylinder in terms of its radius r.
- **b** Find the height of the cylinder if its surface area is 91.6 m^2.



8 The slant height of a hollow cone is three times its radius.

- **a** Write an expression for the outer surface area of the cone in terms of its radius r.
- **b** Given that the surface area is 21.2 cm^2, find the cone's:
  - **i** slant height
  - **ii** height.



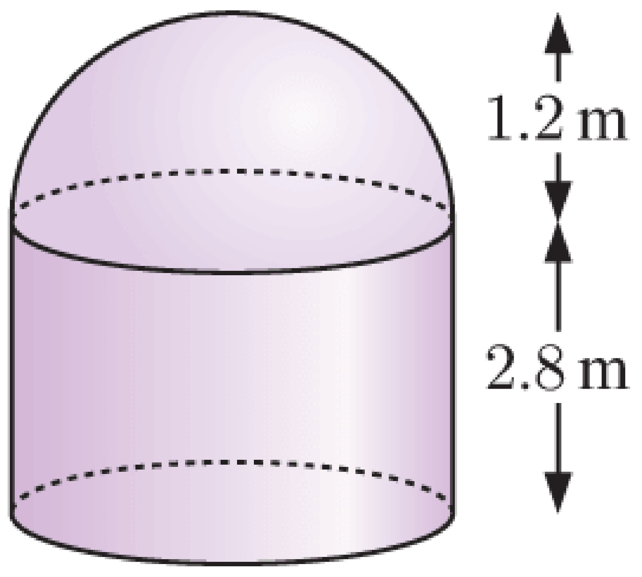
9 Write a formula for the surface area of:

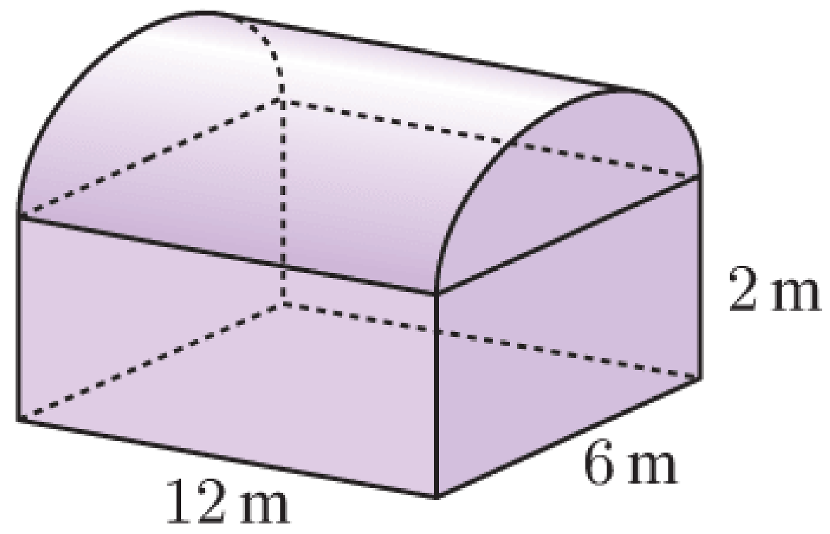
- **a** a cylinder with radius x cm and height $2x$ cm
- **b** a hemisphere with radius r cm
- **c** a cone with radius x cm and height $2x$ cm.

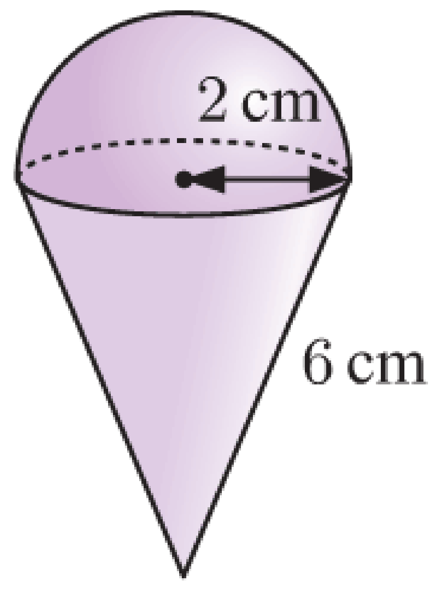
10 Find:

- **a** the radius of a sphere with surface area $64\pi\text{ cm}^2$
- **b** the height of a solid cylinder with radius 6.3 cm and surface area 1243 cm^2
- **c** the radius of a cone with slant height 143 mm and surface area $60\,000\text{ mm}^2$.

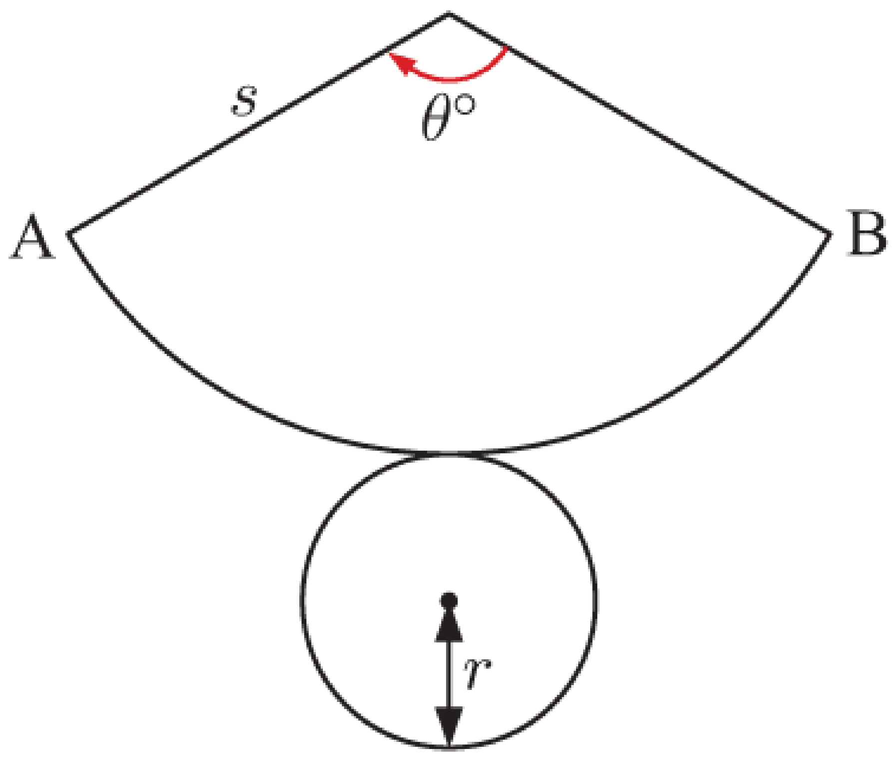
11 Find, correct to 1 decimal place, the surface area of each solid:

a


b


c


12 The planet Neptune is roughly spherical and has surface area $\approx 7.618 \times 10^9$ km^2. Estimate the radius of Neptune.

13


For the net of a cone alongside, notice that the length of arc AB must equal the circumference of the base circle.

a Write the arc length AB in terms of s and θ.

b Hence write θ in terms of r and s.

c Show that the surface area of the cone is given by $A = \pi rs + \pi r^2$.

C VOLUME

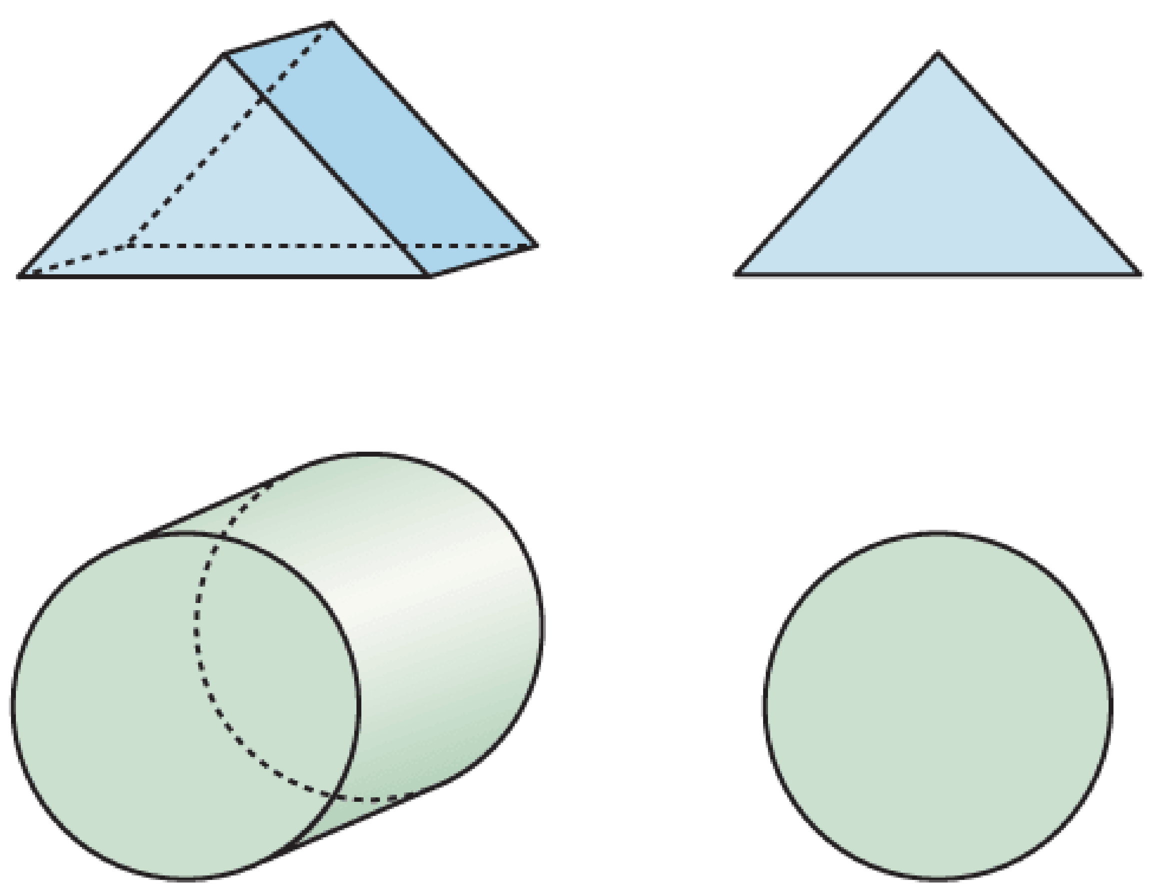
The **volume** of a solid is the amount of space it occupies.

SOLIDS OF UNIFORM CROSS-SECTION

In the triangular prism alongside, any vertical slice parallel to the front triangular face will be the same size and shape as that face. Solids like this are called *solids of uniform cross-section*. The cross-section in this case is a triangle.

Another example is a cylinder which has a circular cross-section.

Solid **Cross-section**

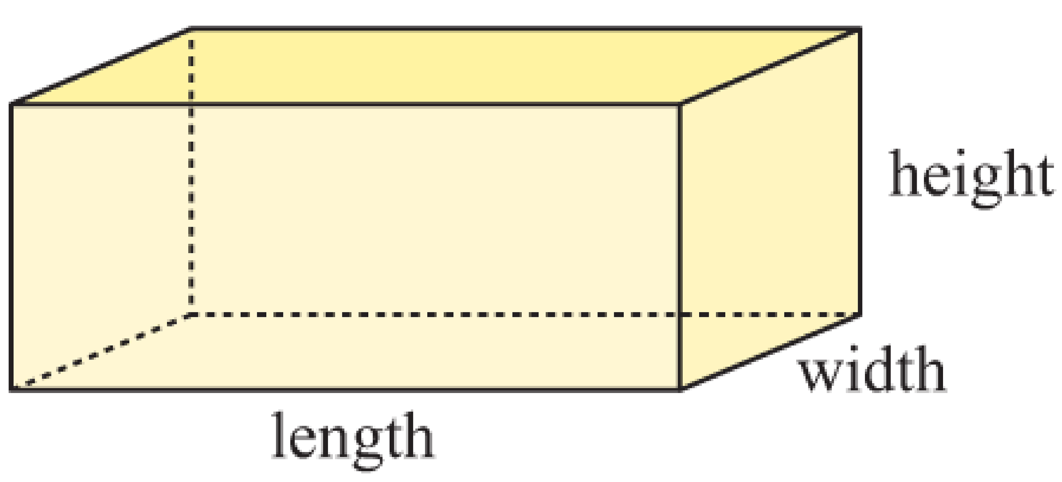


For any solid of uniform cross-section:

Volume = area of cross-section × length

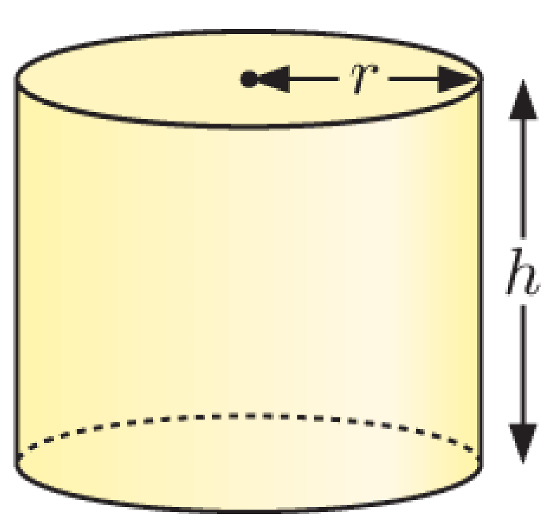
In particular, we can define formulae for the volume of:

- rectangular prisms



Volume = length × width × height

- cylinders

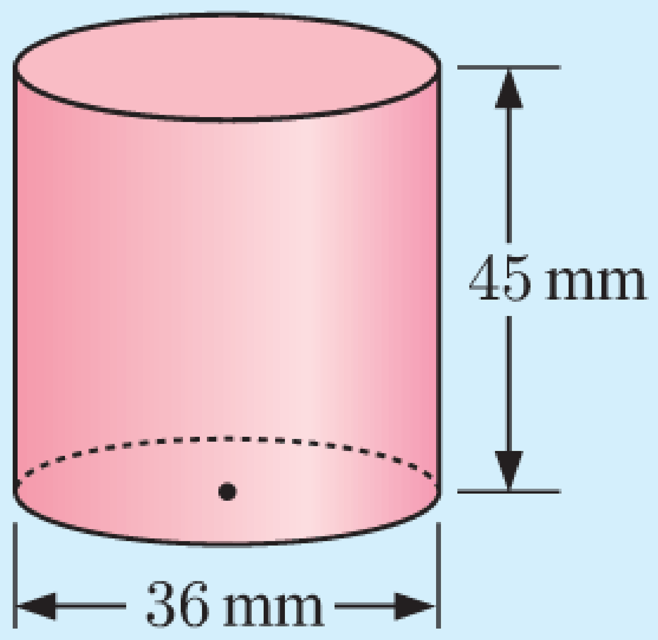


Volume $= \pi r^2 h$

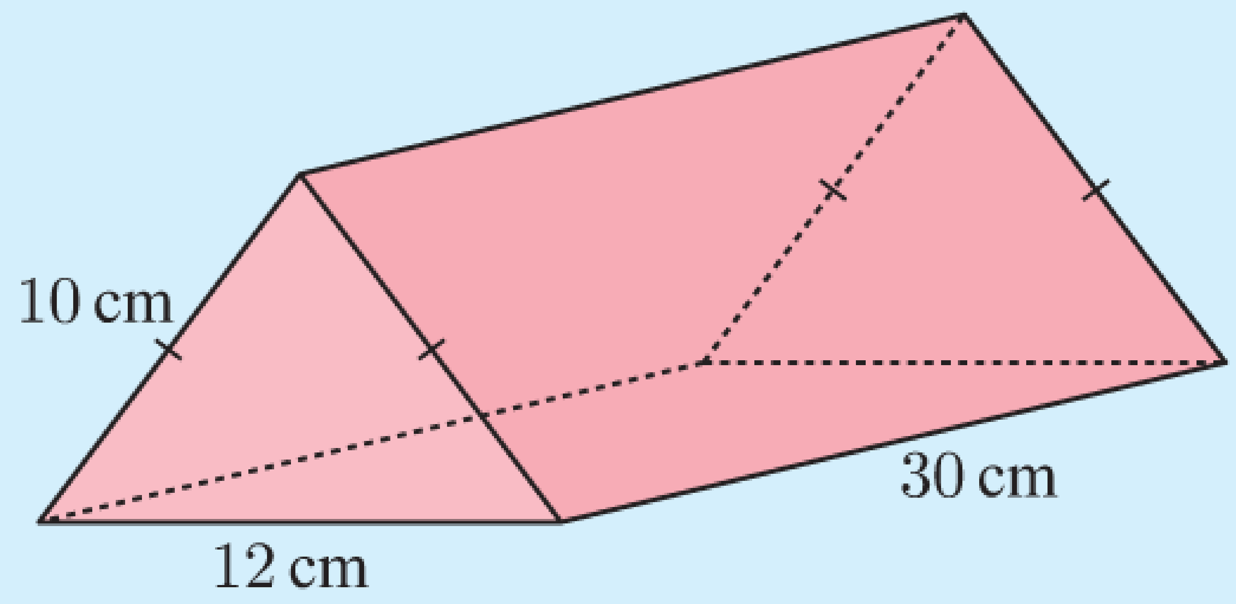
Example 5 **Self Tutor**

Find the volume of:

a



b

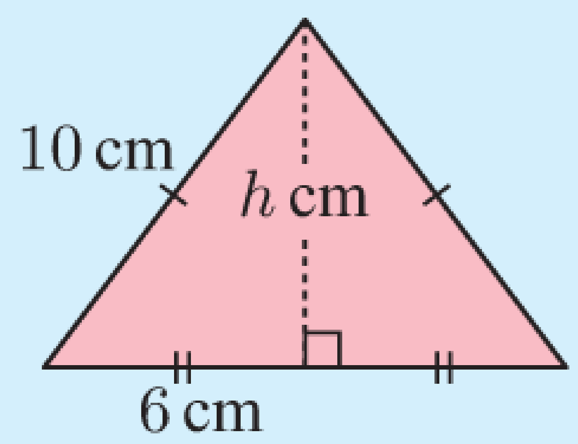


a $V = \pi r^2 h$

$= \pi \times 18^2 \times 45 \text{ mm}^3$

$\approx 45\,800 \text{ mm}^3$

b



Let the prism have height h cm.

