Although e has many important properties, it is after all just a number. Therefore, the standard rules of arithmetic and exponents still apply.

Exercise 2C

1. Find the values of the following to 3 significant figures.

   (a) (i) $e+1$ (ii) $e-4$

   (b) (i) $3e$ (ii) $\frac{e}{2}$

   (c) (i) e^2 (ii) e^{-3}

   (d) (i) $5e^{0.5}$ (ii) $\frac{3}{e^7}$

2. Evaluate $\sqrt[6]{(\pi^4+\pi^5)}$. What do you notice about the result?

3. Expand $\left(e^2+\frac{2}{e^2}\right)^2$. *[4 marks]*

2D Introduction to logarithms

In this section we shall look at an operation which reverses the effect of exponentiating (raising to a power) and allows us to find an unknown power. If you are asked to solve

$$x^2 = 3 \text{ for } x \geq 0$$

then you can can either find a decimal approximation (for example by using a calculator or trial and improvement) or use the square root symbol to write

$$x = \sqrt{3}$$

This statement just says that 'x is the positive value which when squared gives 3'.

Similarly, to solve

$$10^x = 50$$

we could use trial and improvement to seek a decimal value:

$$10^1 = 10$$

$$10^2 = 100$$

See the Supplementary sheet 2 'Logarithmic scales and log-log graphs' on the CD-ROM if you are interested in discovering logarithms for yourself.

So x must be between 1 and 2:

$$10^{1.5} = 31.6$$

$$10^{1.6} = 39.8$$

$$10^{1.7} = 50.1$$

So the answer is around 1.7.

Just as we can use the square root to answer the question 'what is the number which when squared gives this value?', there is also a function that can be used to answer the question 'what is the number which when put as the exponent of 10 gives this value?' This function is called a base-10 **logarithm**, written $\log_{10}$.

In the above example, we can write the solution as $x = \log_{10} 50$. More generally, the equation $y = 10^x$ can be re-expressed as $x = \log_{10} y$. In fact, the base need not be 10, but could be any positive value other than 1.

The symbol ⇔ means that if the left-hand side is true then so is the right-hand side, and if the right-hand side is true then so is the left-hand side. When it appears between two statements, it means that the statements are equivalent and you can switch between them.

KEY POINT 2.13

$$b = a^x \Leftrightarrow x = \log_a b$$

It is worth noting that the two most common bases have abbreviations for their logarithms. Since we use a decimal system of counting, 10 is the default base for a logarithm, so $\log_{10} x$ is usually written simply as $\log x$ and is called the 'common logarithm'. Also, the number e that we met in section 2C is considered the 'natural' base, so the base-e logarithm is called the 'natural logarithm' and is denoted by $\ln x$.

KEY POINT 2.14

$\log_{10} x$ is often written as $\log x$

$\log_e x$ is often written as $\ln x$

Since taking a logarithm reverses the process of exponentiating, we have the following facts:

KEY POINT 2.15

$\log_a(a^x) = x$

$a^{\log_a x} = x$

EXAM HINT

Log x and ln x have a button on graphical calculators ('log' and 'ln') that you can use to evaluate. If the base is not 10 or e however, you will have to use the principles of Key point 2.15. Or, use the change-of-base rule in Key point 2.22.

These are referred to as the cancellation principles. This sort of 'cancellation', similar to stating that (for positive x) $\sqrt[n]{x^n} = x = \left(\sqrt[n]{x}\right)^n$, is often useful when simplifying logarithm expressions; but remember that you can only do such cancellations when the base of the logarithm and the base of the exponential match and are immediately adjacent in the expression.

The cancellation principles can be combined with the rules of exponents to derive an interesting relationship between the base-e exponential function and any other exponential function. From the second cancellation principle it follows that $e^{\ln a} = a$. By raising both sides to the power x and using the rule of exponents $(b^y)^x = b^{yx}$ (Key point 2.5), we obtain the following useful formula.

A related change-of-base rule for logarithms is given in Key point 2.22.

When we study rates of change in chapter 12, we will need to use base e for exponential functions.

KEY POINT 2.16

$$e^{x\ln a} = a^x$$

This says that we can always change the base of an exponential function to e.

Worked example 2.7

Evaluate

(a) $\log_5 625$ (b) $\log_8 16$

Express the argument of the logarithm in exponent form with the same base.

(a) $\log_5 625 = \log_5 5^4$

Apply the cancellation principle $\log_a(a^x) = x$.

