14.1 MULTIPLICATION PRINCIPLE

14.1.1 DEFINITION

Permutations

Permutations represents a counting process where the **order must be taken into account**.

For example, the number of permutations of the letters A, B, C and D, if only two are taken at a time, can be enumerated as

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, BC

That is, AC is a different permutation from CA (different order).

Instead of **permutation** the term **arrangement** is often used.

This definitions lead to a number of **Counting Principles** which we now look at.

14.1.2 MULTIPLICATION PRINCIPLE

Rule 1: If any one of n different mutually exclusive and exhaustive events can occur on each of k trials, the number of possible outcomes is equal to n^k.

For example, if a die is rolled twice, there are a total of $6^2 = 36$ possible outcomes.

Rule 2: If there are n_1 events on the first trial, n_2 events on the second trial, and so on, and finally, n_k events on the kth trial, then the number of possible outcomes is equal to $n_1 \times n_2 \times \ldots \times n_k$.

For example, if a person has three different coloured pairs of pants, four different shirts, five different ties and three different coloured pairs of socks, the total number of different ways that this person can dress is equal to $3 \times 4 \times 5 \times 3 = 180$ ways.

Rule 3: The total number of ways that n different objects can be arranged in order is equal to $n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1$.

Because of the common usage of this expression, we use the factorial notation. That is, we write

$$n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1$$

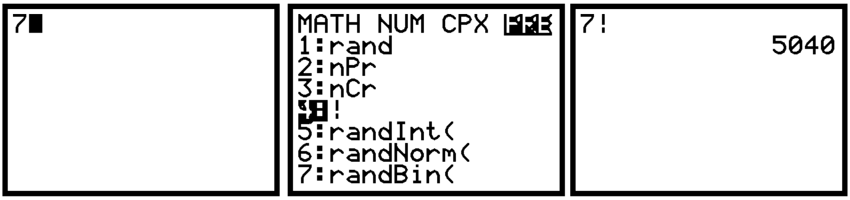
which is read as ***n* factorial**. Notice also that 0! is defined as 1, i.e., 0! = 1.

For example, in how many ways can 4 boys and 3 girls be seated on a park bench? In this case any one of the seven children can be seated at one end, meaning that the adjacent position can be filled by any one of the remaining six children, similarly, the next adjacent seat can be occupied by any one of the remaining 5 children, and so on . . . .

Therefore, in total there are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = 5040$ possible arrangements.

Using the TI–83, we have:

Enter **7**: Select **MATH** and **PRB**: Select option **4: !** and press **ENTER:**

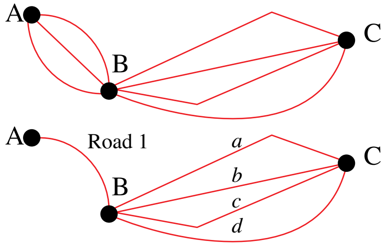


EXAMPLE 14.1

John wishes to get from town A to town C via town B. There are three roads connecting town A to town B and 4 roads connecting town B to town C. In how many different ways can John get from town A to town C?

Solution

We start by visualising this situation:



Consider the case where John uses Road 1 first.

The possibilities are:

Road 1 then a,
Road 1 then b,
Road 1 then c,
Road 1 then d.

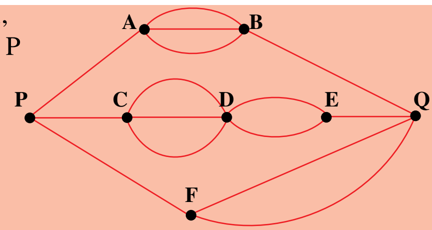
That is, there are 4 possible routes.

Then, for each possible road from A to B there are another 4 leading from B to C.

All in all, there are $4 + 4 + 4 = 3 \times 4 = 12$ different ways John can get from A to C via B.

EXAMPLE 14.2

Using the following street network, in how many different ways can a person get from point P to point Q if they can only move from left to right



Solution

In travelling from P to Q there are:

$3 = 1 \times 3 \times 1$ paths (along P to A to B to Q)
$6 = 1 \times 3 \times 2 \times 1$ paths (along P to C to D to E to Q)
$2 = 1 \times 2$ paths (along P to F to Q)

In total there are $3 + 6 + 2 = 11$ paths

EXAMPLE 14.3

A golfer has 3 drivers, 4 tees and 5 golf balls. In how many ways can the golfer take his first hit.

Solution

We think of this problem as follows:

The golfer has 3 possible drivers to use and so the first task can be carried out in 3 ways.
The golfer has 4 possible tees to use and so the second task can be carried out in 4 ways.
The golfer has 5 golf balls to use and so the third task can be carried out in 5 ways.

Using the multiplication principle, there are a total of $3 \times 4 \times 5 = 60$ ways to take the first hit.