$h^2 + 6^2 = 10^2$ {Pythagoras}

$\therefore\ h^2 + 36 = 100$

$\therefore\ h^2 = 64$

$\therefore\ h = 8$ {as $h > 0$}

Volume = area of cross-section × length

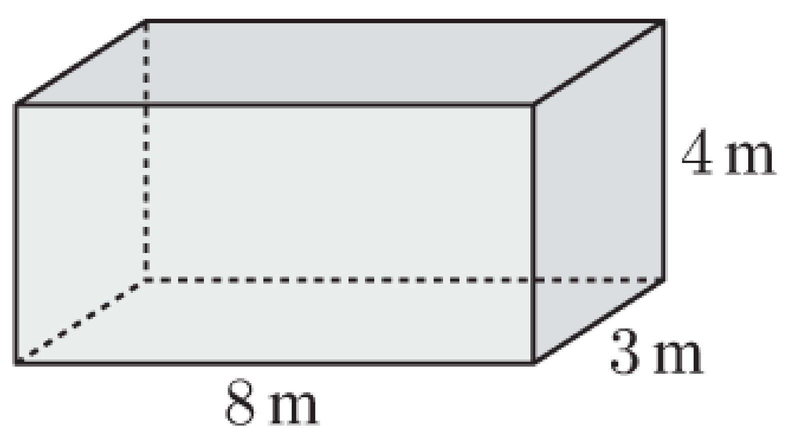
$= (\frac{1}{2} \times 12 \times 8) \times 30 \text{ cm}^3$

$= 1440 \text{ cm}^3$

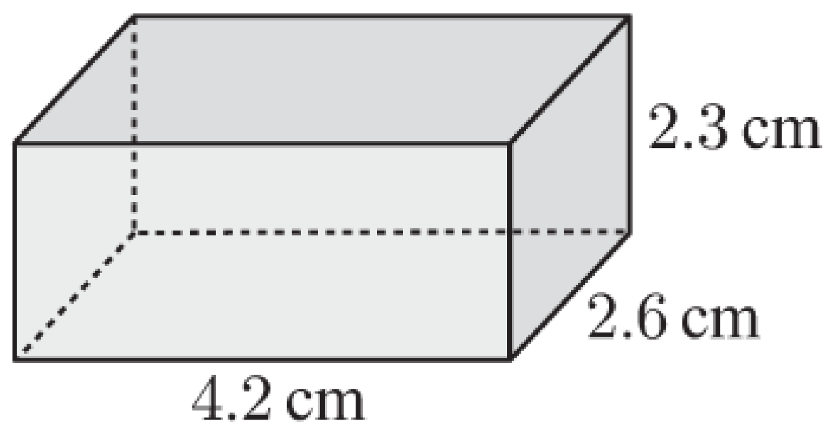
EXERCISE 6C.1

1 Find the volume of:

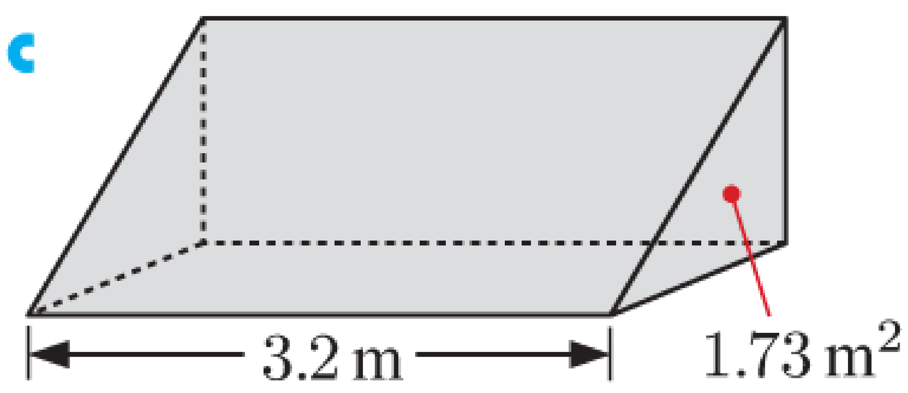
a



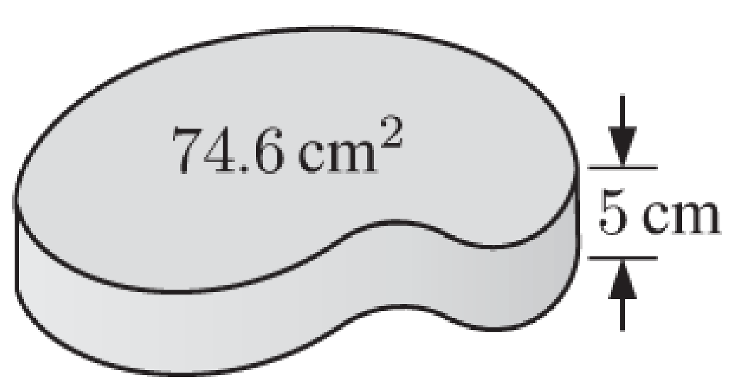
b



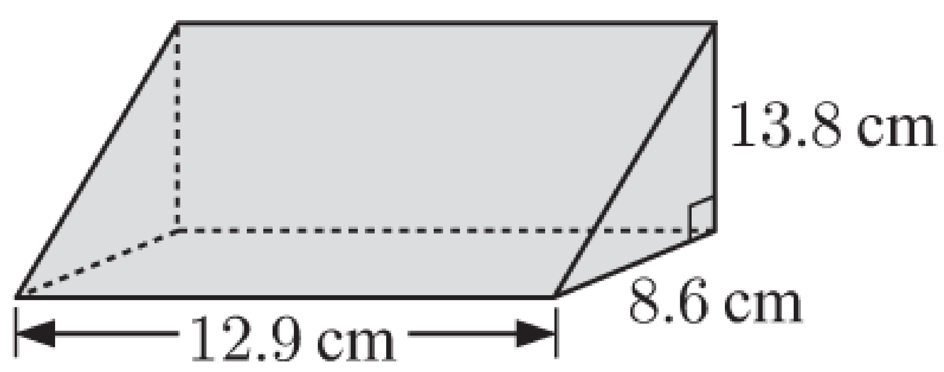
c



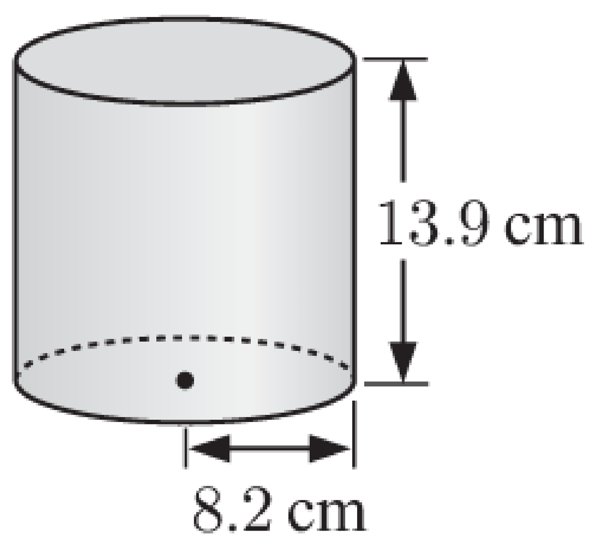
d



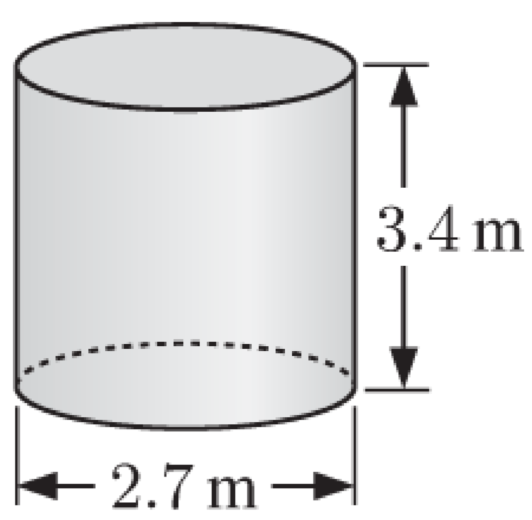
e



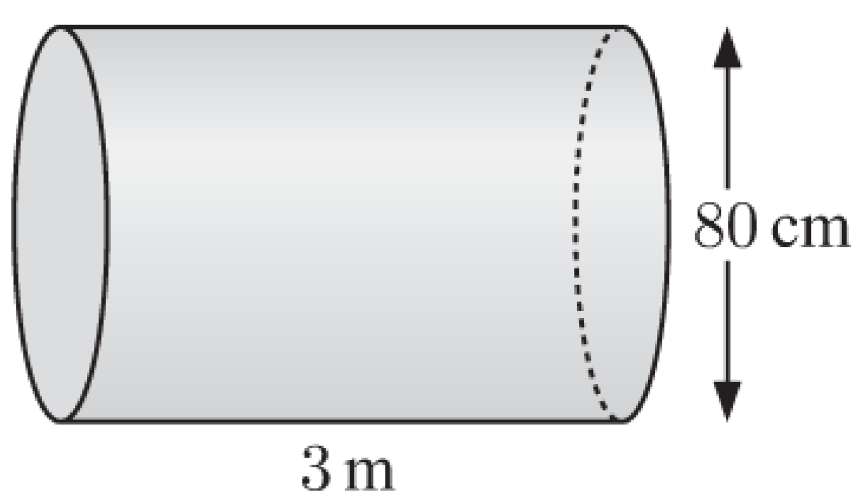
f



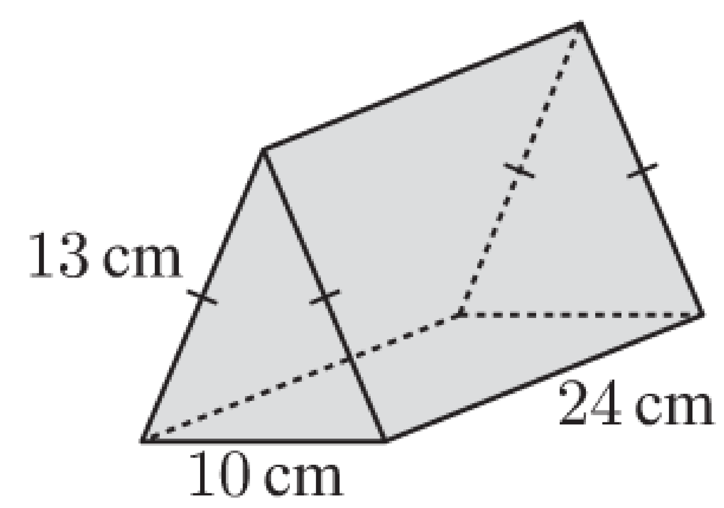
g



h



i



2 A circular cake tin has radius 20 cm and height 7 cm. When cake mix was added to the tin, its depth was 2 cm. After the cake was cooked it rose to 1.5 cm below the top of the tin.

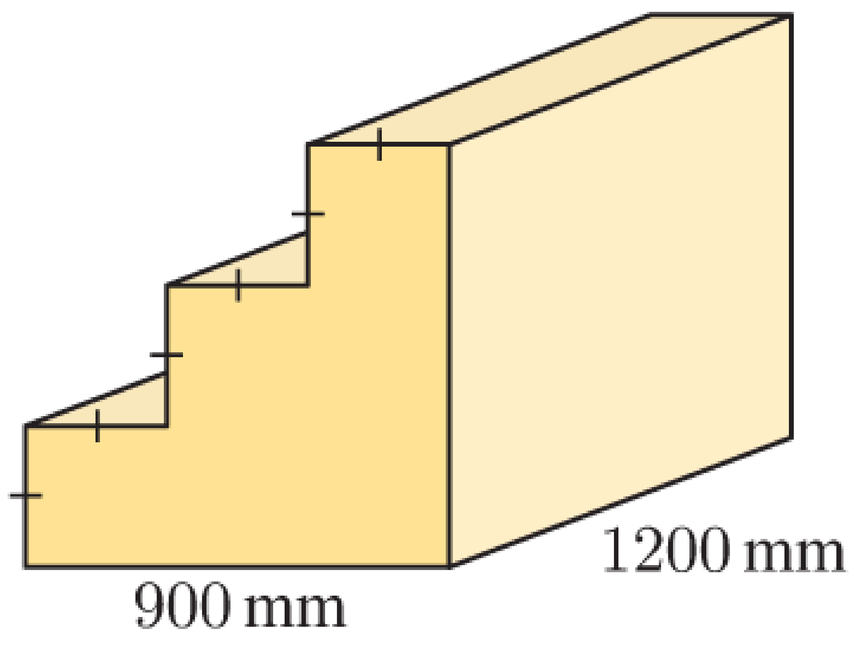
a Sketch these two situations.

b Find the volume of: **i** the cake mix **ii** the cooked cake.

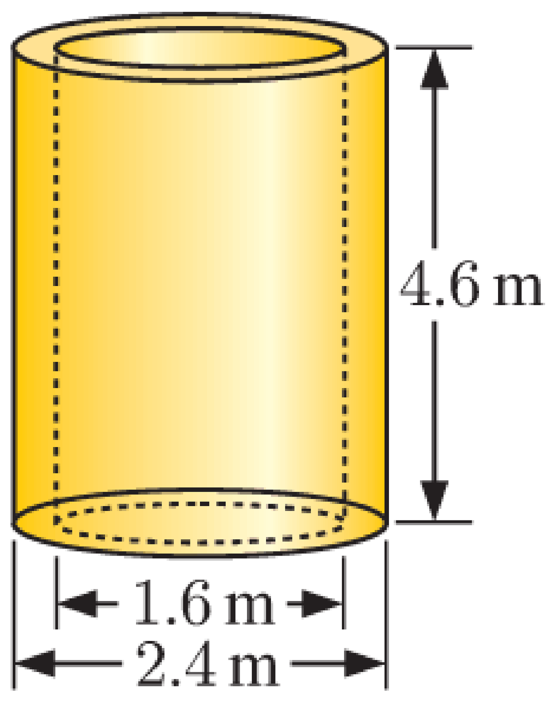
c What was the percentage increase in the volume of the cake while it cooked?

3 Find the volume of:

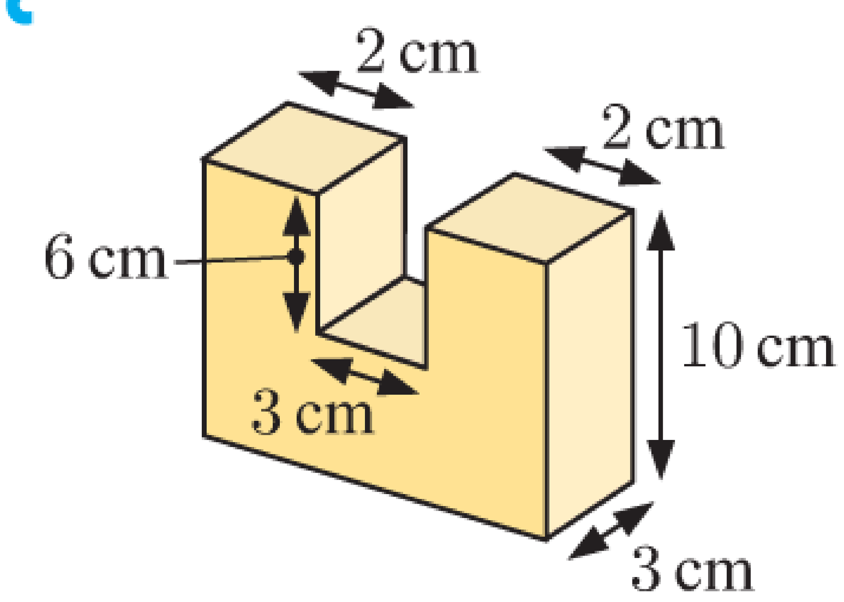
a



b



c

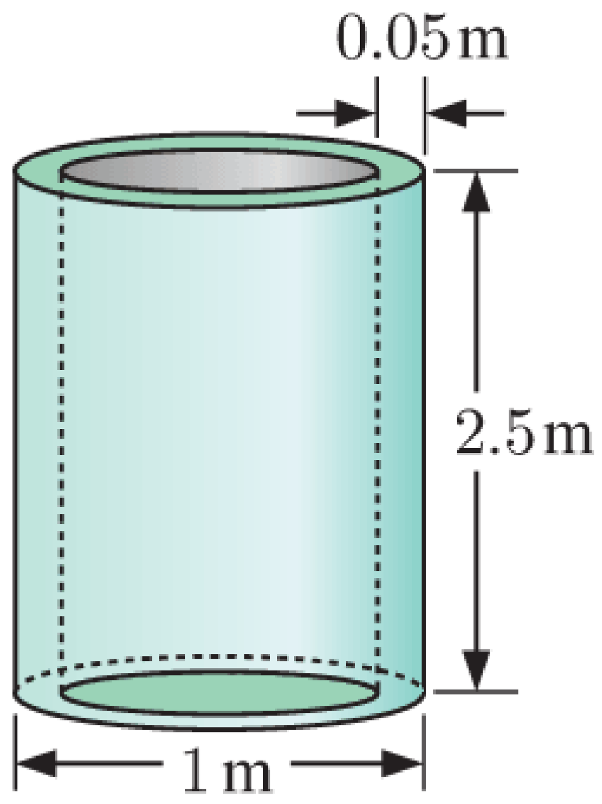


4 The Water Supply department uses huge concrete pipes to drain stormwater.

a Find the external radius of a pipe.

b Find the internal radius of a pipe.

c Find the volume of concrete necessary to make one pipe.



5 A rectangular garage floor 9.2 m by 6.5 m is to be concreted to a depth of 120 mm.

a What volume of concrete is required?

b Concrete costs \$135 per m^3, and is only supplied in multiples of 0.2 m^3. How much will the concrete cost?

6 A concrete path 1 m wide and 10 cm deep is placed around a circular lighthouse of diameter 12 m.

a Draw an overhead view of the situation.

b Find the surface area of the concrete.

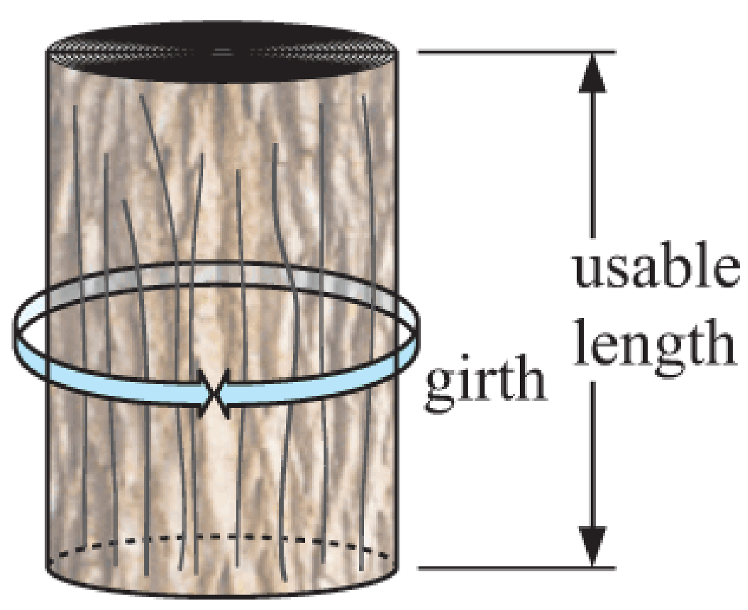
c Find the volume of concrete required for the path.

7 In the timber industry, treefellers need to calculate the volume of usable timber in a tree. They use the following approximation for the tree's volume:

$V \approx 0.06 \times g^2 \times l$ where $V =$ volume (in m^3)

$g =$ approximate girth (in m)

and $l =$ usable length (in m).



a Estimate the volume of usable timber in a tree with an average girth of 3.8 m and usable length 9.9 m.

b For a cylinder with circumference g, and height l, show that $V = \frac{1}{4\pi} g^2 \times l$.

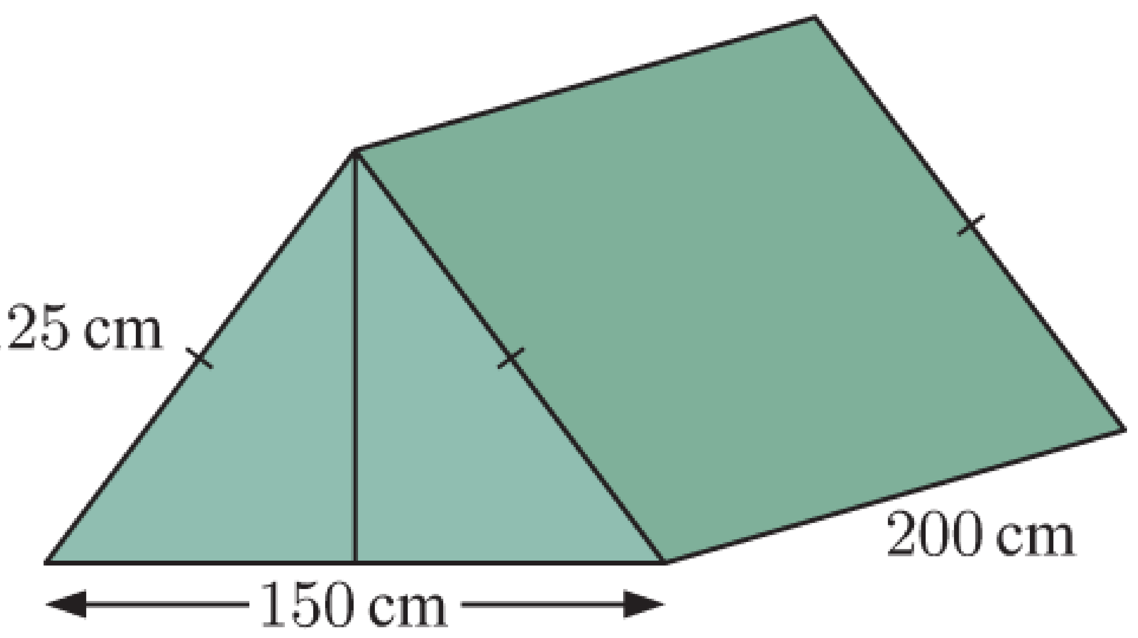
c Compare the volumes predicted by these two formulae, and explain the difference between them.

