1 Counting principles

In this chapter you will learn:

- how to break down complicated questions into parts that are easier to count, and then combine them together
- how to count the number of ways to arrange a set of objects
- the algebraic properties of a useful new tool called the factorial function
- in how many ways you can choose objects from a group
- strategies for applying these tools to harder problems.

Introductory problem

If a computer can print a line containing all 26 letters of the alphabet in 0.01 seconds, estimate how long it would take to print all possible permutations of the alphabet.

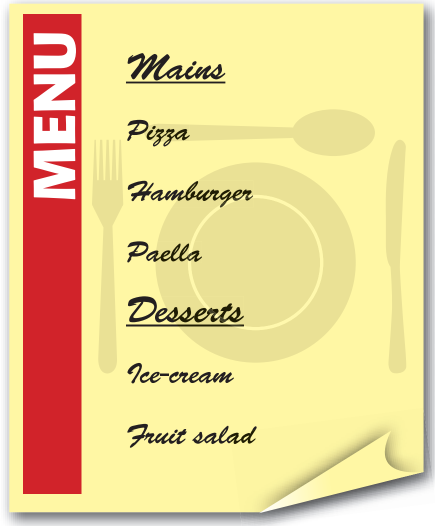
Counting is one of the first things we learn in mathematics and at first it seems very simple. If you were asked to count how many people there are in your school, this would not be too tricky. If you were asked how many chess matches would need to be played if everyone were to play everyone else, this would be a little more complicated. If you were asked how many different football teams could be chosen, you might find that the numbers become far too large to count without coming up with some clever tricks. This chapter aims to help develop strategies for counting in such difficult situations.

Counting sometimes gets extremely difficult. Are there more whole numbers or odd numbers; fractions or decimals? Have a look at the work of Georg Cantor, and the result may surprise you!

1A The product principle and the addition principle

Counting very small groups is easy. So, we need to break down more complicated problems into counting small groups. But how do we then combine these together to come up with an answer to the overall problem? The answer lies in using the product principle and the addition principle, which can be illustrated using the following menu.

The word analysis literally means 'breaking up'. When a problem is analysed it is broken down into simpler parts. One of the purposes of studying mathematics is to develop an analytical mind, which is considered very useful in many different disciplines.



Anna would like to order a main course *and* a dessert. She can choose one of three main courses and one of two desserts. How many different choices could she make? Bob would like to order *either* a main course *or* a dessert. He can choose one of the three main courses or one of the two desserts; how many different orders can he make?

$n(A)$ means the size of the set of options of A. *See Prior Learning Section G on the CD-ROM.*

We can use the notation $n(A)$ to represent the number of ways of making a choice about A.

The **product principle** tells us that when we want to select one option from A *and* one option from B we *multiply* the individual possibilities together.

KEY POINT 1.1

The Product principle (AND rule)

The number of ways of in which both choice A *and* choice B can be made is the product of the number of options for A and the number of options for B.

$n(A \text{ AND } B) = n(A) \times n(B)$

The **addition principle** tells us that when we wish to select one option from A *or* one option from B we *add* the individual possibilities together.

The addition principle has one essential restriction. You can only use it if there is no overlap between the choices for A and the choices for B. For example, you cannot apply the addition principle to counting the number of ways of getting an odd number or a prime number on a die. If there is no overlap between the choices for A and for B, the two events are **mutually exclusive**.

KEY POINT 1.2

The Addition Principle (OR rule)

The number of ways of in which either choice A *or* choice B can be made is the sum of the number of options for A and the number of options for B.

If A and B are mutually exclusive then
$n(A \text{ OR } B) = n(A) + n(B)$

The hardest part of applying either the addition or product principle is breaking the problem down and deciding which principle to use. You must make sure that you have included all of the cases and checked that they are mutually exclusive. It is often useful to rewrite questions to emphasise what is required, 'AND' or 'OR'.

Worked example 1.1

An examination has ten questions in section A and four questions in section B. How many different ways are there to choose questions if you must:

(a) choose one question from each section?
(b) choose a question from either section A or section B?