$= 4$

The argument of the logarithm, 16, is not a power of the base 8, but both 8 and 16 are powers of 2.

(b) $\log_8(16) = \log_8(2^4)$

continued . . .

Using a rule of exponents, convert 2^4 to an exponent of $8 = 2^3$.

$= \log_8\left(2^{3\times\frac{4}{3}}\right)$

$= \log_8\left(8^{\frac{4}{3}}\right)$

Apply the cancellation principle $\log_a(a^x) = x$.

$= \frac{4}{3}$

Whenever you raise a positive number to a power, whether positive or negative, the result is always positive. Therefore a question such as 'to what power do you raise 10 to get −3?' has no answer.

KEY POINT 2.17

You cannot take the logarithm of a negative number or zero.

Exercise 2D

1. Evaluate the following:

(a) (i) $\log_3 27$ (ii) $\log_4 16$

(b) (i) $\log_5 5$ (ii) $\log_3 3$

(c) (i) $\log_{12} 1$ (ii) $\log_{15} 1$

(d) (i) $\log_3 \frac{1}{3}$ (ii) $\log_4 \frac{1}{64}$

(e) (i) $\log_4 2$ (ii) $\log_{27} 3$

(f) (i) $\log_8 \sqrt{8}$ (ii) $\log_2 \sqrt{2}$

(g) (i) $\log_8 4$ (ii) $\log_{81} 27$

(h) (i) $\log_{25} 125$ (ii) $\log_{16} 32$

(i) (i) $\log_4 2\sqrt{2}$ (ii) $\log_9 81\sqrt{3}$

(j) (i) $\log_{25} 0.2$ (ii) $\log_4 0.5$

2. Use a calculator to evaluate each of the following, giving your answer correct to 3 significant figures.

(a) (i) $\log 50$ (ii) $\log\left(\frac{1}{4}\right)$

(b) (i) $\ln 0.1$ (ii) $\ln 10$

3. Simplify the following expressions:

(a) (i) $7\log x - 2\log x$ (ii) $2\log x + 3\log x$

(b) (i) $(\log x - 1)(\log y + 3)$ (ii) $(\log x + 2)^2$

(c) (i) $\dfrac{\log a + \log b}{\log a \log b}$ (ii) $\dfrac{(\log a)^2 - 1}{\log a - 1}$

See Prior Learning section O on the CD-ROM if you need to brush up on simplifying fractions.

4. Make x the subject of the following:

(a) (i) $\log_3 x = y$ (ii) $\log_4 x = 2y$

(b) (i) $\log_a x = 1 + y$ (ii) $\log_a x = y^2$

(c) (i) $\log_x 3y = 3$ (ii) $\log_x y = 2$

EXAM HINT

Remember that 'log x' is just another value so can be treated the same way as any variable.

5. Find the value of x in each of the following:

(a) (i) $\log_2 x = 32$ (ii) $\log_2 x = 4$

(b) (i) $\log_5 25 = 5x$ (ii) $\log_{49} 7 = 2x$

(c) (i) $\log_x 36 = 2$ (ii) $\log_x 10 = \dfrac{1}{2}$

6. Solve the equation $\log_{10}(9x+1) = 3$. *[4 marks]*

7. Solve the equation $\log_8 \sqrt{1-x} = \dfrac{1}{3}$. *[4 marks]*

8. Find the exact solution to the equation $\ln(3x-1) = 2$. *[5 marks]*

9. Find all values of x which satisfy $(\log_3 x)^2 = 4$. *[5 marks]*

10. Solve the equation $3(1 + \log x) = 6 + \log x$. *[5 marks]*

11. Solve the equation $\log_x 4 = 9$. *[4 marks]*

12. Solve the simultaneous equations

$$\log_3 x + \log_5 y = 6$$
$$\log_3 x - \log_5 y = 2$$

[6 marks]

13. The Richter scale is a way of measuring the strength of earthquakes. An increase of one unit on the Richter scale corresponds to an increase by a factor of 10 in the strength of the earthquake. What would be the Richter level of an earthquake which is twice as strong as a level 5.2 earthquake? *[5 marks]*

2E Laws of logarithms

Just as there are rules to follow when performing arithmetic with exponents, so there are corresponding rules which apply to logarithms.

See Fill-in proof 12 'Differentiating logarithmic functions graphically' on the CD-ROM for how these rules for logarithms can be derived from the laws of exponents.