8 1000 km of black plastic cylindrical water piping with internal diameter 13 mm and walls of thickness 2 mm is required for a major irrigation project. The piping is made from bulk plastic which weighs 0.86 tonnes per cubic metre. How many tonnes of black plastic are required?

9 I am currently building a new rectangular garden which is 8.6 m by 2.4 m, and 15 cm deep. I have decided to purchase some soil from the local garden supplier, and will load it into my trailer which measures 2.2 m $\times$ 1.8 m $\times$ 60 cm. I will fill the trailer to within 20 cm from the top.

- **a** How many trailer loads of soil will I need?
- **b** Each load of soil costs \$87.30. What will the total cost of the soil be?
- **c** I decide to put bark on top of the soil in the garden. Each load covers 11 m^2 of garden bed.
  - **i** How many loads of bark will I need?
  - **ii** Each load of bark costs \$47.95. What is the total cost of the bark?
- **d** Calculate the total cost of establishing the garden.

10

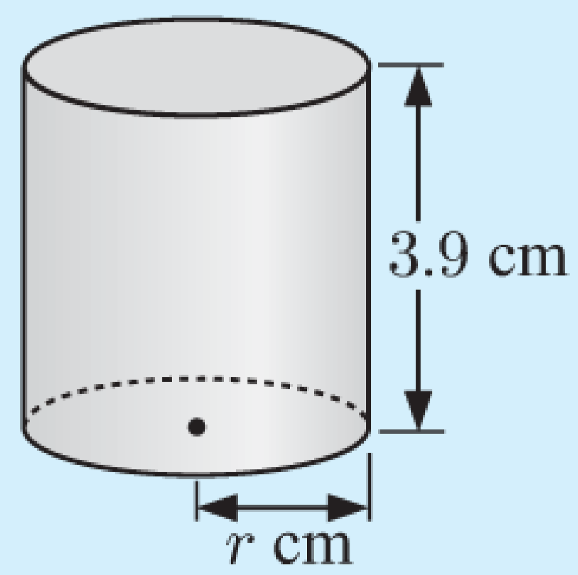


A scout's tent is 150 cm wide and 200 cm long. It has the shape of an isosceles triangular prism as shown.

- **a** Find the height of each vertical support post.
- **b** Find the volume of the tent.
- **c** Find the total area of the canvas in the tent, including the ends and floor.

Example 6 — Self Tutor

Find, to 3 significant figures, the radius of a cylinder with height 3.9 cm and volume 54.03 cm^3.



$V = 54.03$ cm^3

$\therefore \pi \times r^2 \times 3.9 = 54.03$ {V = area of cross-section $\times$ length}

$\therefore r^2 = \dfrac{54.03}{\pi \times 3.9}$ {dividing both sides by $\pi \times 3.9$}

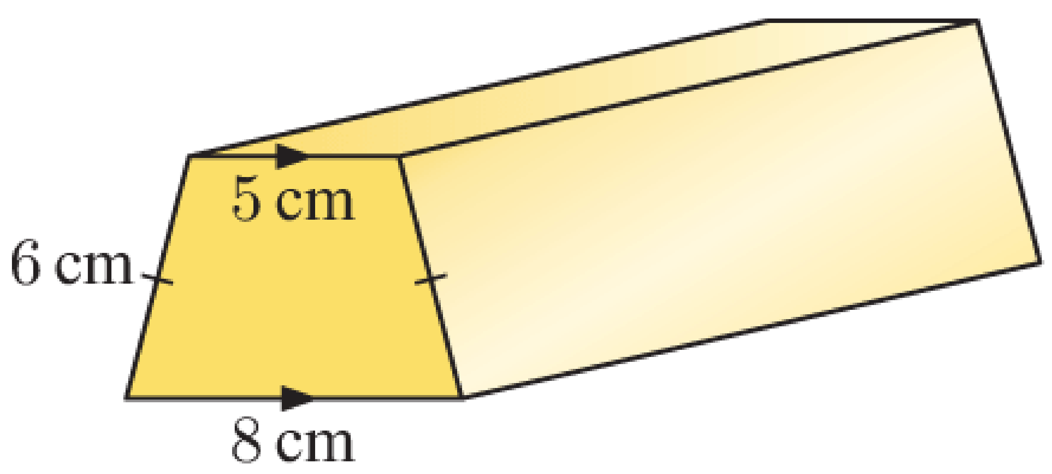
$\therefore r = \sqrt{\dfrac{54.03}{\pi \times 3.9}} \approx 2.10$ {as $r > 0$}

The radius is approximately 2.10 cm.

11 Find:

- **a** the height of a rectangular prism with base 5 cm by 3 cm and volume 40 cm^3
- **b** the side length of a cube of butter with volume 34.01 cm^3
- **c** the radius of a steel cylinder with height 4.6 cm and volume 43.75 cm^3.

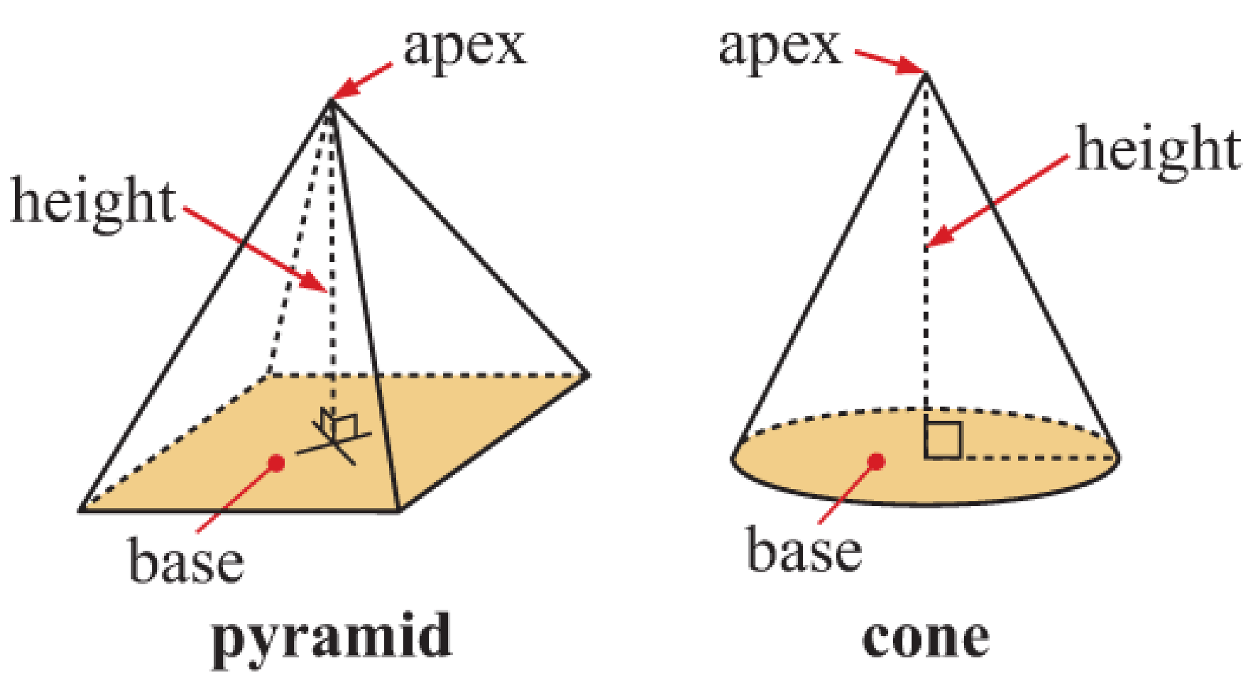
12



A gold bar has the trapezoidal cross-section shown. Its volume is 480 cm^3.
Find the length of the bar.

TAPERED SOLIDS

For pyramids and cones, the cross-section is not uniform. Rather, the cross-sections are a set of similar shapes which get smaller and smaller as we approach the apex. We call these **tapered solids**.



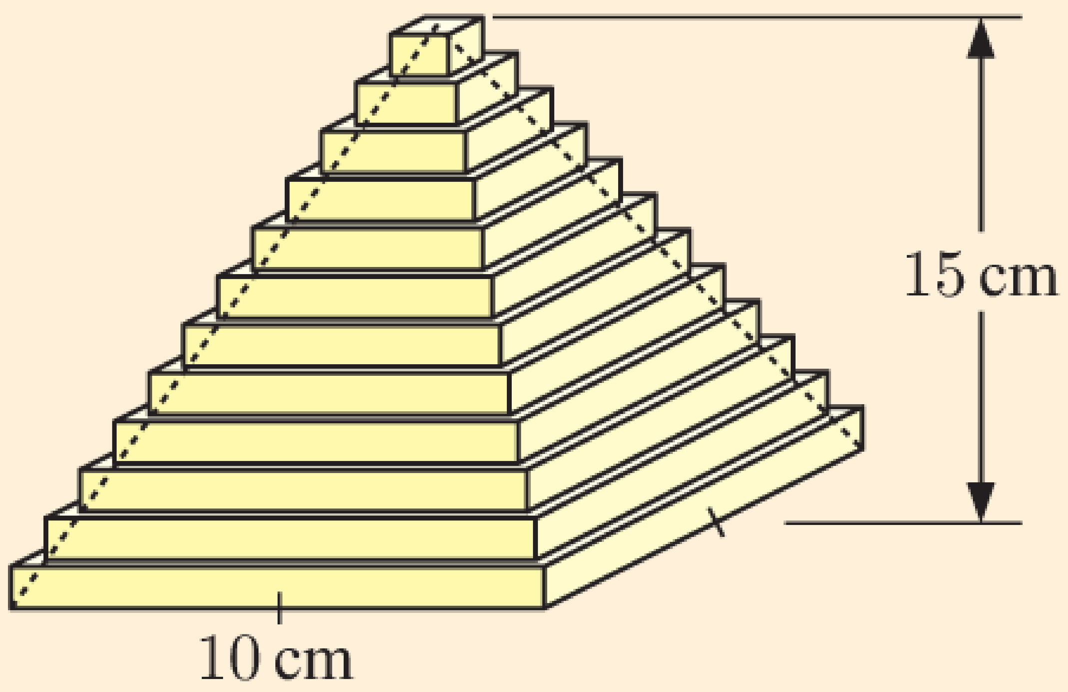
INVESTIGATION THE VOLUME OF TAPERED SOLIDS

Having seen formulae for the volume of solids of uniform cross-section, we now seek to establish formulae for tapered solids including pyramids and cones.

What to do:

1 **a** Find the volume V_p of a rectangular prism whose base is a 10 cm × 10 cm square, and whose height is 15 cm.

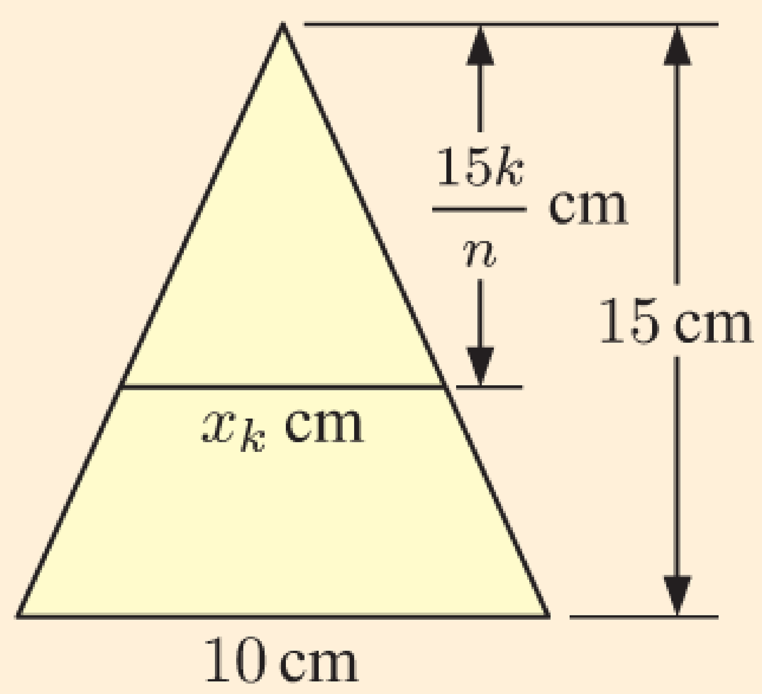
b Consider a pyramid with the same square base and same height as the rectangular prism in **a**. The pyramid can be approximated using a set of rectangular prisms with equal thickness, and each with a square base, as shown.



Suppose there are n prisms in our approximation.

i Explain why the height of each prism is $\frac{15}{n}$ cm.

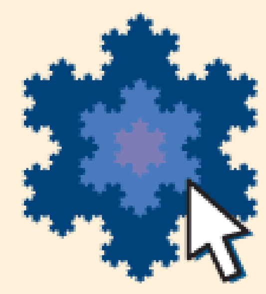
ii We suppose the base of each prism is the cross-section of the actual pyramid at the corresponding height. We start at the apex and move down.
We suppose the kth prism has base x_k cm × x_k cm.
Use the diagram alongside to explain why $x_k = \frac{10k}{n}$.



iii Hence explain why the volume of the kth prism is equal to $\frac{V_p k^2}{n^3}$ cm^3, where V_p is the volume of the corresponding solid with uniform cross-section which you found in **a**.

c Load the spreadsheet which will calculate the sum of the volumes of the n prisms, for values up to $n = 100\,000$.

SPREADSHEET



i Check that the volume V_p of the corresponding solid with uniform cross-section is correct.

ii Discuss with your class how the formulae in the spreadsheet work.

iii Check that for $n = 1$, the approximation is simply the corresponding solid with uniform cross-section.

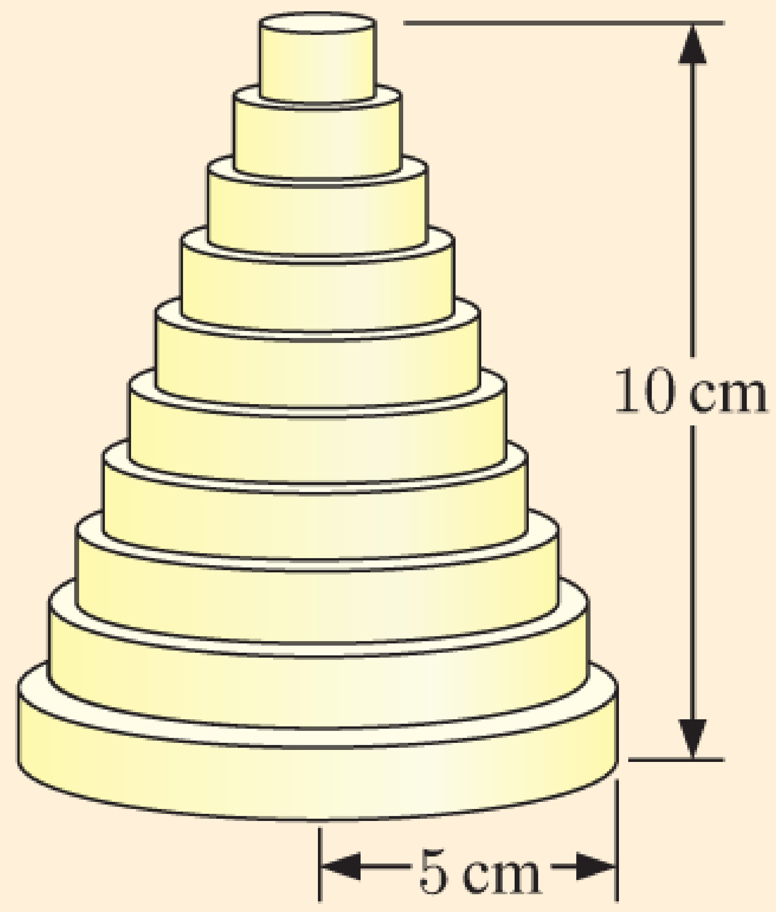
iv Find the approximate volume of the pyramid manually for $n = 5$. Use the spreadsheet to check your answer.

v Construct a table of values for the approximate volume of the pyramid. Use $n = 10$, 100, 1000, 10 000, and 100 000.

vi What do you think the actual volume of the pyramid is? What *fraction* of the corresponding solid with uniform cross-section is this?

2 Repeat **1** for a different sized square-based pyramid. Remember that you will need to change the appropriate "volume of corresponding solid of uniform cross-section" cell in the spreadsheet. Comment on your results.

3 **a** Find, in terms of π, the volume V_c of a cylinder with base radius 5 cm and height 10 cm.



b Consider a cone with the same base and height as the cylinder in **a**. The cone can be approximated using a set of cylinders with equal thickness, as shown.

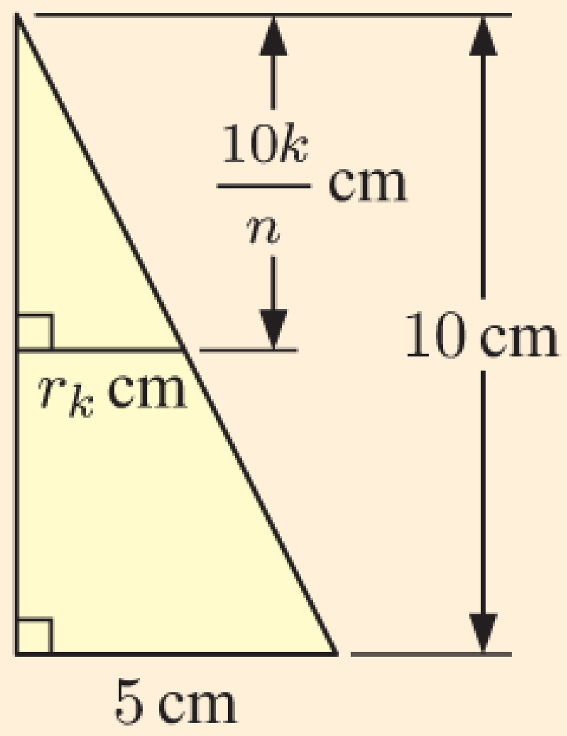
Suppose there are n cylinders in our approximation.

i Explain why the height of each cylinder is $\frac{10}{n}$ cm.

ii We suppose the base of each cylinder is the cross-section of the actual cone at the corresponding height. We start at the apex and move down.

We suppose the kth cylinder has base radius r_k cm.

Use the diagram alongside to explain why $r_k = \frac{5k}{n}$.



iii Hence explain why the volume of the kth cylinder is equal to $\frac{V_c k^2}{n^3}$ cm^3, where V_c is the volume of the corresponding solid with uniform cross-section which you found in **a**.

c Load the spreadsheet which will calculate the sum of the volumes of the n cylinders, for values up to $n = 100\,000$. Note that volumes are given in lots of π cm^3.

SPREADSHEET

i Check that the volume V_c of the corresponding solid with uniform cross-section is correct.

ii Check that for $n = 1$, the approximation is simply the corresponding solid with uniform cross-section.

iii Construct a table of values for the approximate volume of the cone. Use $n = 10$, 100, 1000, 10 000, and 100 000.

iv What do you think the actual volume of the cone is? What *fraction* of the corresponding solid with uniform cross-section is this?

4 Repeat **3** for a different sized cone. Remember that you will need to change the appropriate "volume of corresponding solid of uniform cross-section" cell in the spreadsheet. Comment on your results.

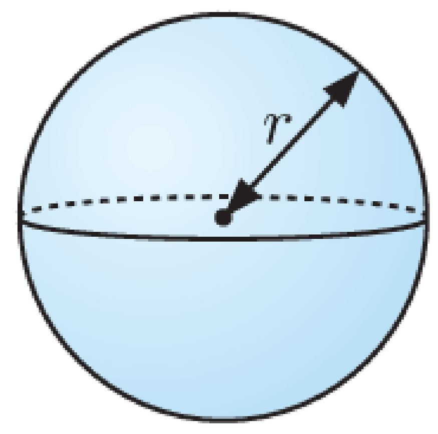
From the **Investigation**, you should have established that the volume of any tapered solid is given by:

$$\textbf{Volume} = \tfrac{1}{3}(\textbf{area of base} \times \textbf{height})$$

It is a third of the volume of the solid of uniform cross-section with the same base and height.