14.1.3 PERMUTATIONS

Based on the definition given in §14.1.1 we have the following rule:

Rule 4: **Permutations**:
The total number of ways of **arranging** n objects, taking r at a time is given by

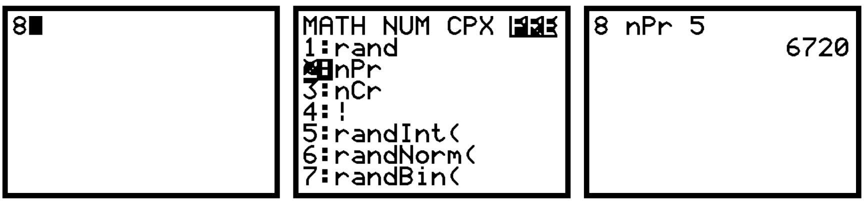
$$\frac{n!}{(n-r)!}$$

Notation: We use the notation ${}^{n}P_{r}$ (read as "n–p–r') to denote $\frac{n!}{(n-r)!}$.

That is, ${}^{n}P_{r} = \frac{n!}{(n-r)!}$.

For example, the total number of arrangements of 8 books on a bookshelf if only 5 are used is given by ${}^{8}P_{5} = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720$.

When using the TI–83, we can either use the same approach as in the previous example or use the **nPr** function:
Type the first number, **8**, then select **MATH** and **PRB**, then select option **2:nPr**, enter the second number, **5**, and then press **ENTER**:



EXAMPLE 14.4

In how many ways can 5 boys be arranged in a row
(a) using three boys at a time?
(b) using 5 boys at a time?

Solution

We have 5 boys to be arranged in a row with certain constraints.

(a) The constraint is that we can only use 3 boys at a time. In other words, we want the number of arrangements (permutations) of 5 objects taken 3 at a time.
From rule 4: $n = 5, r = 3,$

$$\text{Therefore, number of arrangements} = {}^5P_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$$

(b) This time we want the number of arrangements of 5 boys taking all 5 at a time.
From rule 4: $n = 5, r = 5,$

$$\text{Therefore, number of arrangements} = {}^5P_5 = \frac{5!}{(5-5)!} = \frac{120}{0!} = 120$$

Box method

Problems like Example 14.4 can be solved using a method known as "the box method". In that particular example, part (a) can be considered as filling three boxes (with only one object per box) using 5 objects:

Box 1	Box 2	Box 3

The first box can be filled in 5 different ways (as there are 5 possibilities available). Therefore we 'place 5' in box 1:

Box 1	Box 2	Box 3
5		

Now, as we have used up one of the objects (occupying box 1), we have 4 objects left that can be used to fill the second box. So, we 'place 4' in box 2:

Box 1	Box 2	Box 3
5	4	

At this stage we are left with three objects (as two of them have been used). Meaning that there are 3 possible ways in which the third box can be filled. So, we 'place 3' in box 3:

Box 1	Box 2	Box 3
5	4	3

This is equivalent to saying, that we can carry out the first task in 5 different ways, the second task in 4 different ways and the third task in 3 different ways. Therefore, using the multiplication principle we have that the total number of arrangements is $5 \times 4 \times 3 = 60$.

Comparing this to the expression ${}^5P_3 = \frac{5!}{(5-3)!}$ we have $\frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$

$$= 5 \times 4 \times 3$$
$$= 60$$

i.e., the last step in the evaluation process is the same as the step used in the 'box method'.

EXAMPLE 14.5

Vehicle licence plates consist of two letters from a 26–letter alphabet, followed by a three–digit number whose first digit cannot be zero. How many different licence plates can there be?

Solution

We have a situation where there are five positions to be filled:

Letter	Letter	Number	Number	Number

That is, the first position must be occupied by one of 26 letters, similarly, the second position must be occupied by one of 26 letters. The first number must be made up of one of nine different digits (as zero must be excluded), whilst the other two positions have 10 digits that can be used. Therefore, using Rule 2, we have:

Total number of arrangements = $26 \times 26 \times 9 \times 10 \times 10 = 608400$.

EXAMPLE 14.6

How many 5-digit numbers greater than 40 000 can be formed from the digits 0, 1, 2, 3, 4, and 5 if

(a) there is no repetition of digits allowed?

(b) repetition of digits is allowed?

Solution

(a) Consider the five boxes

Box 1	Box 2	Box 3	Box 4	Box 5

Only the digits 4 and 5 can occupy the first box (so as to obtain a number greater than 40 000). So there are 2 ways to fill box 1:

Box 1	Box 2	Box 3	Box 4	Box 5
2				

Box 2 can now be filled using any of the remaining 5 digits. So, there are 5 ways of filling box 2:

Box 1	Box 2	Box 3	Box 4	Box 5
2	5			

We now have 4 digits left to be used. So, there are 4 ways of filling box 3:

Box 1	Box 2	Box 3	Box 4	Box 5
2	5	4		

Continuing in this manner we have:

Box 1	Box 2	Box 3	Box 4	Box 5
2	5	4	3	2

Then, using the multiplication principle we have $2 \times 5 \times 4 \times 3 \times 2 = 240$ arrangements.