KEY POINT 2.18

The logarithm of a *product* is the *sum* of the logarithms.

$$\log_a xy = \log_a x + \log_a y$$

For example, you can check that $\log_2 32 = \log_2 8 + \log_2 4$.

KEY POINT 2.19

The logarithm of a *quotient* is the *difference* of the logarithms.

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

For example, $\log 6 = \log 42 - \log 7$.

KEY POINT 2.20

The logarithm of an *exponent* is the *multiple* of the logarithm.

$$\log_a x^r = r \log_a x$$

For example, $\log_5 8 = \log_5 (2^3) = 3\log_5 2$.

A special case of this is the logarithm of a reciprocal:

$$\log x^{-1} = -\log x$$

We know that $a^0 = 1$ irrespective of a. We can express this in terms of logarithms:

KEY POINT 2.21

The logarithm of 1 is always 0, irrespective of the base.

$$\log_a 1 = 0$$

EXAM HINT

It is just as important to know what you *cannot* do with logarithms. One very common mistake is to rewrite $\log(x+y)$ as $\log x + \log y$ or as $\log x \times \log y$.

We can use the laws of logarithms to manipulate expressions and solve equations involving logarithms, as the next two examples illustrate.

Worked example 2.8

If $x = \log_{10} a$ and $y = \log_{10} b$, express $\log_{10} \frac{100a^2}{b}$ in terms of x, y and integers.

Use laws of logs to isolate $\log_{10} a$ and $\log_{10} b$ in the given expression. First, use the law about the logarithm of a fraction.

$$\log_{10} \frac{100a^2}{b} = \log_{10}(100a^2) - \log_{10} b$$

Then use the law about the log of a product.

$$= \log_{10} 100 + \log_{10} a^2 - \log_{10} b$$

For the second term use the law about the log of an exponent.

$$= \log_{10} 100 + 2\log_{10} a - \log_{10} b$$

Finally, simplify (by evaluating) $\log_{10} 100$.

$$= 2 + 2\log_{10} a - \log_{10} b$$
$$= 2 + 2x - y$$

Worked example 2.9

Solve the equation $\log_2 x + \log_2(x+4) = 5$.

Rewrite the left-hand side as a single logarithm.

$$\log_2 x + \log_2(x+4) = 5$$
$$\Leftrightarrow \quad \log_2(x(x+4)) = 5$$

Undo the logarithm by exponentiating both sides; the base must be the same as the base of the logarithm, 2.

$$2^{\log_2(x(x+4))} = 2^5$$

Apply the cancellation principle to the left-hand side.

$$x(x+4) = 32$$
$$x^2 + 4x = 32$$

This is a quadratic equation, which can be factorised.

$$x^2 + 4x - 32 = 0$$
$$(x+8)(x-4) = 0$$
$$x = -8 \text{ or } x = 4$$

Check the validity of the solutions by putting them into the original equation.

When $x = -8$:
LHS $= \log_2(-8) + \log_2(-4)$, and since we cannot take logarithms of negative numbers, this solution does not work.
When $x = 4$:
LHS $= \log_2 4 + \log_2 8$
$= 2 + 3 = 5 =$ RHS
So $x = 4$ is the only solution of the given equation.

EXAM HINT

Checking your answers is an important part of solving mathematical problems, and involves more than looking for arithmetic errors. As this example shows, false solutions may arise through correct algebraic manipulations.

Although we have discussed logarithms with a general base a, your calculator may only have buttons for the common logarithm and the natural logarithm ($\log x$ and $\ln x$). To use a calculator to evaluate a logarithm with base other than 10 or e (for example, $\log_5 20$), we use the following **change-of-base rule** for logarithms.

KEY POINT 2.22

Change-of-base rule for logarithms:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

So, we can calculate $\log_5 20$ using the logarithm with base 10 as follows:

$$\log_5 20 = \frac{\log 20}{\log 5} = 1.86 \text{ (3 SF)}$$

The change-of-base rule is useful for more than just evaluating logarithms.

Worked example 2.10

Solve the equation $\log_3 x + \log_9 x = 2$.

We want to have logarithms involving just one base, so use the change-of-base rule to turn the log with base 9 into a log with base 3.

$$\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{2}$$

Therefore the equation is

$$\log_3 x + \frac{\log_3 x}{2} = 2$$

Collect the logs together.

$$\Leftrightarrow \frac{3}{2}\log_3 x = 2$$

$$\Leftrightarrow \log_3 x = \frac{4}{3}$$

Exponentiate both sides with base 3.