SPHERES

In previous years you should have seen that the volume of a sphere with radius r is given by

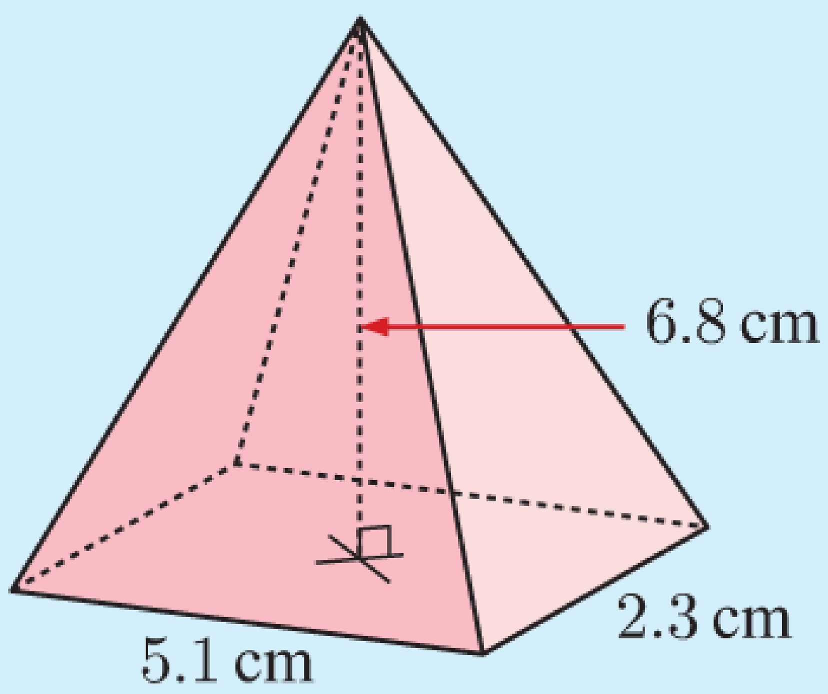


$$\textbf{Volume} = \tfrac{4}{3}\pi r^3$$

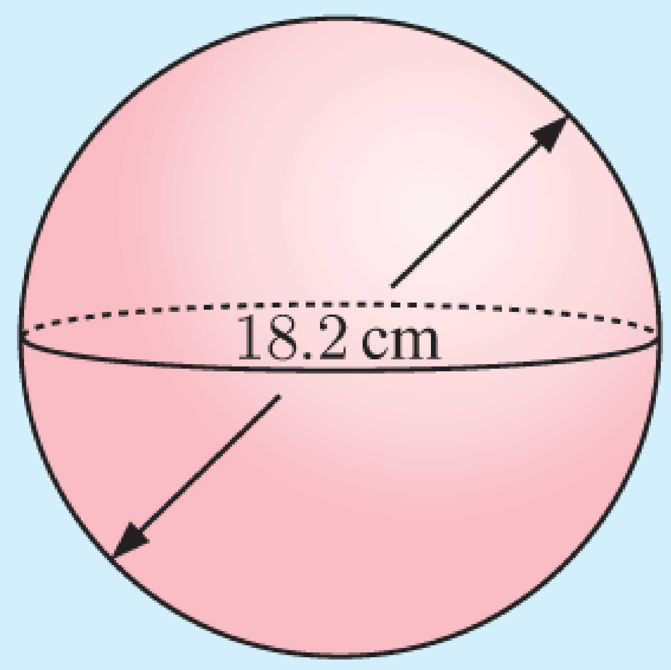
Example 7 Self Tutor

Find the volume of each solid:

a



b



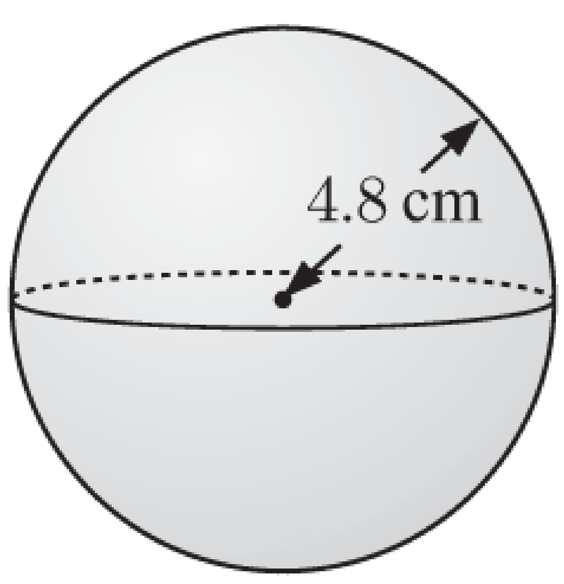
a
$$\begin{aligned} V &= \tfrac{1}{3}(\text{area of base} \times \text{height}) \\ &= \tfrac{1}{3}(\text{length} \times \text{width} \times \text{height}) \\ &= \tfrac{1}{3}(5.1 \times 2.3 \times 6.8) \text{ cm}^3 \\ &\approx 26.6 \text{ cm}^3 \end{aligned}$$

b
$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3}\pi \left(\frac{18.2}{2}\right)^3 \\ &\approx 3160 \text{ cm}^3 \end{aligned}$$

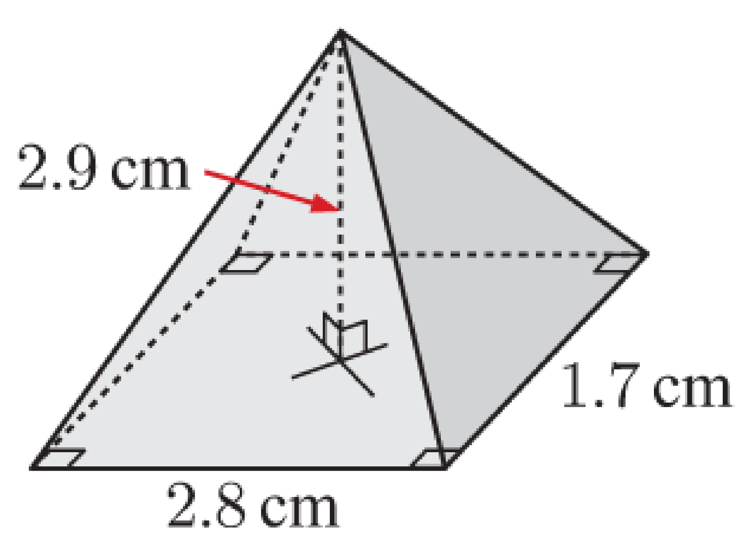
EXERCISE 6C.2

1 Find the volume of:

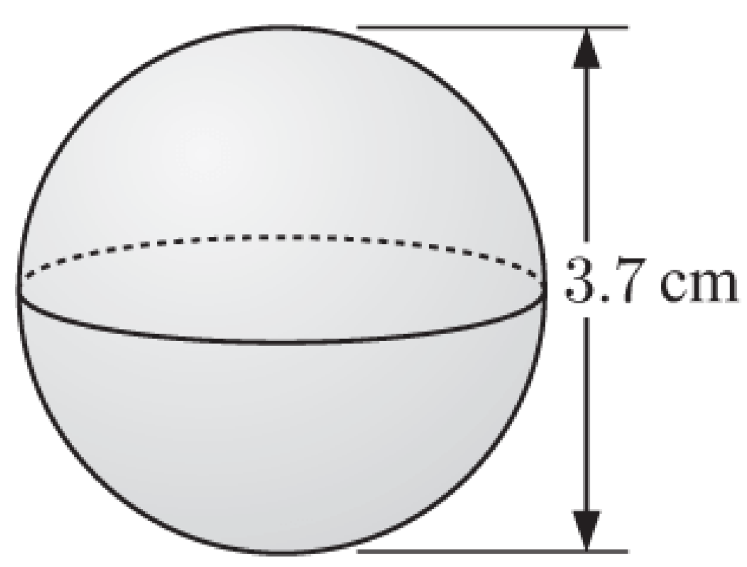
a



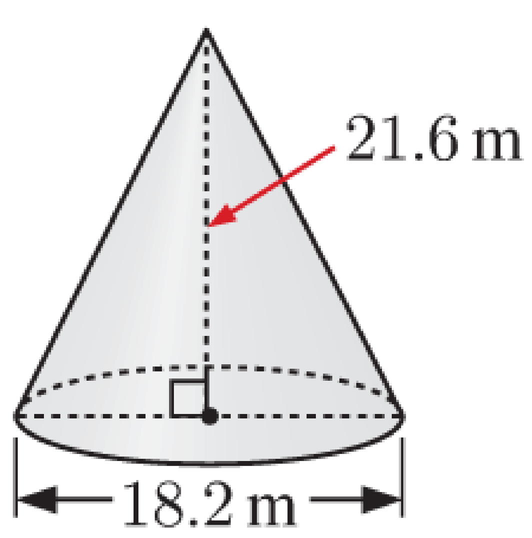
b



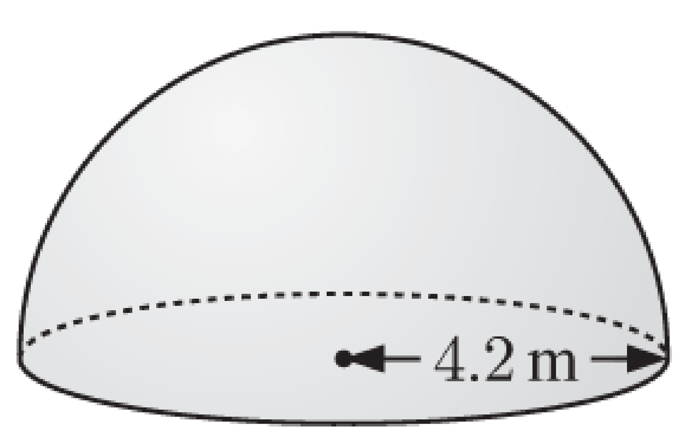
c



d



e



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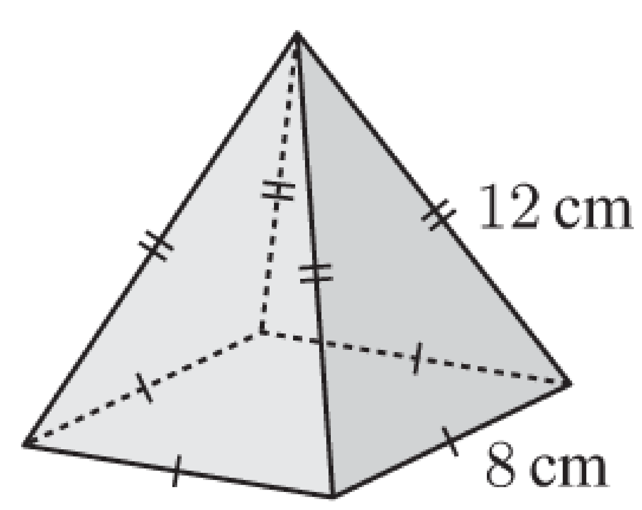
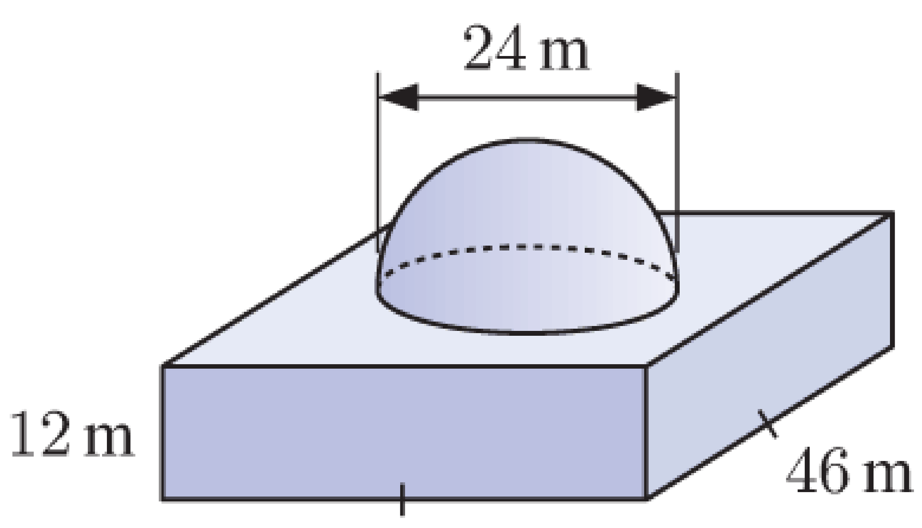
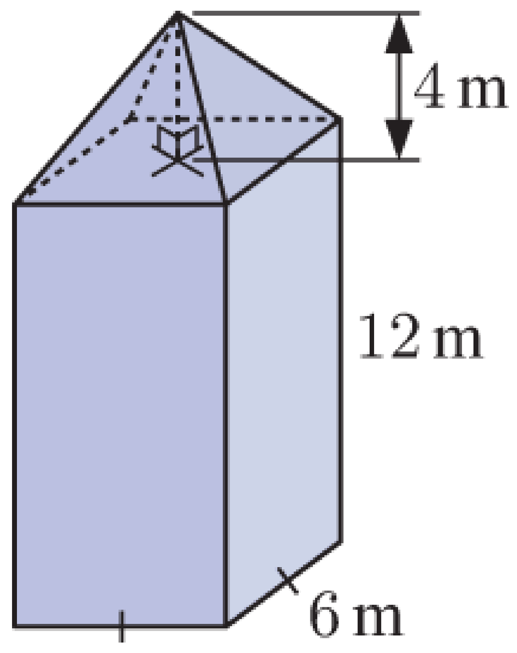


2 Find the volume of:

a



b



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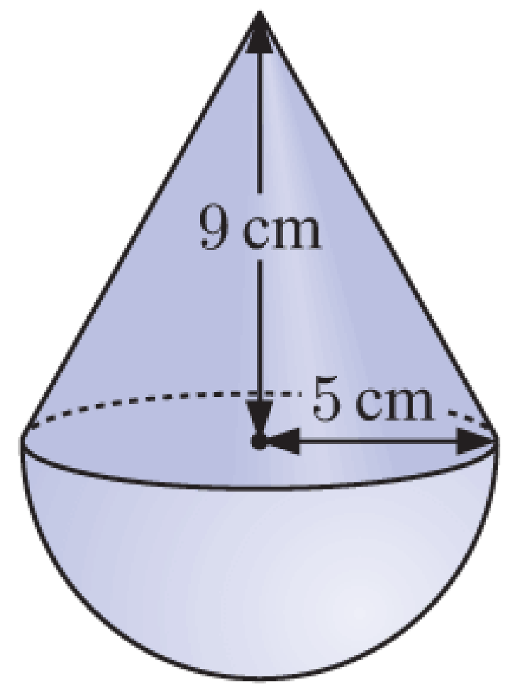
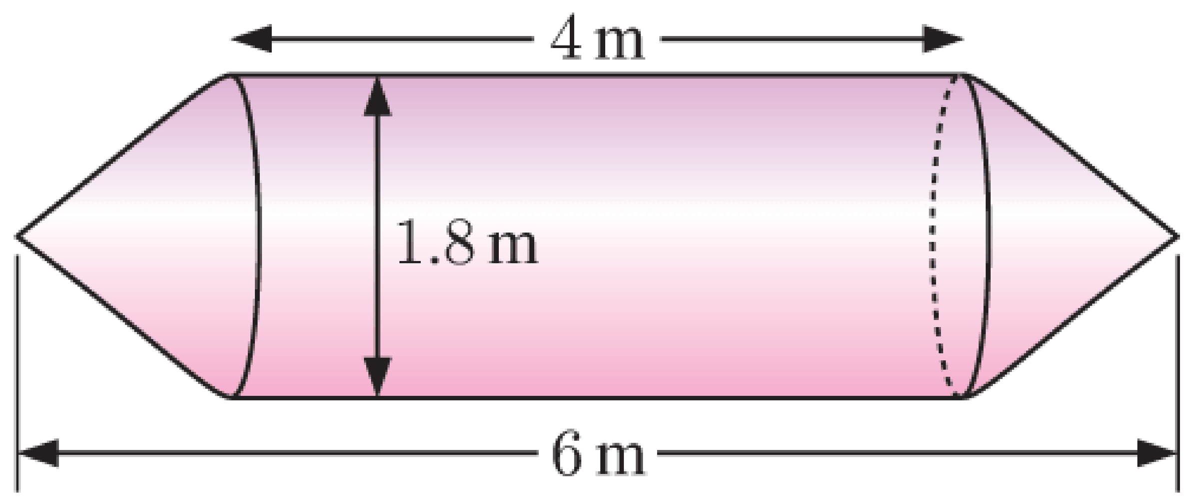


3 A ready mixed concrete tanker is to be constructed from steel as a cylinder with conical ends.



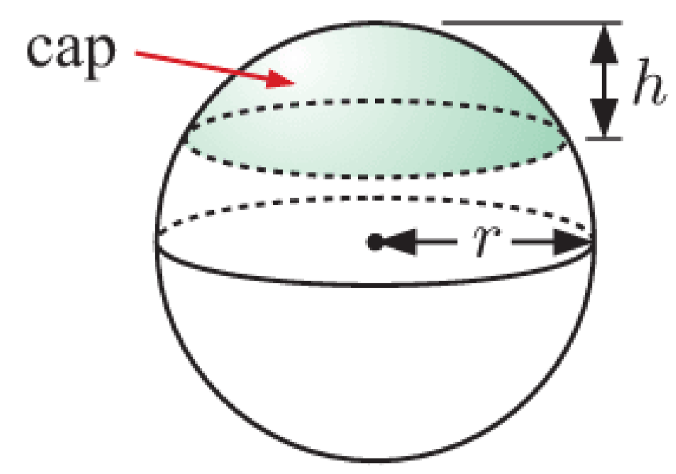
- **a** Calculate the total volume of concrete that can be held in the tanker.
- **b** How *long* would the tanker be if the ends were hemispheres instead of cones, but the cylindrical section remained the same?
- **c** How much more or less concrete would fit in the tanker if the ends were hemispheres instead of cones?
- **d** Show that the surface area of the tanker:
  - **i** with conical ends is about 30 m^2
  - **ii** with hemispherical ends is about 33 m^2.
- **e** Overall, which do you think is the better design for the tanker? Give reasons for your answer.

4 Find:

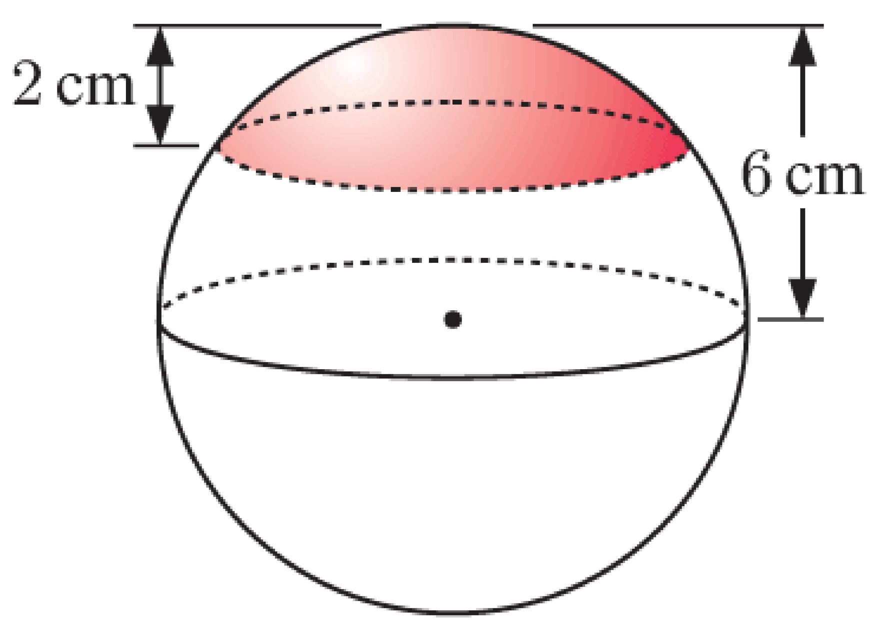
- **a** the height of a glass cone with base radius 12.3 cm and volume 706 cm^3
- **b** the radius of a spherical weather balloon with volume 73.62 m^3
- **c** the base radius of a cone with height 6.2 cm and volume 203.9 cm^3.

5 A cylinder of resin has height equal to its diameter. Some of it is used to form a cone with the same height and diameter as the cylinder. Show that the remainder is the exact amount needed to form a sphere with the same diameter.