Otherwise, we could have relied on rule 4 and obtained $2 \times {}^5P_4 = 2 \times 120 = 240$

(b) As in part (a), only the digits 4 and 5 can occupy the first box.
If repetition is allowed, then boxes 2 to 5 can each be filled using any of the 6 digits:

Box 1	Box 2	Box 3	Box 4	Box 5
2	6	6	6	6

Using the multiplication principle there are $2 \times 6 \times 6 \times 6 \times 6 = 2592$ arrangements.

However, one of these arrangements will also include the number 40 000. Therefore, the number of 5 digit numbers greater than 40 000 (when repetition is allowed) is given by 2592 – 1 = 2591.

EXAMPLE 14.7

Find n if ${}^nP_3 = 60$.

Solution

$${}^nP_3 = 60 \Leftrightarrow \frac{n!}{(n-3)!} = 60$$

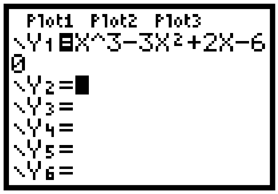
$$\Leftrightarrow \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 60$$

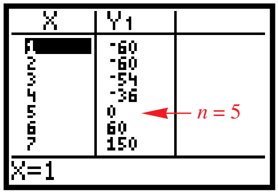
$$\Leftrightarrow n(n-1)(n-2) = 60$$

$$\Leftrightarrow n^3 - 3n^2 + 2n = 60$$

$$\Leftrightarrow n^3 - 3n^2 + 2n - 60 = 0$$

Using the TI–83 to solve this polynomial, we have:





i.e., $n = 5$.

EXAMPLE 14.8

How many different arrangements of the letters of the word HIPPOPOTAMUS are there?

Solution

The word ‘HIPPOPOTAMUS’ is made up of 12 letters, unfortunately, they are not all different! Meaning that although we can swap the three P’s with each other, the word will remain the same.

Now, the total number of times we can re–arrange the Ps (and not alter the word) is 3! = 6 times (as there are three Ps). Therefore, if we ‘blindly’ use Rule 2, we will have increased the number of arrangements 6 fold.

Therefore, we will need to divide the total number of ways of arranging 12 objects by 6.

That is, $\frac{12!}{3!} = 79833600$.

However, we also have 2 Os, and so, the same argument holds. So that in fact, we now have a total of $\frac{12!}{3! \times 2!} = 39916800$ arrangements.

This example is a special case of **permutations with repetitions**:

Rule 5: The number of permutations of n objects of which n_1 are identical, n_2 are identical, . . . , n_k are identical is given by $\frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$.

EXERCISES 14.1

1. A, B and C are three towns. There are 5 roads linking towns A and B and 3 roads linking towns B and C. How many different paths are there from town A to town C via town B?

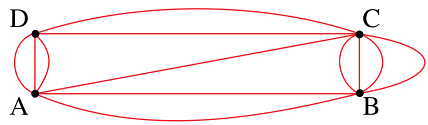
2. In how many ways can 5 letters be mailed if there are
(a) 2 mail boxes available?
(b) 4 mail boxes available?

3. There are 4 letters to be placed in 4 letter boxes. In how many ways can these letters be mailed if
(a) only one letter per box is allowed?
(b) there are no restrictions on the number of letters per box?

4. Consider the cubic polynomial $p(x) = ax^3 + bx^2 - 5x + c$.
(a) If the coefficients, a, b and c come from the set $\{-3, -1, 1, 3\}$, find the number of possible cubics if no repetitions are allowed.
(b) Find the number of cubics if the coefficients now come from $\{-3, -1, 0, 1, 3\}$ (again without repetitions).

5. The diagram alongside shows the possible routes linking towns A, B, C and D.

A person leaves town A for town C. How many different routes can be taken if the person is always heading towards town C



6. In how many different ways can Susan get dressed if she has 3 skirts, 5 blouses, 6 pairs of socks and 3 pairs of shoes to chose from?

7. In how many different ways can 5 different books be arranged on a shelf?

8. In how many ways can 8 different boxes be arranged taking 3 at a time?

9. How many different signals can be formed using 3 flags from 5 different flags.

10. Three Italian, two Chemistry and four Physics books are to be arranged on a shelf. In how many ways can this be done
(a) if there are no restrictions?
(b) if the Chemistry books must remain together?
(c) if the books must stay together by subject?

11. Find n if ${}^nP_2 = 380$.

12. 5 boys and 5 girls, which include a brother-sister, pair are to be arranged in a straight line. Find the number of possible arrangements if
(a) there are no restrictions.
(b) the tallest must be at one end and the shortest at the other end.
(c) the brother and sister must be i. together ii. separated