Describe the problem accurately	(a) Choose one question from A (10 ways) AND one from B (4 ways)
'AND' means we should apply the product principle: $n(A) \times n(B)$	Number of ways $= 10 \times 4$ $= 40$
Describe the problem accurately	(b) Choose one question from A (10 ways) OR one from B (4 ways)
'OR' means we should apply the addition principle: $n(A) + n(B)$	Number of ways $= 10 + 4$ $= 14$

In the example above we cannot answer a question twice so there are no repeated objects; however, this is not always the case.

Worked example 1.2

In a class there are awards for best mathematician, best sportsman and nicest person. Students can receive more than one award. In how many ways can the awards be distributed if there are twelve people in the class?

Describe the problem accurately

Choose one of 12 people for the best mathematician (12 ways)
AND
one of the 12 for best sportsmen (12 ways)
AND
one of the 12 for nicest person. (12 ways)

Apply the product principle

$12 \times 12 \times 12 = 1728$

This leads us to a general idea.

KEY POINT 1.3

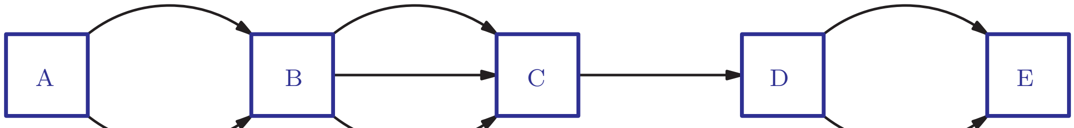
The number of ways of selecting something r times from n objects is n^r.

Exercise 1A

1. If there are 10 ways of doing A, 3 ways of doing B and 19 ways of doing C, how many ways are there of doing
   (a) (i) both A and B? (ii) both B and C?
   (b) (i) either A or B? (ii) either A or C?

2. If there are 4 ways of doing A, 7 ways of doing B and 5 ways of doing C, how many ways are there of doing
   (a) all of A, B and C?
   (b) exactly one of A, B or C?

3. How many different paths are there
   (a) from A to C?
   (b) from C to E?
   (c) from A to E?



4. Jamil is planting out his garden and needs one new rose bush and some dahlias. There are 12 types of rose and 4 varieties of dahlia in his local nursery. How many possible selections does he have to choose from? *[3 marks]*

5. A lunchtime menu at a restaurant offers 5 starters, 6 main courses and 3 desserts. How many different choices of meal can you make if you would like

(a) a starter, a main course and a dessert?

(b) a main course and either a starter or a dessert?

(c) any two different courses? *[6 marks]*

6. Five men and three women would like to represent their club in a tennis tournament. In how many ways can one mixed doubles pair be chosen? *[3 marks]*

7. A mathematics team consists of one student from each of years 7, 8, 9 and 10. There are 58 students in year 7, 68 in year 8, 61 in year 9 and 65 in year 10.

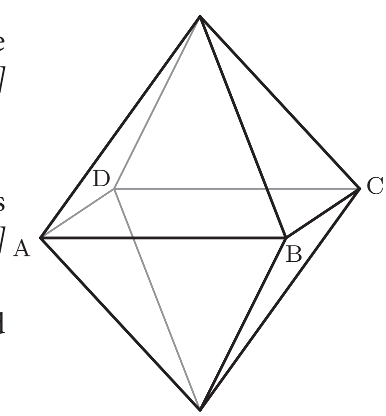
(a) How many ways are there of picking the team?

Year 10 is split into three classes: 10A (21 students), 10B (23 students) and 10C (21 students).

(b) If students from 10B cannot participate in the competition, how many ways are there of picking the team? *[4 marks]*

8. Student passwords consist of three letters chosen from A to Z, followed by four digits chosen from 1–9. Repeated characters are allowed. How many possible passwords are there? *[4 marks]*

9. A beetle walks along the edges from the base to the tip of an octahedral sculpture, visiting exactly two of the middle vertices (A, B, C or D). How many possible routes are there? *[6 marks]*



10. Professor Square has 15 different ties (seven blue, three red and five green), four waistcoats (red, black, blue and brown) and 12 different shirts (three each of red, pink, white and blue). He always wears a shirt, a tie and a waistcoat.