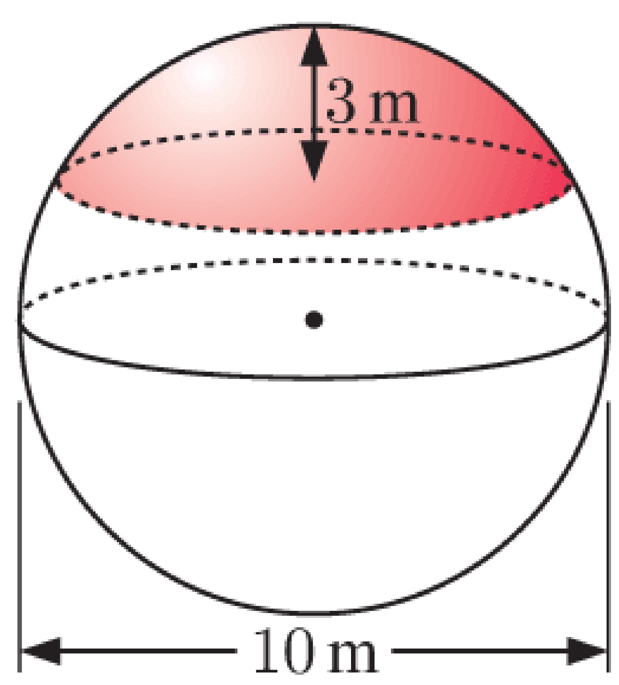
6 For a sphere of radius r, the volume of the **cap** of height h is $V = \frac{\pi h^2}{3}(3r - h)$.



- **a** Find the volumes of these caps:
  - **i**



  - **ii**



- **b** Write an expression for the volume of the cap in the case that $h = r$. Compare this volume with the volume of the sphere. Explain your result.

ACTIVITY 1 DENSITY

The **density** of a substance is its mass per unit volume.

$$\textbf{density} = \frac{\textbf{mass}}{\textbf{volume}}$$

One **gram** is the mass of one cubic centimetre of pure water at 4°C. The density of pure water at 4°C is therefore $\frac{1\text{ g}}{1\text{ cm}^3} = 1\text{ g cm}^{-3}$.

Some densities of common substances are shown in the table:

Substance	*Density* (g cm^{-3})
pine wood	0.41
paper	0.80
oil	0.92
water	1.00
steel	8.05
copper	8.96
lead	11.34

What to do:

1 Find the density of:

a a metal rod with mass 10 g and volume 2 cm^3

b a cube of chocolate with side length 2 cm and mass 10.6 g

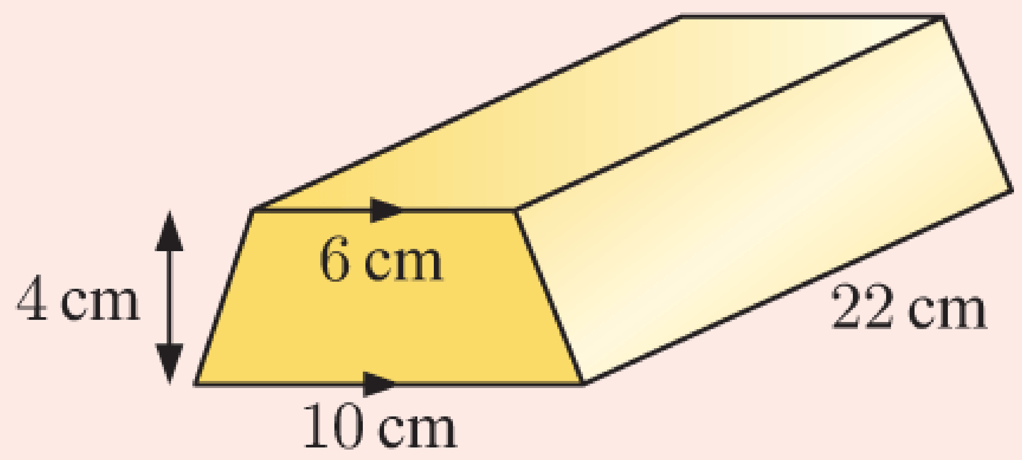
c a glass marble with radius 4.5 mm and mass 1.03 g.

2 Rearrange the density formula to make: **a** mass the subject **b** volume the subject.

3 Find the volume of 80 g of salt with density 2.16 g cm^{-3}.

4 Find the mass of a copper wire with radius 1 mm and length 250 m.

5 The gold bar shown has mass 13.60 kg. Find the density of gold.



6 Jonathon has a steel ball bearing with radius 1.4 cm, and a lead sphere with radius 1.2 cm. Which sphere weighs more, and by what percentage?

7 Oil and water are *immiscible*, which means they do not mix. Does oil float on water, or water float on oil? Explain your answer.

8 **a** In general, as a substance is heated, it expands. What happens to the density of the substance?

b Water is unusual in that its solid state is less dense than its liquid state. How do we observe this in the world around us?

9 Determine the total mass of stone required to build a square-based pyramid with all edges of length 200 m. The density of the stone is 2.25 tonnes per m^3.

10 The planet Uranus is approximately spherical with radius 2.536×10^7 m and mass 8.681×10^{25} kg.

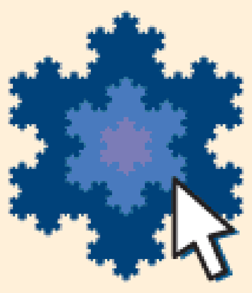
a Estimate the volume of Uranus. **b** Hence find its density.

PROJECT HOW BIG IS THE MOUNTAIN?

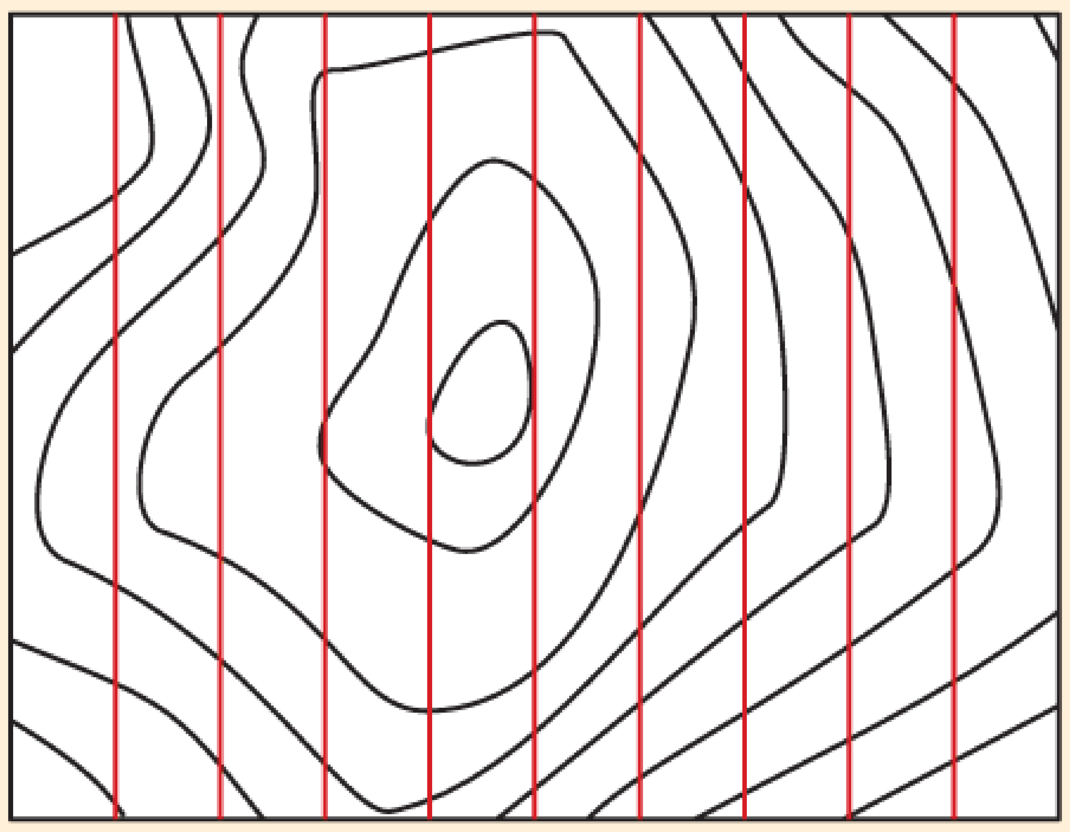
Choose an iconic mountain of the world. Your task is to estimate its volume.

To achieve this task, you will need:

- a topographic map of the mountain
- knowledge of Simpson's rule.



What to do:



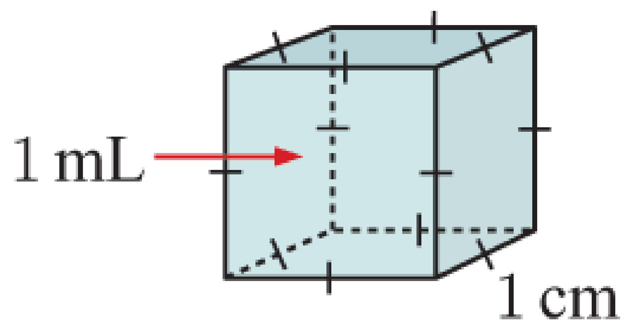
1. **a** Use Simpson's rule to estimate the cross-sectional area of the mountain at each contour level.
   b Hence estimate the volume of the mountain added at each change in altitude level.
   c Use a solid of uniform cross-section to estimate the volume of the mountain from your lowest chosen contour down to sea level.
   d Sum your results to estimate the total volume of the mountain.
   e Discuss the assumptions you made in your calculations.
2. **a** Make regular slices across your contour map and use Simpson's rule to estimate the area of each slice.
   b Hence estimate the volume of the mountain for each interval between the slices.
   c Sum your results to give the total volume of the mountain.
   d Discuss the assumptions you made in your calculations.
3. **a** Overlay a fine grid on top of the topographical map. Use the contours to estimate the altitude at each vertex point of the grid. Hence estimate the average altitude of each grid square.
   b Hence estimate the volume of the mountain under each grid square.
   c Sum your results to give the total volume of the mountain.
   d Discuss the assumptions you have made in your calculations.
4. Compare the estimates you have obtained for the volume of the mountain.
   a What assumptions do you need to make in order to compare them fairly?
   b Which method do you think is the:
   i most elegant **ii** most accurate **iii** easiest to automate using software?
5. Can you suggest a more accurate method for estimating the volume of the mountain? Explain why you believe it is more accurate, and perform calculations.
6. Research the composition of your chosen mountain and use the information to estimate its mass.
7. If you measured the volume of a mountain down to the base plane around it rather than to sea level, what is the "biggest" mountain on Earth?

D CAPACITY

The **capacity** of a container is the quantity of fluid it is capable of holding.

Notice that the term "capacity" belongs to the container rather than the fluid itself. The capacity of the container tells us what *volume* of fluid fits inside it. The units of volume and capacity are therefore linked:

1 mL of water occupies 1 cm^3 of space.



Volume		*Capacity*
1 cm^3	$\equiv$	1 mL
1000 cm^3	$\equiv$	1 L
1 m^3	$\equiv$	1 kL
1 m^3	$\equiv$	1000 L

DISCUSSION

- In common language, are the terms *volume* and *capacity* used correctly?
- Which of the following statements are technically correct? Which are commonly accepted in language, even though they are not technically correct?
  - The jug has capacity 600 mL.
  - The jug can hold 600 mL of water.
  - The volume of the jug is 600 cm^3.
  - The jug can hold 600 cm^3 of water.
  - I am going to the supermarket to buy a 2 L bottle of milk.
  - I am going to the supermarket to buy 2 L of milk.

Example 8 Self Tutor

Find the volume of liquid which will fit in a container with capacity:

a 9.6 L **b** 3240 L

a 9.6 L
$= (9.6 \times 1000)$ cm^3
$= 9600$ cm^3

b 3240 L
$= (3240 \div 1000)$ m^3
$= 3.24$ m^3

EXERCISE 6D

1 Find the volume of liquid which will fit in a container with capacity:

a 800 mL **b** 12 L **c** 4.6 kL **d** 3200 mL.

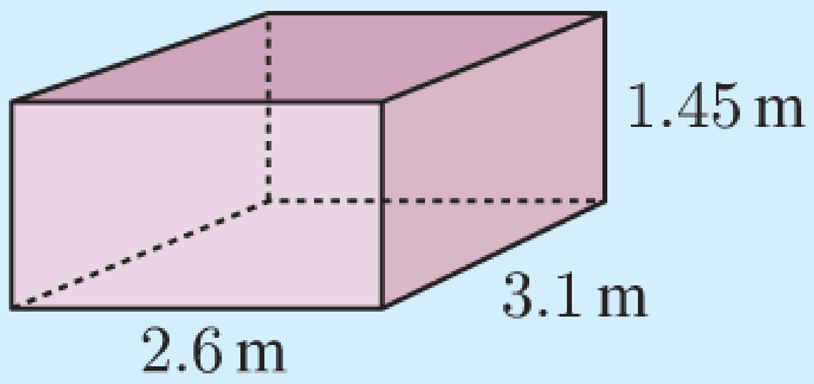
2 Find the capacity of a container needed to hold a fluid with volume:

a 8.4 cm^3 **b** 1800 cm^3 **c** 1.8 m^3 **d** 7154 m^3.

3 Find, in m^3, the volume of gas which will fit in a container with capacity 3.85×10^4 L.

Example 9 — Self Tutor

Find the capacity of a 2.6 m by 3.1 m by 1.45 m tank.



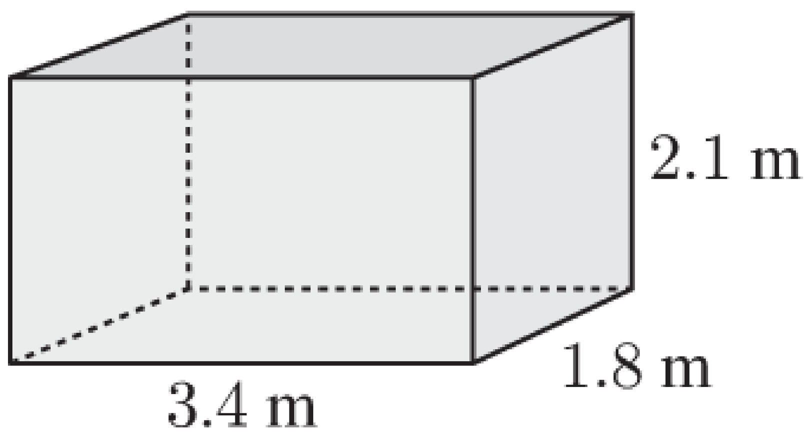
V = length × width × height

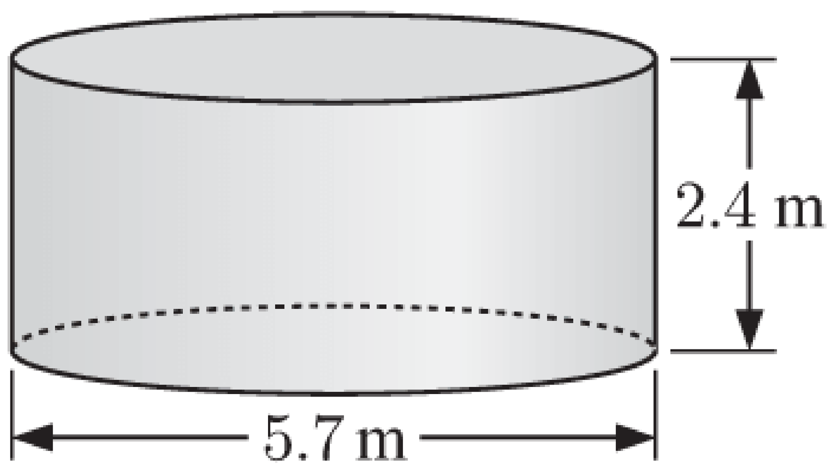
$= 2.6 \times 3.1 \times 1.45 \text{ m}^3$

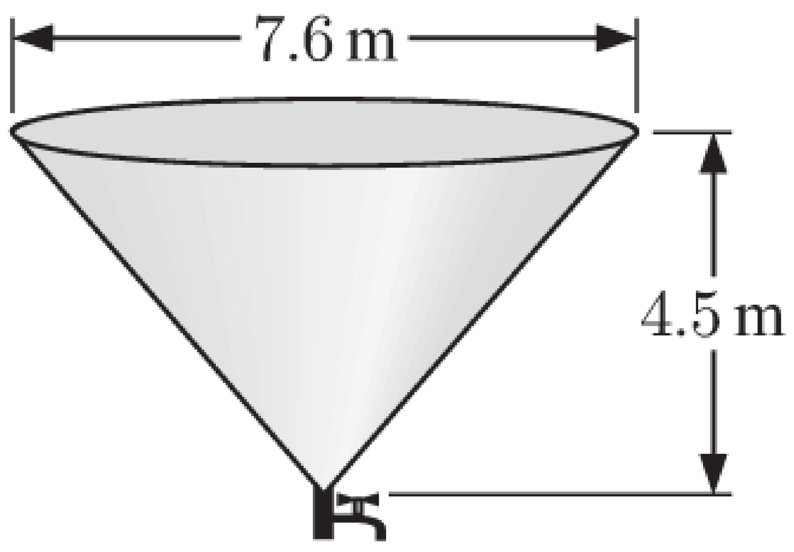
$= 11.687 \text{ m}^3$

The tank's capacity is 11.687 kL.