Hence

$$x = 3^{\frac{4}{3}} = 4.33 \text{ (3SF)}$$

Exercise 2E

1. Given that $b>0$, simplify each of the following.
   (a) (i) $\log_b b^4$ (ii) $\log_b \sqrt{b}$
   (b) (i) $\log_{\sqrt{b}} b^3$ (ii) $\log_b b^2 - \log_{b^2} b$

2. If $x=\log a$, $y=\log b$ and $z=\log c$, express the following in terms of x, y and z.
   (a) (i) $\log(bc)$ (ii) $\log\left(\frac{c}{a}\right)$
   (b) (i) $\log a^3$ (ii) $\log b^5$
   (c) (i) $\log cb^7$ (ii) $\log a^2b$
   (d) (i) $\log\left(\frac{ab^2}{c}\right)$ (ii) $\log\left(\frac{a^2}{bc^3}\right)$
   (e) (i) $\log\left(\frac{100}{bc^5}\right)$ (ii) $\log(5b)+\log(2c^2)$

3. Solve the following equations for x.
   (a) (i) $\log_3\left(\frac{2+x}{2-x}\right)=3$ (ii) $\log_2(7x+4)=5$
   (b) (i) $\log_3 x-\log_3(x-6)=1$ (ii) $\log_8 x-2\log_8\left(\frac{1}{x}\right)=1$
   (c) (i) $\log_2 x+1=\log_4 x$ (ii) $\log_8 x+\log_2 x=4$
   (d) (i) $\log_4 x+\log_8 x=2$ (ii) $\log_{16} x-\log_{32} x=0.5$
   (e) (i) $\log_3(x-7)+\log_3(x+1)=2$
   (ii) $2\log(x-2)-\log(x)=0$
   (f) (i) $\log(x^2+1)=1+2\log(x)$
   (ii) $\log(3x+6)=\log(3)+1$

4. Find the exact solution to the equation $2\ln x+\ln 9=3$, giving your answer in the form Ae^B where A and B are rational numbers. *[5 marks]*

5. If $a=\ln 2$ and $b=\ln 5$, write the following in terms of a and b.
   (a) $\ln 50$
   (b) $\ln 0.16$ *[6 marks]*

6. Solve $\log_2 x=\log_x 2$. *[5 marks]*

7. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z.

(a) $\log a^3 - 2\log ab^2$

(b) $\log(4b) + 2\log(5ac)$ *[8 marks]*

8. Evaluate $\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \log\frac{4}{3} + \cdots + \log\frac{8}{9} + \log\frac{9}{10}$. *[4 marks]*

9. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z.

(a) $\log_a a^2 b$

(b) $\log_{ab} ac^2$ *[6 marks]*

10. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z.

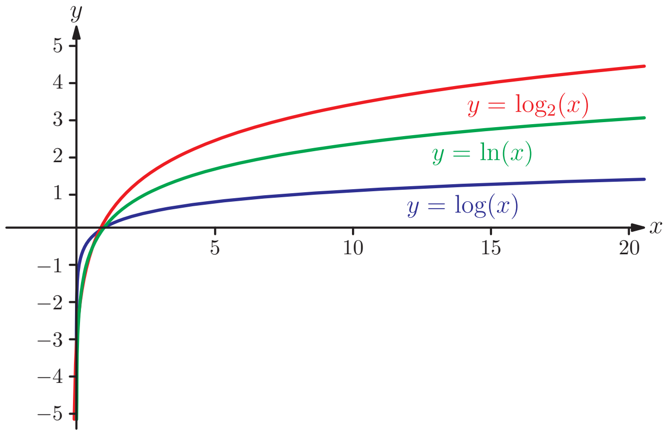
(a) $\log_b\left(\frac{a}{bc}\right)$

(b) $\log_{ab}(b^a)$ *[7 marks]*

2F Graphs of logarithms

Let us now look at the graph of the logarithm function and the various properties of logarithms that we can deduce from it. Here are the graphs of $y = \log x$, $y = \log_2 x$ and $y = \ln x$.

In chapter 5 you will see how this type of change in the function causes a vertical stretch of the graph.



Given the change-of-base rule from section E (Key point 2.22), it is not surprising that these curves all have a similar shape: since $\log_2 x = \frac{\log x}{\log 2}$ and $\ln x = \frac{\log x}{\log e}$, each of the logarithm functions is a multiple of the common logarithm function.