(a) How many different outfits can he make?

Professor Square never wears any outfit that combines red with pink.

(b) How many different outfits can he make with this limitation? *[6 marks]*

11. How many different three digit numbers can be formed using the digits 1,2,3,5,7

(a) once only?

(b) if digits can be repeated? *[6 marks]*

12. In how many ways can

(a) four toys be put into three boxes?

(b) three toys be put into five boxes? *[6 marks]*

If you look in other textbooks you may see permutations referred to in other ways. This is an example of a word that can take slightly different meanings in different countries. The definition here is the one used in the International Baccalaureate® (IB).

$n!$ occurs in many other mathematical situations. You will see it in the Poisson distribution (see Section 23D), and if you study option 9 on calculus, you will see that it is also important in a method for approximating functions called Taylor series.

1B Counting arrangements

The word 'ARTS' and the word 'STAR' both contain the same letters, but arranged in a different order. They are both arrangements (also known as **permutations**) of the letters R, A, T and S. We can count the number of different permutations.

There are four possibilities for the first letter, then for each choice of the first letter there are three options for the second letter (because one of the letters has already been used). This leaves two options for the third letter and then the final letter is fixed. Using the 'AND rule' the number of possible permutations is $4 \times 3 \times 2 \times 1 = 24$.

The number of permutations of n different objects is equal to the product of all positive integers less than or equal to n. This **expression** is abbreviated to $n!$ (pronounced 'n factorial').

KEY POINT 1.4

The number of ways of arranging n objects is $n!$

$n! = n(n-1)(n-2)\ldots \times 2 \times 1$

Worked example 1.3

A test has 12 questions. How many different arrangements of the questions are possible?

Describe the problem accurately

Permute (arrange) 12 items

Number of permutations $= 12!$

$= 479\,001\,600$

In examination questions you might have to combine the idea of permutations with the product and addition principles.

Worked example 1.4

A seven-digit number is formed by using each of the digits 1–7 exactly once. How many such numbers are even?

Describe the problem accurately: even numbers end in 2, 4 or 6	Pick the final digit to be even (3 ways)
Only 6 digits left to arrange	AND then permute the remaining 6 digits (6! ways)
Apply the product principle	$3 \times 720 = 2160$ possible even numbers

This example shows a very common situation where there is a constraint, in this case we have to end with an even digit. It can be more efficient to fix each part of the constraint separately, instead of searching all the possibilities for the ones which are allowed.

EXAM HINT

Factorials get very large very quickly. Although you should know factorials up to 6!, most of the time you will use your calculator. See Calculator skills sheet 3 on the CD-ROM.

Worked example 1.5

How many permutations of the word SQUARE start with three vowels?

Describe the problem accurately	Permute the three vowels at the beginning (3! ways) AND Permute the three consonants at the end (3! ways)
Apply the product principle	Number of ways $= 3! \times 3! = 36$

Exercise 1B

1. Evaluate:

   (a) (i) 5!   (ii) 6!

   (b) (i) $2 \times 4!$ (ii) $3 \times 5!$

   (c) (i) $6! - 5!$ (ii) $6! - 4 \times 5!$

2. Evaluate:

   (a) (i) 8!   (ii) 11!

   (b) (i) $9 \times 5!$ (ii) $9! \times 5$

   (c) (i) $12! - 10!$ (ii) $9! - 7!$

3. Find the number of ways of arranging:

   (a) 6 objects   (b) 8 objects   (c) 26 objects.

4. (a) How many ways are there of arranging seven textbooks on a shelf?

   (b) In how many of those arrangements is the single biggest textbook not at either end? *[5 marks]*

5. (a) How many five-digit numbers can be formed by using each of the digits 1–5 exactly once?