4 Find the capacity (in kL) of the following tanks:

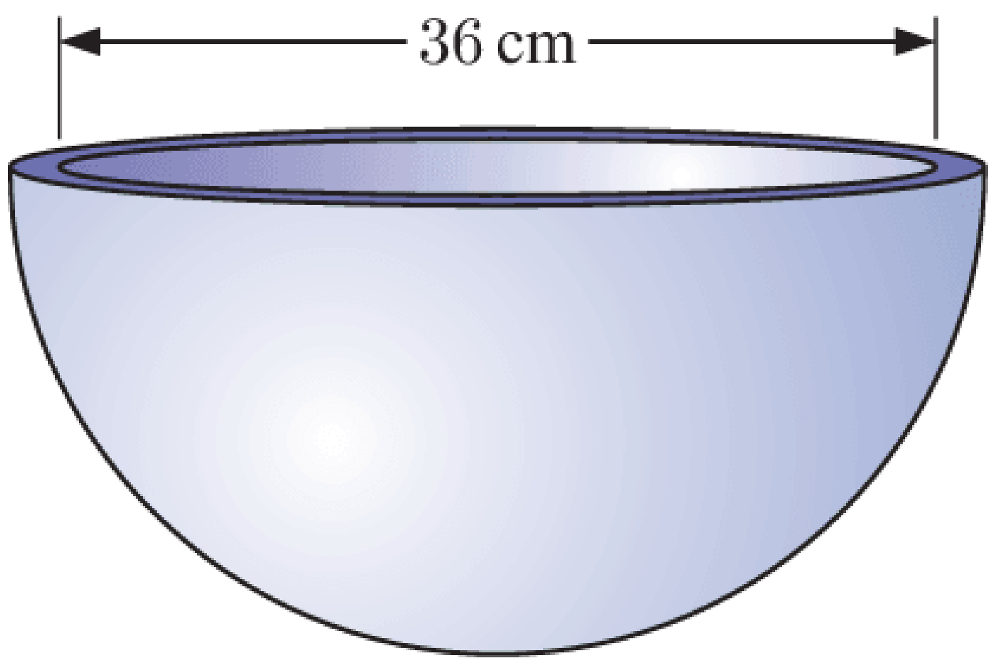
a 

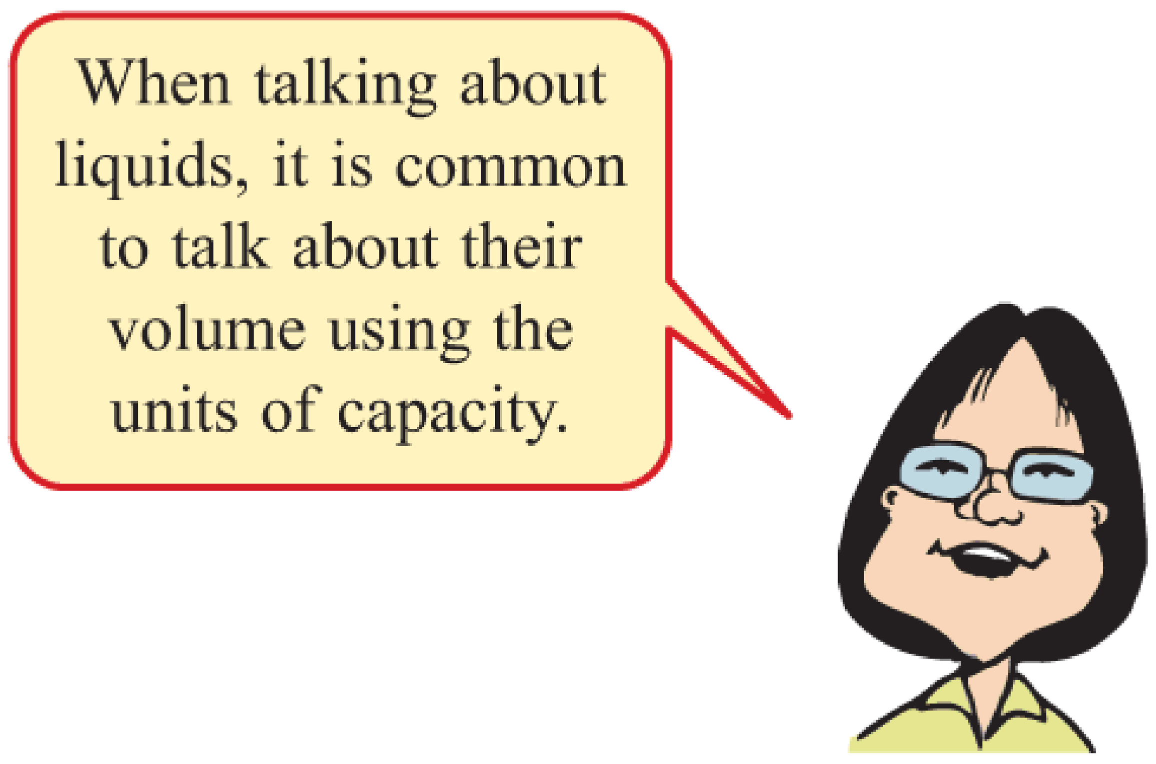
b 

c 

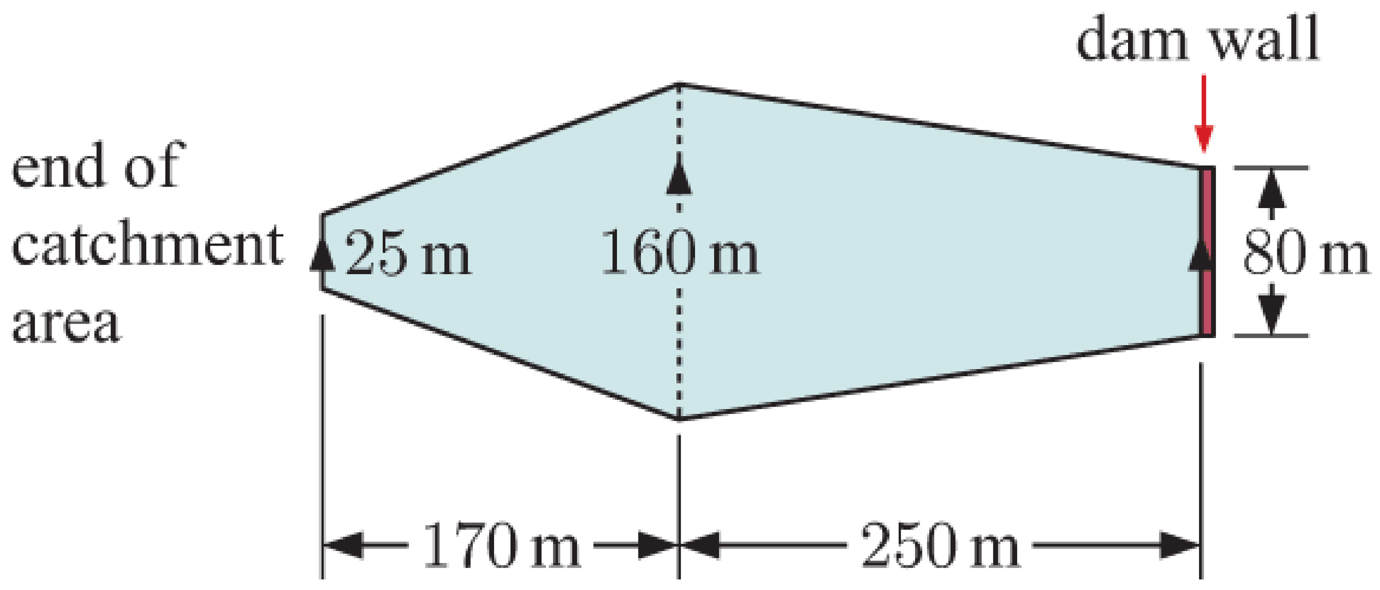
5 Find the volume of soup that will fit in this hemispherical pot. Give your answer in:

a cm^3 **b** L.





6 A dam wall is built at the narrow point of a river to create a small reservoir. When full, the reservoir has an average depth of 13 m, and has the shape shown in the diagram. Find the capacity of the reservoir.



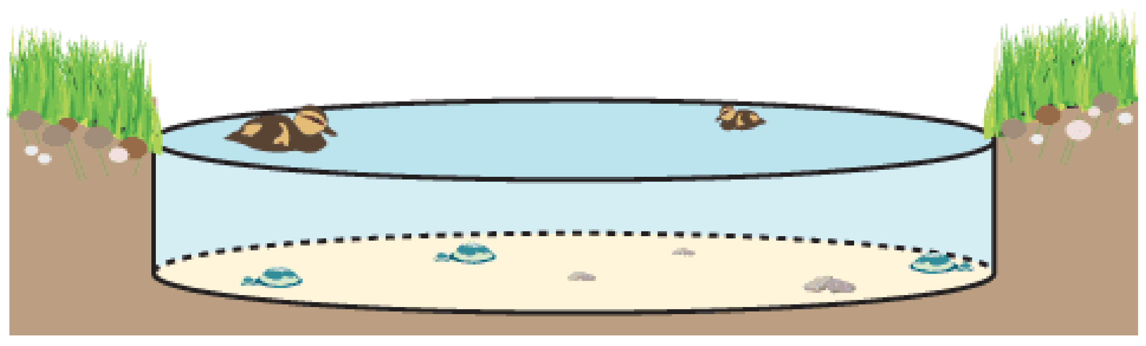
7 Jam is packed into cylindrical tins which have radius 4.5 cm and height 15 cm. The mixing vat is also cylindrical with cross-sectional area 1.2 m^2 and height 4.1 m.

a Find the capacity of each tin. **b** Find the capacity of the mixing vat.

c How many tins of jam could be filled from one vat?

d If the jam is sold at $3.50 per tin, what is the value of one vat of jam?

8 Answer the **Opening Problem** on page **146**.

9 

The circular pond in the park near my house has radius 2.4 m. It has just been filled with 10 kL of water. How deep is the pond?

Example 10 **Self Tutor**

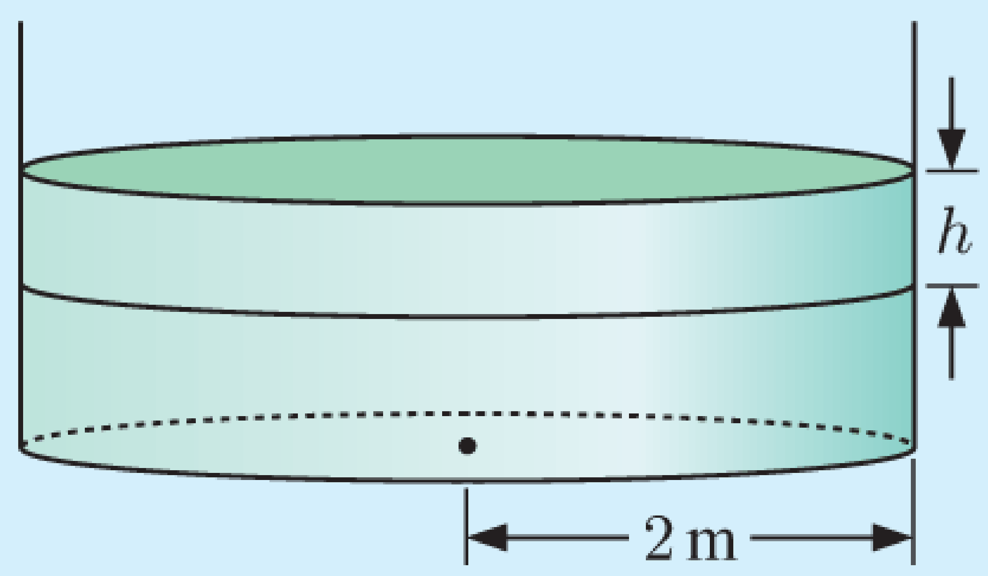
17.3 mm of rain falls on a flat rectangular shed roof which has length 10 m and width 6.5 m. All of the water goes into a cylindrical tank with base diameter 4 m. By how many millimetres does the water level in the tank rise?

For the roof: The dimensions of the roof are in m, so we convert 17.3 mm to metres.

17.3 mm $= (17.3 \div 1000)$ m $= 0.0173$ m

The volume of water collected by the roof $=$ area of roof $\times$ depth
$= 10 \times 6.5 \times 0.0173 \text{ m}^3$
$= 1.1245 \text{ m}^3$

For the tank: The volume added to the tank
$=$ area of base $\times$ height
$= \pi \times 2^2 \times h \text{ m}^3 = 4\pi \times h \text{ m}^3$



The volume added to the tank must equal the volume which falls on the roof, so

$4\pi \times h = 1.1245$

$\therefore \quad h = \frac{1.1245}{4\pi}$ {dividing both sides by 4π}

$\therefore \quad h \approx 0.0895$ m

$\therefore$ the water level rises by about 89.5 mm.

10 The base of a house has area 110 m^2. One night 12 mm of rain falls on the roof. All of the water goes into a tank which has base diameter 4 m.

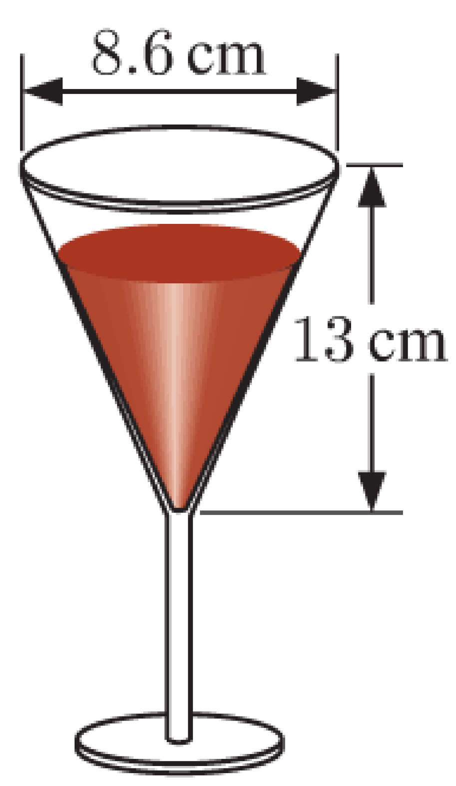
a Find the volume of water which fell on the roof.

b How many kL of water entered the tank?

c By how much did the water level in the tank rise?

11 The design department of a fish canning company wants to change the size of their cylindrical tins. The original tin is 15 cm high and 7.2 cm in diameter. The new tin is to have approximately the same volume, but its diameter will be 10 cm. How high must it be, to the nearest mm?

12 A conical wine glass has the dimensions shown.



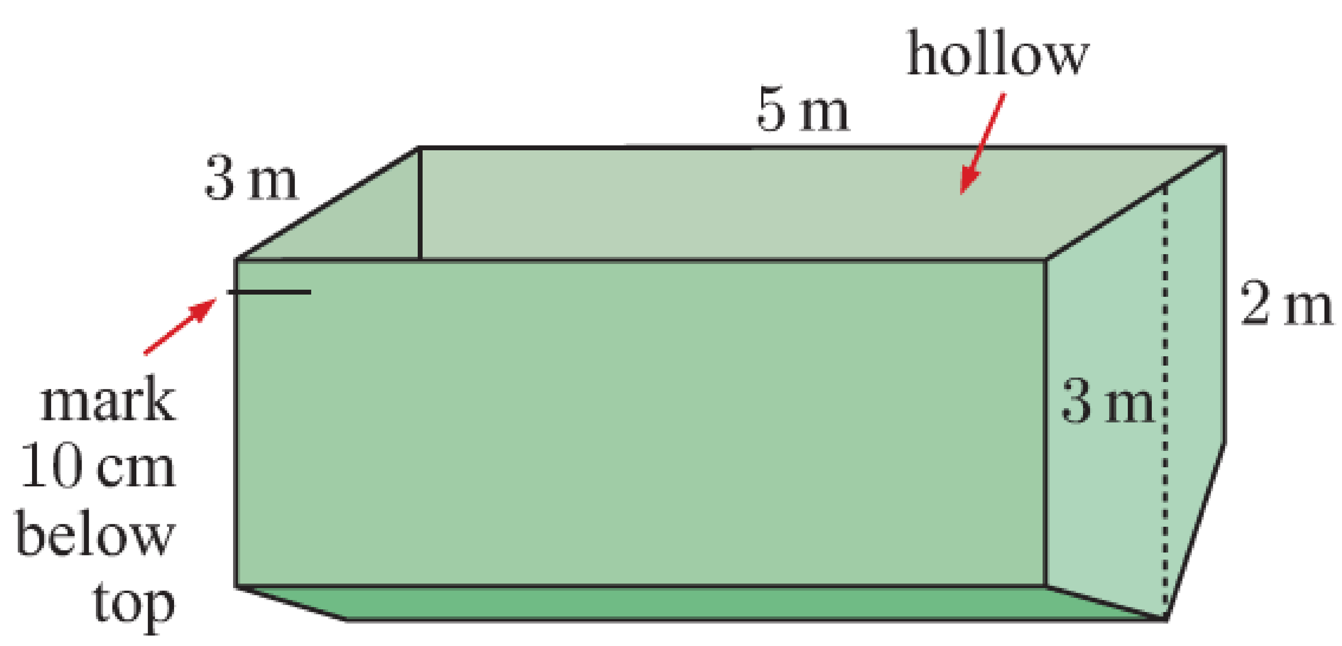
a Find the capacity of the glass.

b Suppose the glass is 75% full.

i How many mL of wine does it contain?

ii If the wine is poured into a cylindrical glass of the same diameter, how high will it rise?

13 A fleet of trucks have containers with the shape illustrated. Wheat is transported in these containers, and its level must not exceed a mark 10 cm below the top. How many truck loads of wheat are necessary to fill a cylindrical silo with internal diameter 8 m and height 25 m?

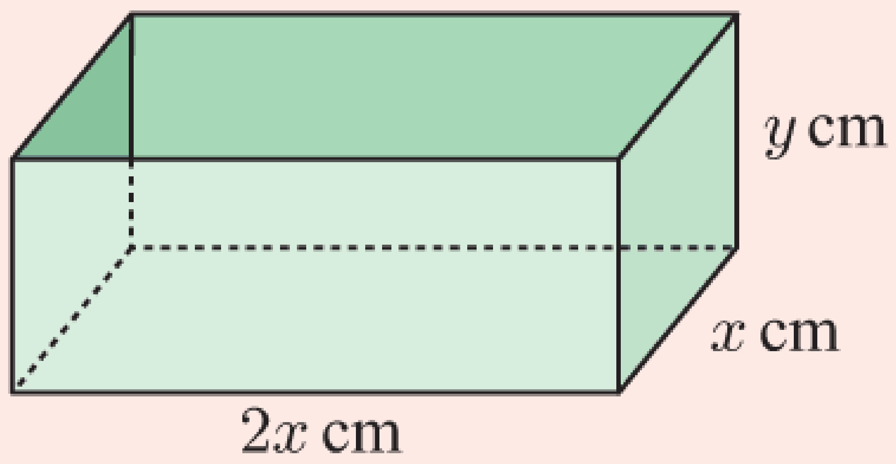


ACTIVITY 2 MINIMISING MATERIAL

Your boss asks you to design a rectangular box-shaped container which is open at the top and contains exactly 1 litre of fluid. The base measurements must be in the ratio $2:1$. She intends to manufacture millions of these containers, and wishes to keep manufacturing costs to a minimum. She therefore insists that the least amount of material is used.

What to do:

1 The base is to be in the ratio $2:1$, so we let the dimensions be x cm and $2x$ cm. The height is also unknown, so we let it be y cm. As the values of x and y vary, the container changes size.

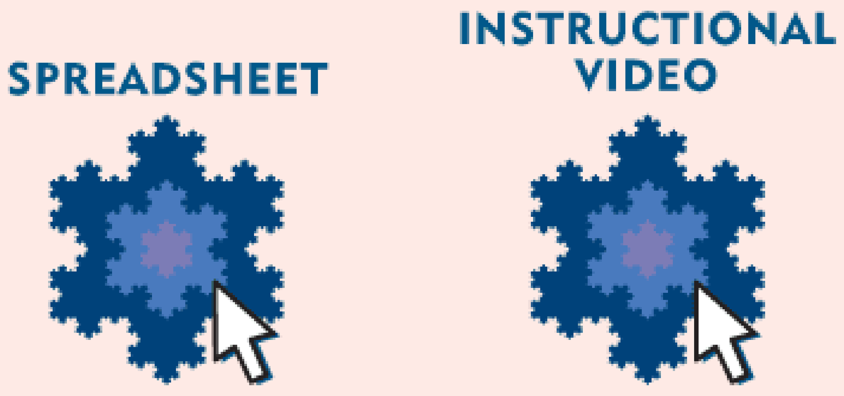


Explain why:

a the volume $V = 2x^2y$ **b** $2x^2y = 1000$ **c** $y = \frac{500}{x^2}$

2 Show that the surface area is given by $A = 2x^2 + 6xy$.

3 Construct a spreadsheet which calculates the surface area for $x = 1, 2, 3, 4, ...$



	A	B	C
1	x values	y values	A values
2	1	=500/A2^2	=2*A2^2+6*A2*B2
3	=A2+1	↓	↓
4	↓	↓	↓
5		fill down	

4 Find the smallest value of A, and the value of x which produces it. Hence write down the dimensions of the box your boss desires.