   (b) How many of those numbers are divisible by 5? *[5 marks]*

6. A class of 16 pupils and their teacher are queuing outside a cinema.

   (a) How many different arrangements are there?

   (b) How many different arrangements are there if the teacher has to stand at the front? *[5 marks]*

7. A group of nine pupils (five boys and four girls) are lining up for a photograph, with all the girls in the front row and all the boys at the back. How many different arrangements are there? *[5 marks]*

8. (a) How many six-digit numbers can be made by using each of the digits 1–6 exactly once?

   (b) How many of those numbers are smaller than 300 000? *[5 marks]*

9. A class of 30 pupils are lining up in three rows of ten for a class photograph.
How many different arrangements are possible? *[6 marks]*

10. A baby has nine different toy animals. Five of them are red and four of them are blue. She arranges them in a line so that the colours are arranged symmetrically. How many different arrangements are possible? *[7 marks]*

1C Algebra of factorials

To solve more complicated counting problems we often need to simplify expressions involving factorials. This is done using the formula for factorials, which you saw in Key point 1.4:

$$n! = n(n-1)(n-2)\ldots\times 2\times 1$$

Worked example 1.6

(a) Evaluate $9! \div 6!$

(b) Simplify $\frac{n!}{(n-3)!}$

(c) Write $10 \times 11 \times 12$ as a ratio of two factorials.

Write in full and look for common factors in denominator and numerator

(a) $\frac{9\times 8\times 7\times 6\times 5\ldots\times 2\times 1}{6\times 5\ldots\times 2\times 1}$

$= 9\times 8\times 7$

$= 504$

Write in full and look for common factors in denominator and numerator

(b) $\frac{n\times(n-1)\times(n-2)\times(n-3)\ldots\times 2\times 1}{(n-3)\times(n-4)\ldots\times 2\times 1}$

$= n(n-1)(n-2)$

Reverse the ideas from (b)

(c) $10\times 11\times 12 = \frac{1\times 2\ldots\times 9\times 10\times 11\times 12}{1\times 2\ldots\times 9}$

$= \frac{12!}{9!}$

You can usually solve questions involving sums or differences of factorials by looking for common factors of the terms. It is important to understand the link between one factorial and the next:

KEY POINT 1.5

$$(n+1)! = (n+1) \times n!$$

Worked example 1.7

Simplify:

(a) $9! - 7!$ (b) $(n+1)! - n! + (n-1)! = 72 \times 7!$

Link 9! and 7!

(a) $9! = 9 \times 8! = 9 \times 8 \times 7!$

Take out the common factor of 7!

$9! - 7! = 72 \times 7! - 7$
$= (72 - 1) \times 7! = 71 \times 7!$

Link $(n+1)!$, $n!$ and $(n-1)!$

(b) $n! = n \times (n-1)!$
$(n+1)! = (n+1) \times n! = (n+1) \times n \times (n-1)!$
$(n+1)! - n! + (n-1)! = n(n+1)(n-1)! - n(n-1)! + (n-1)!$

Take out the common factor of $(n-1)!$

$= (n-1)!(n^2 + n - n + 1) = (n-1)!(n^2 + 1)$

0! is defined to be 1. Why might this be?

$\frac{1}{2}!$ is $\frac{\sqrt{\pi}}{2}$

To explore why you need to look at the Gamma function.

Being able to simplify expressions involving $n!$ is useful because $n!$ becomes very large very quickly. Often factorials cannot be evaluated even using a calculator. For example, a standard scientific calculator can only calculate up to $69! = 1.71 \times 10^{98}$ (to 3 significant figures).

EXAM HINT

Remember that you will lose one mark per paper in the IB if you give any answers to less than three significant figures (3SF) (unless the answer is exact of course!). See Prior learning Section B on the CD-ROM if you need a reminder about significant figures.

Exercise 1C

1. Fully simplify the following fractions:

(a) (i) $\frac{7!}{6}$ (ii) $\frac{12!}{11!}$

(b) (i) $\frac{8!}{6!}$ (ii) $\frac{11!}{8!}$