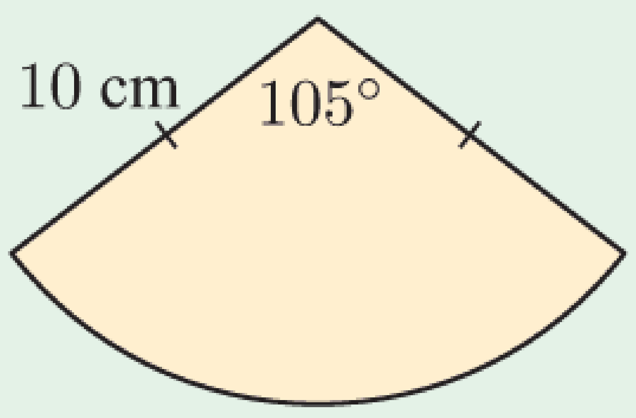
REVIEW SET 6A

1 For the given sector, find to 3 significant figures:

a the length of the arc

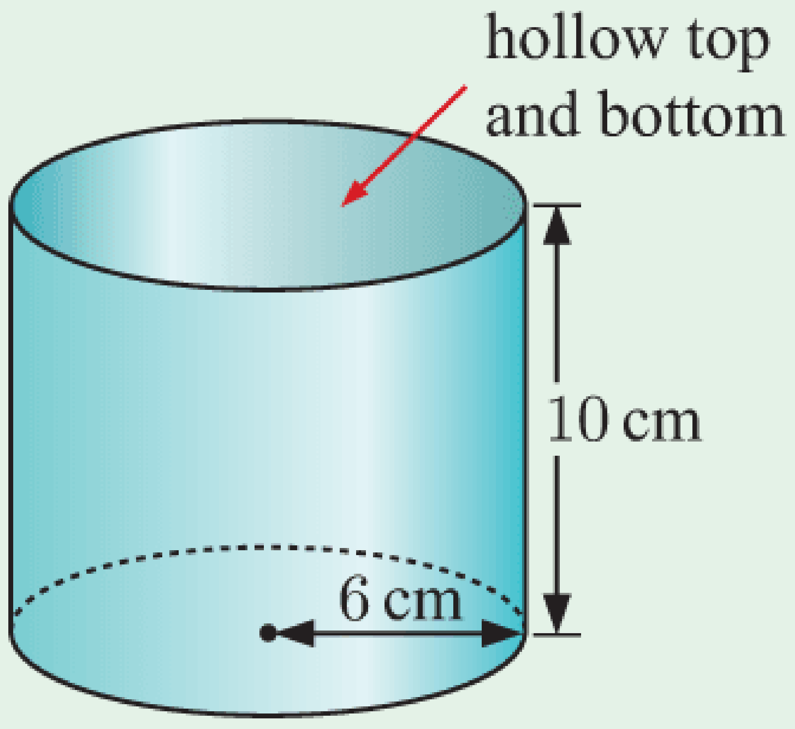
b the perimeter of the sector

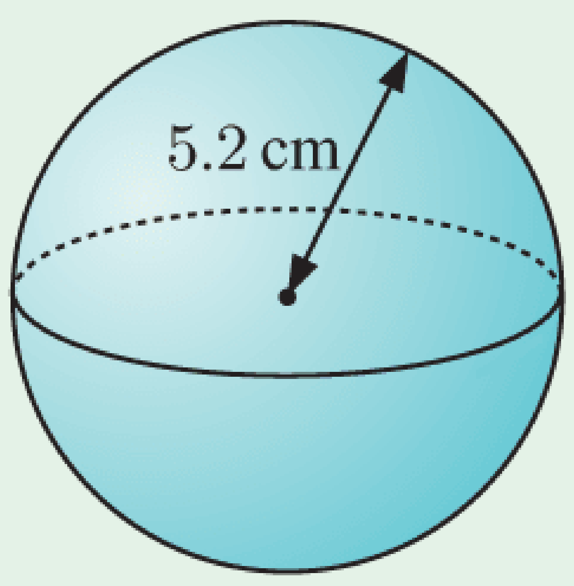
c the area of the sector.

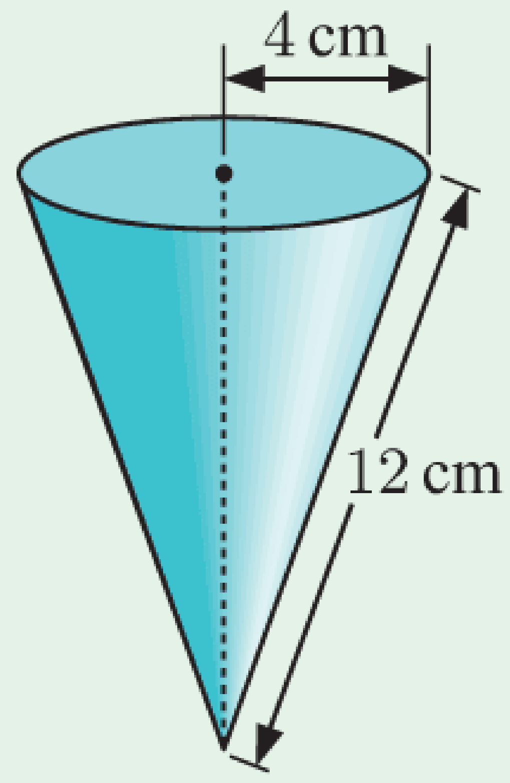


2 Find the radius of a sector with angle $80°$ and area 24π cm^2.

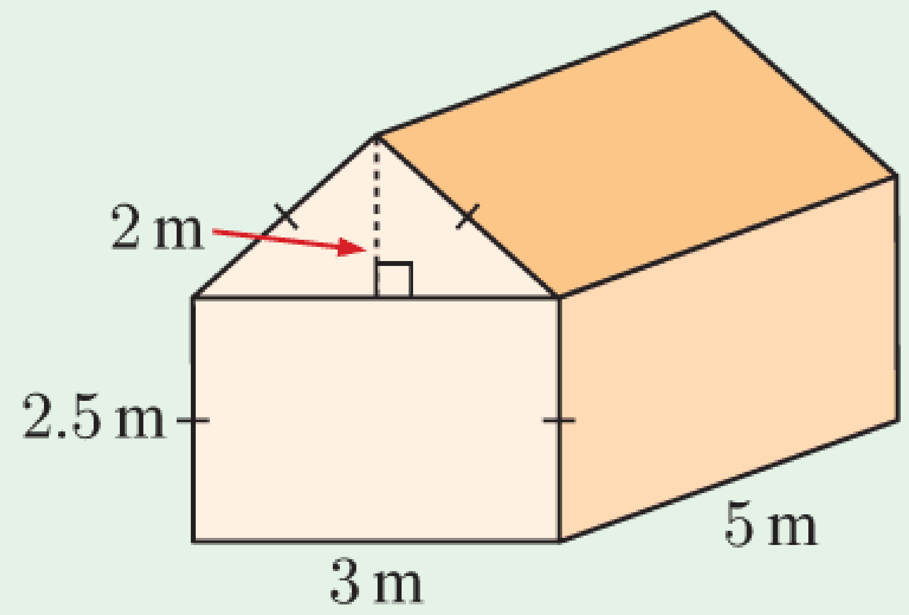
3 Find, to 1 decimal place, the outer surface area of:

a 

b 

c 

4 A tool shed with the dimensions illustrated is to be painted with two coats of zinc-alum. Each litre of zinc-alum covers 5 m^2 and costs \$8.25. It must be purchased in whole litres.



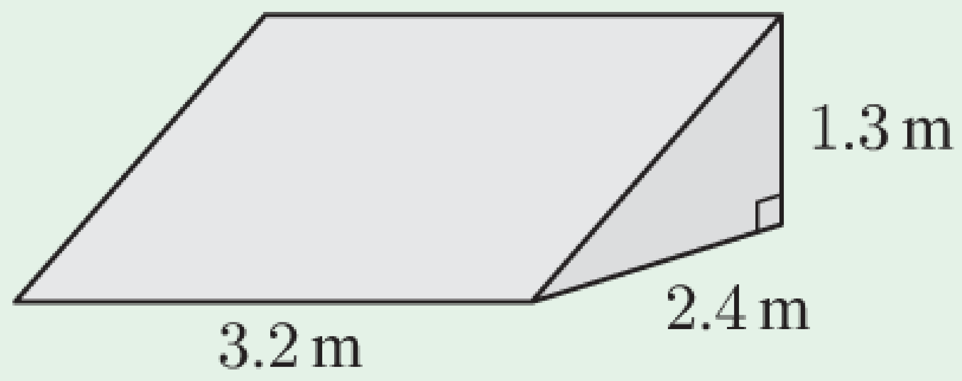
a Find the area to be painted, including the roof.

b Find the total cost of the zinc-alum.

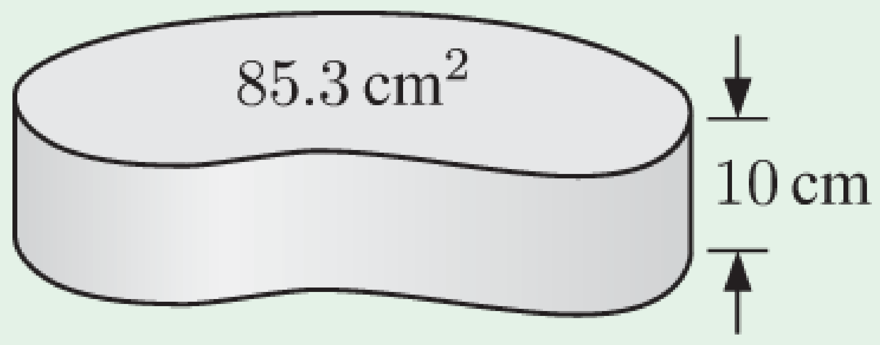
5 A fish farm has *six* netted cylindrical cages open at the top. The cylinders have depth 17 m and diameter 7.5 m. Find the total area of netting in the cages.

6 Calculate, to 3 significant figures, the volume of:

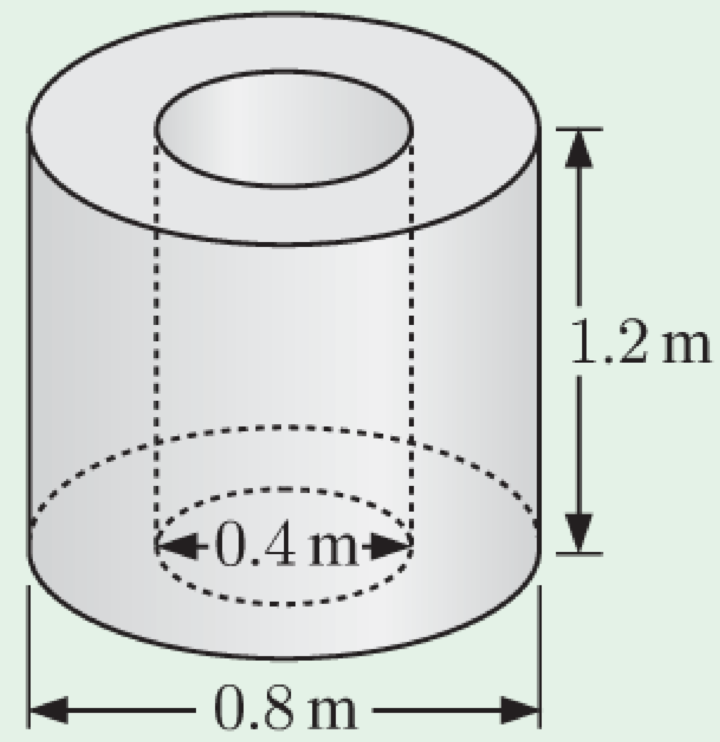
a



b



c



7 Tom has just had a load of sand delivered. The sand is piled in a cone with radius 1.6 m and height 1.2 m. Find the volume of the sand.

8 A manufacturer of spikes has 245 L of molten iron. If each spike contains 15 mL of iron, how many spikes can be made?

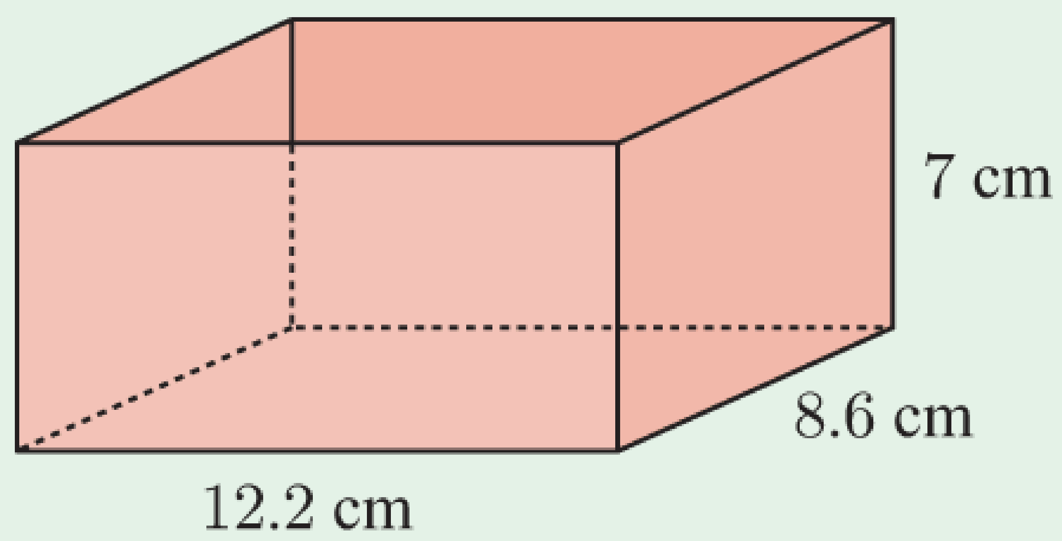
9 A plastic beach ball has radius 27 cm. Find its volume.

10 The capacity of a petrol tank is 65 L. State the volume of petrol required to fill the tank in:

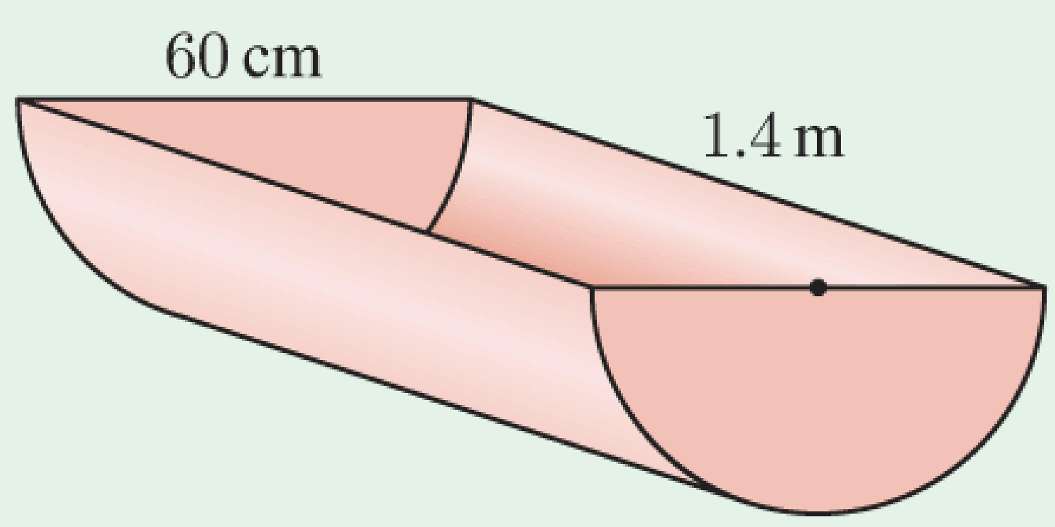
a cm^3 **b** m^3.

11 Find the capacity of:

a

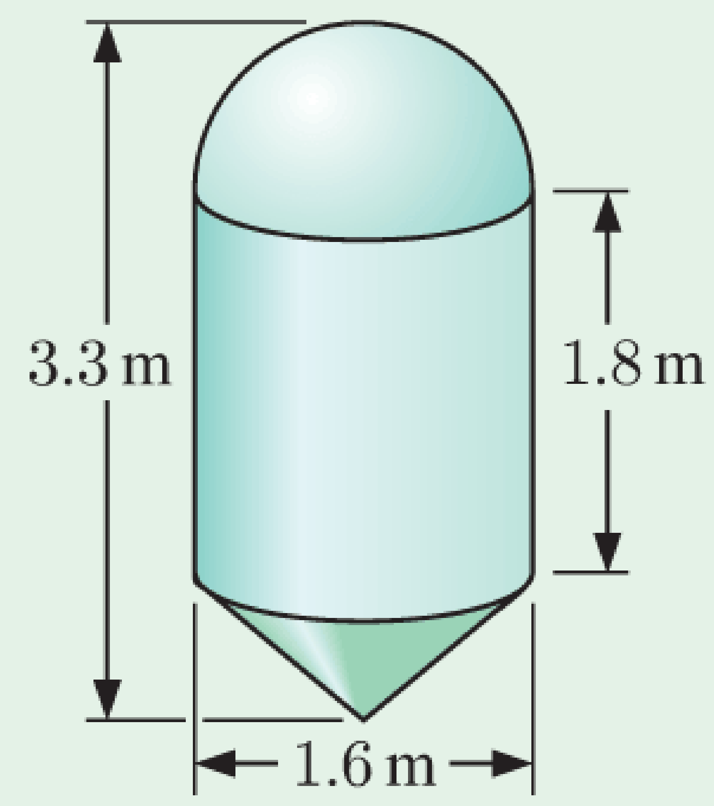


b



12 A rectangular shed has a roof of length 12 m and width 5.5 m. Rainfall from the roof runs into a cylindrical tank with base diameter 4.35 m. If 15.4 mm of rain falls, how many millimetres does the water level in the tank rise?

13 A feed silo is made out of sheet steel using a hemisphere, a cylinder, and a cone.



a Explain why the height of the cone must be 70 cm.

b Hence find the *slant height* of the conical section.

c Calculate the total amount of steel used.

d Show that the silo can hold about 5.2 cubic metres of grain.

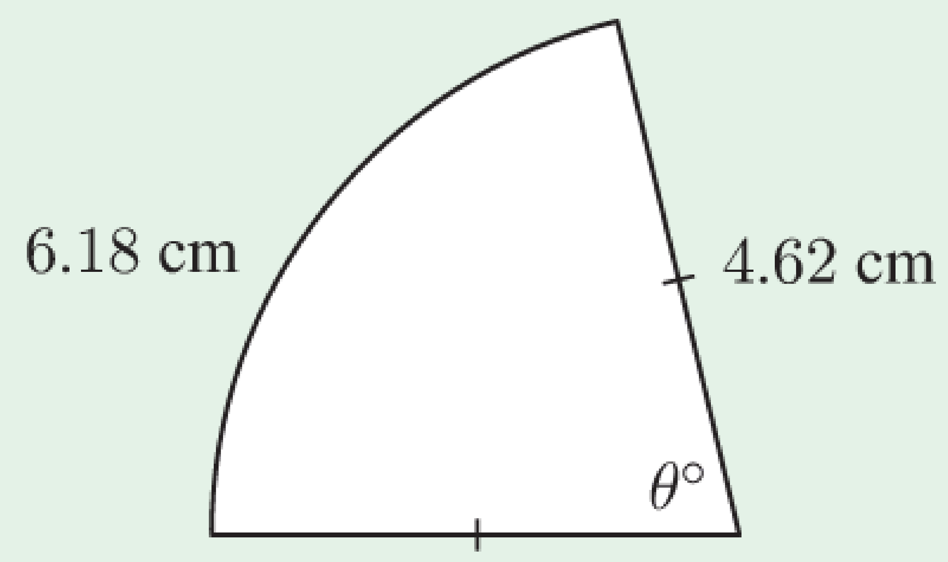
e Write the capacity of the silo in kL.

REVIEW SET 6B

1 For the given sector, find to 3 significant figures:

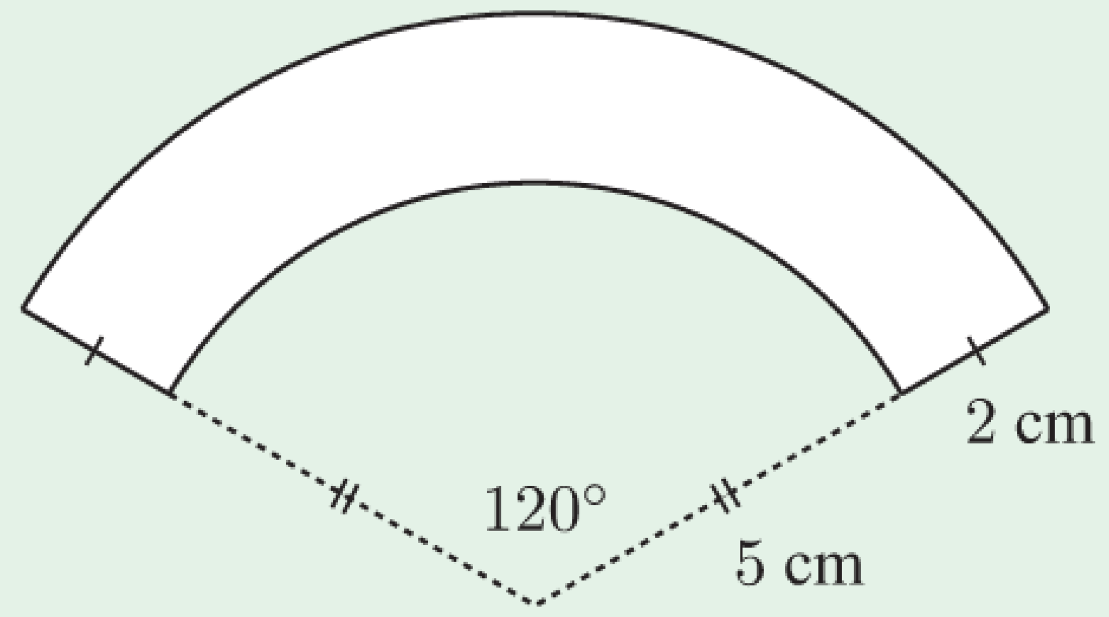
a the angle $\theta°$

b the area of the sector.



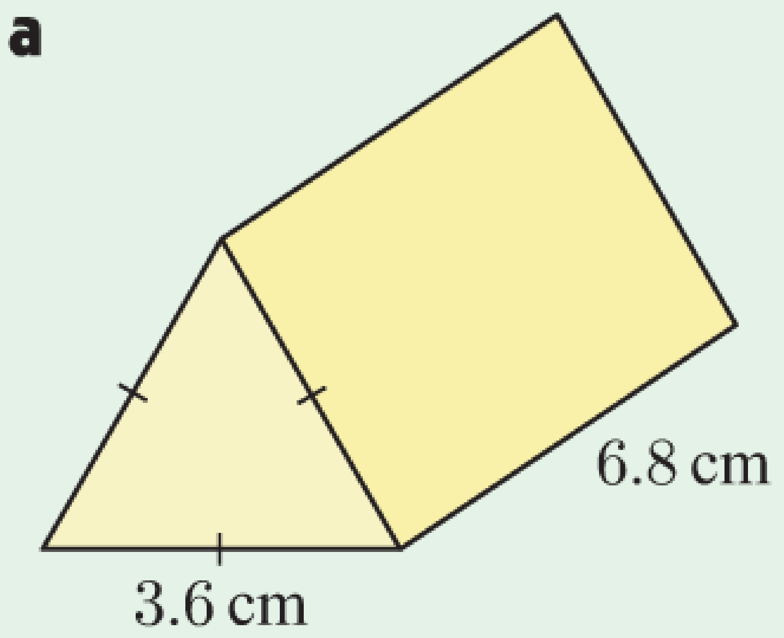
2 For the given figure, find the:

a perimeter **b** area.

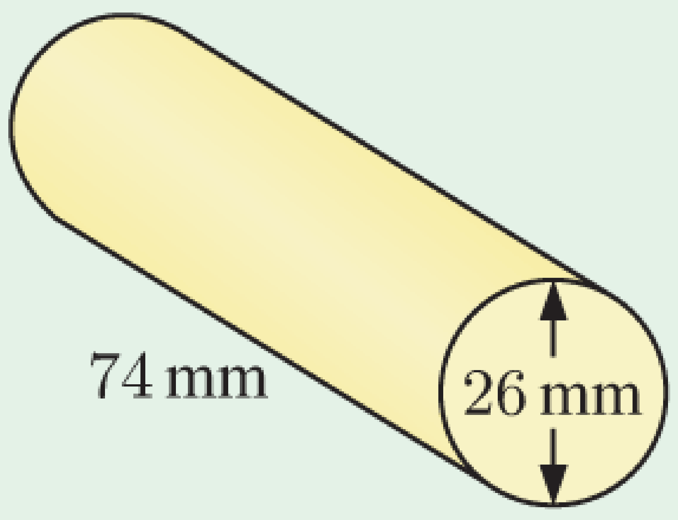


3 Find the surface area of:

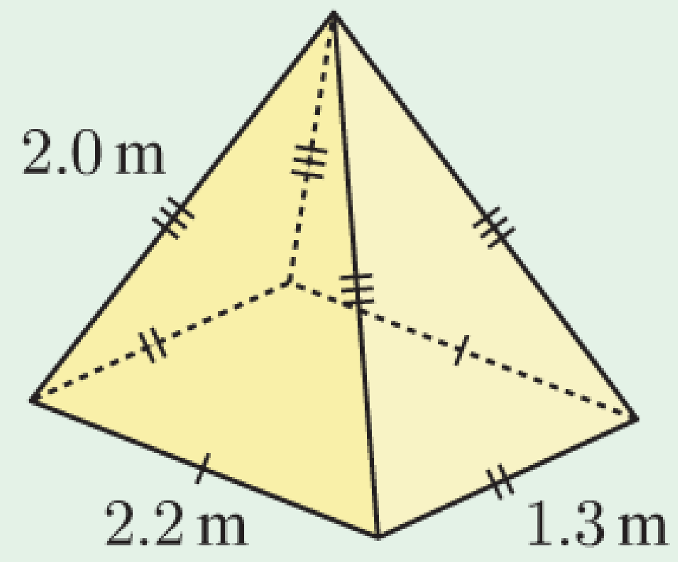
a



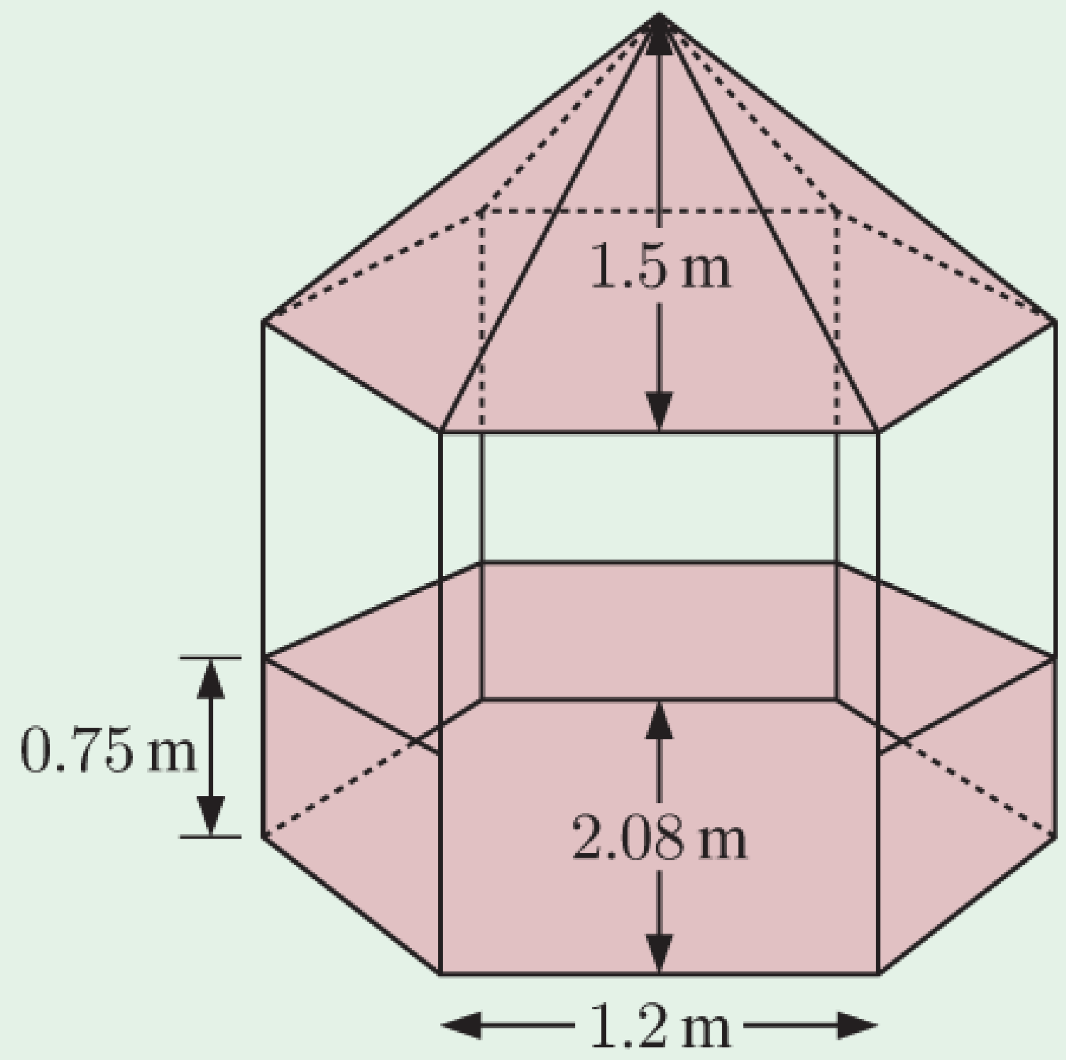
b



c



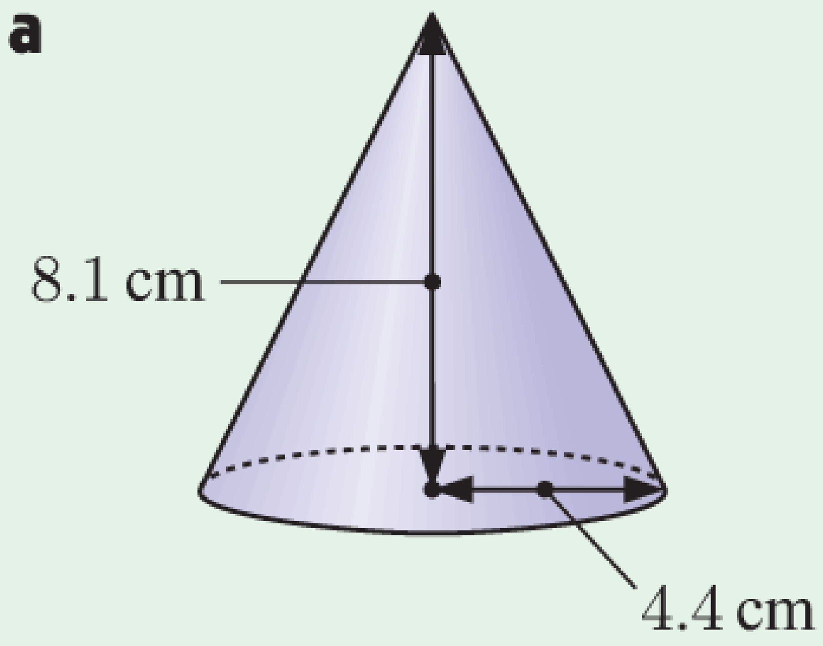
4 The hexagonal gazebo shown has wood panelling for the roof, floor, and part of five of the walls. Find the total surface area of wood panelling in the gazebo. Include the interior as well as the exterior.



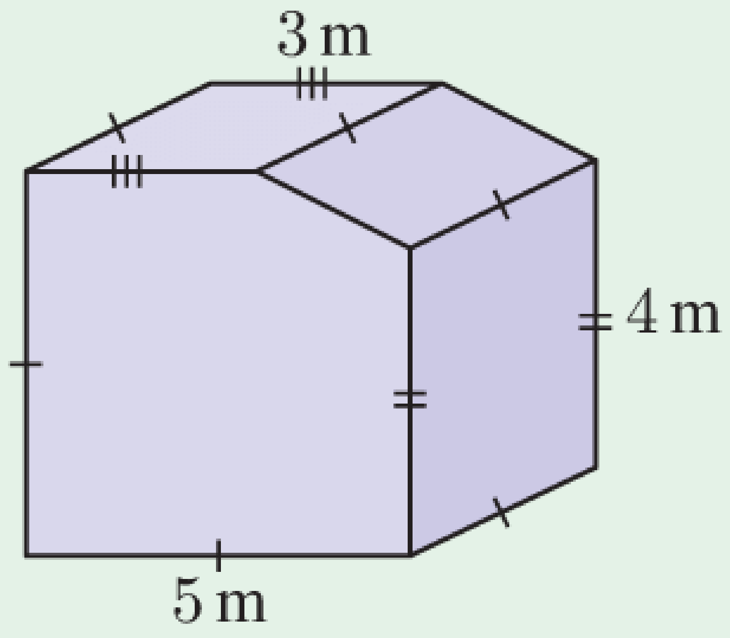
5 I am sending my sister some fragile objects inside a postal cylinder. The cylinder is 325 mm long and has diameter 40 mm. What area of bubble wrap do I need to line its inside walls?

6 Find the volume of:

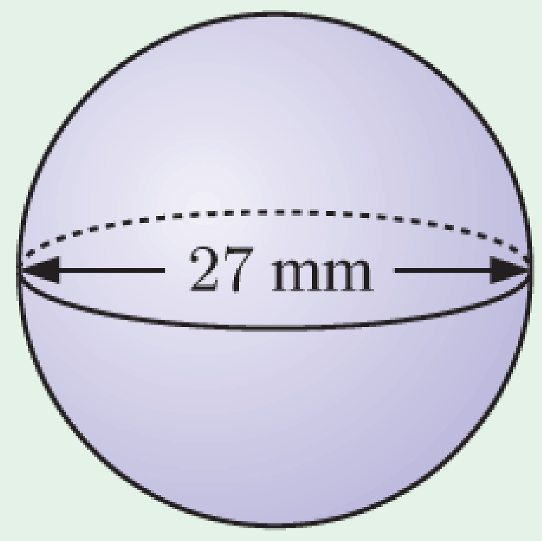
a



b



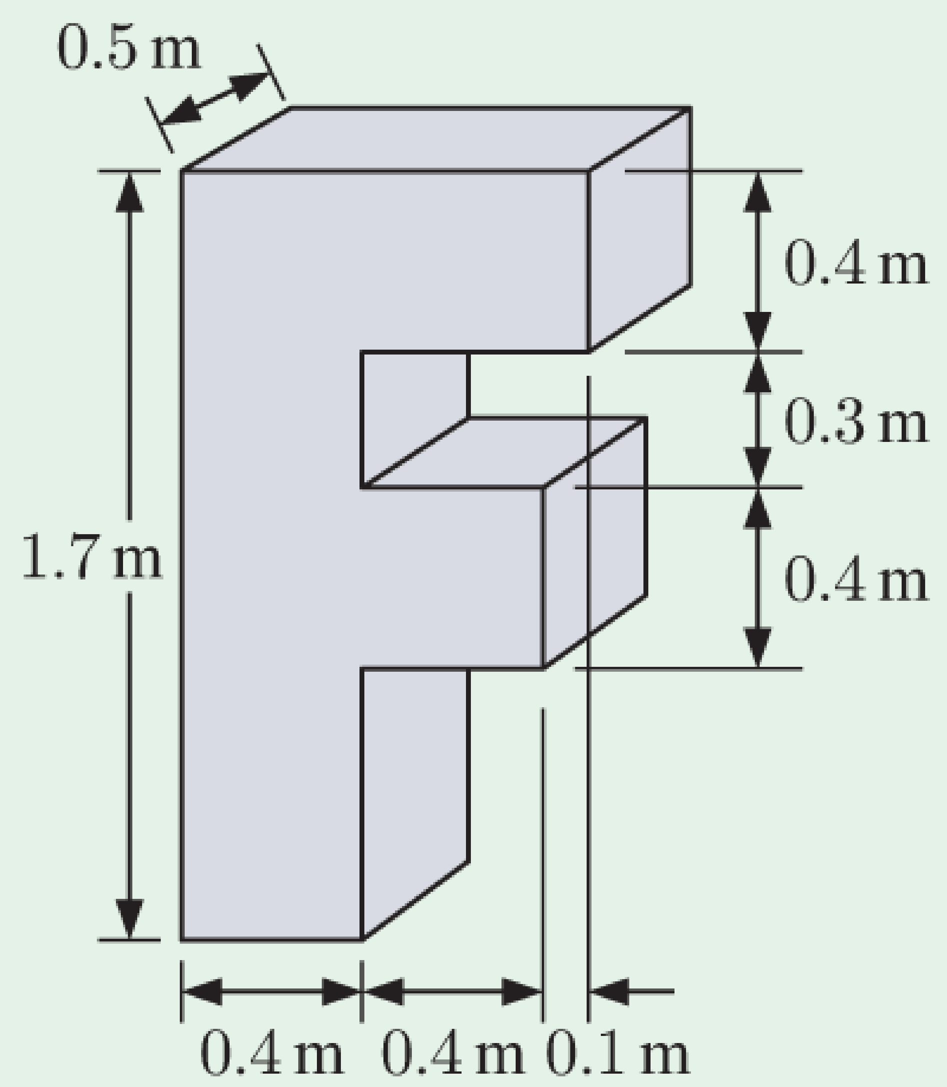
c



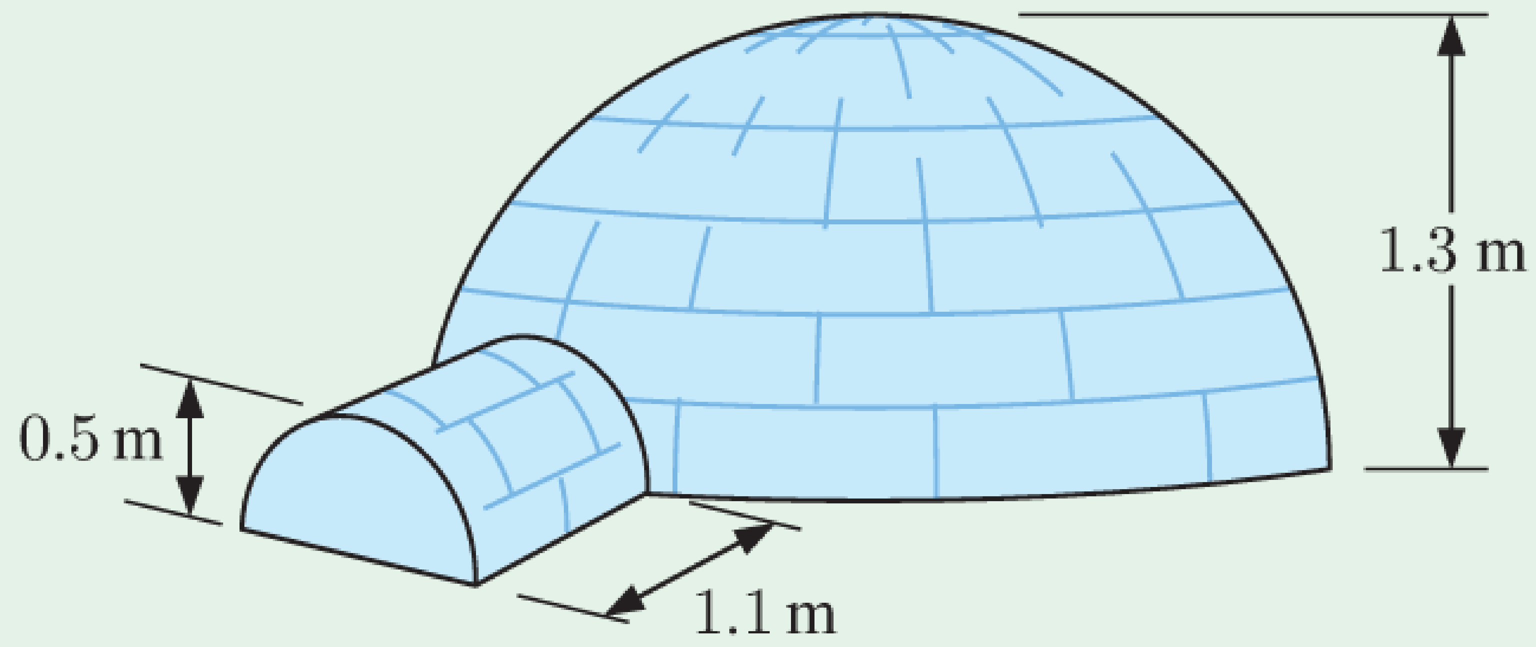
7 Frank wants to have a large F outside his shop for advertising. He designs one with the dimensions shown.

a If the F is made from solid plastic, what volume of plastic is needed?

b If the F is made from fibreglass as a hollow object, what surface area of fibreglass is needed?



8 Find the volume of the igloo:



9 A kitchen bench is a rectangular prism measuring 3845 mm by 1260 mm by 1190 mm. It contains a rectangular sink which is 550 mm wide, 750 mm long, and 195 mm deep. Find the storage capacity of the bench in litres.

10

A gelateria sells gelato in cones with the dimensions shown. The cone is filled completely, with a hemispherical scoop on top.

a Find the volume of gelato sold with each cone.

b How many cones can be sold from 10 L of gelato?

11 A cylindrical drum for storing industrial waste has capacity 10 kL. If the height of the drum is 3 m, find its radius.

12 The Sun is a nearly perfect sphere with radius $\approx 6.955 \times 10^8$ m.
Find, in scientific notation, the Sun's:

a surface area **b** volume.

13 A solid metal spinning top is constructed by joining a hemispherical top to a cone-shaped base.
The radius of both the hemisphere and the base of the cone is 3 cm. The volume of the cone is half that of the hemisphere.
Calculate:

a the volume of the hemispherical top

b the height of the cone-shaped base

c the outer surface area of the spinning top.