# 8.1 ARITHMETIC SEQUENCES & SERIES

## 8.1.1 ARITHMETIC SEQUENCES

A **sequence** is a set of quantities arranged in a definite order.

$1, 2, 3, 4, 5, 6, \ldots\ldots\ldots\ldots$ $\quad -1, 2, -4, 8, -16\ldots.1,$ $\quad 1, 2, 3, 5, 8, 13\ldots.$

are all examples of sequences. When the terms of a sequence are added, we obtain a series. Sequences and series are used to solve a variety of practical problems in, for example, business.

There are two major types of sequence, **arithmetic** and **geometric**. This section will consider arithmetic sequences (also known as arithmetic progressions, or simply A.P). The characteristic of such a sequence is that there is a common difference between successive terms. For example:

1, 3, 5, 7, 9, 11, . . . (the odd numbers) has a first term of 1 and a common difference of 2.
18, 15, 12, 9, 6, . . . has a first term of 18 and a common difference of –3 (sequence is decreasing).

The terms of a sequence are generally labelled $u_1, u_2, u_3, u_4, \ldots u_n$ . The '$n$th term' of a sequence is labelled $u_n$ . In the case of an arithmetic sequence which starts with $a$ and has a **common difference** of $d$, the ***n*th term** can be found using the formula:

$$\boxed{u_n = a + (n-1)d} \text{ where } d = u_2 - u_1 = u_3 - u_2 = \ldots$$

CHAPTER 8

**EXAMPLE 8.1** For the sequence 7, 11, 15, 19, . . . , find the 20th term.

**Solution**

In this case $a = 7$ and $d = 4$ because the sequence starts with a 7 and each term is 4 bigger than the one before it, i.e., $d = 11 - 7 = 4$. Therefore the $n$th term is given by

$$\boxed{u_n = 7 + (n-1) \times 4}$$

That is, $u_n = 4n + 3$

$\therefore u_{20} = 4 \times 20 + 3 = 83$ [$n = 20$ corresponds to the 20th term]

**EXAMPLE 8.2** An arithmetic sequence has a first term of 120 and a 10th term of 57. Find the 15th term.

**Solution**

The data is: $a = 120$ and when $n = 10$, $u_{10} = 57$ [i.e., 10th term is 57].

This gives, $u_{10} = 120 + (10-1)d = 57 \Leftrightarrow 120 + 9d = 57$

$$\therefore d = -7$$

Using $u_n = a + (n-1)d$ , we then have $u_n = 120 + (n-1) \times (-7) = 127 - 7n$ .

Therefore, when $n = 15$, $u_{15} = 127 - 7 \times 15 = 22$.

**EXAMPLE 8.3**

An arithmetic sequence has a 7th term of 16.5 and a 12th term of 24. Find the 24th term.

**Solution**

In this instance we know neither the first term nor the common difference and so we will need to set up equations to be solved simultaneously.

The data is:

$$u_7 = a + 6d = 16.5 \quad -(1)$$
$$u_{12} = a + 11d = 24 \quad -(2)$$

We first solve for '$d$': $(2) - (1)$: $5d = 7.5 \Leftrightarrow d = 1.5$

Substituting into (1): $a + 6 \times 1.5 = 16.5 \Leftrightarrow a = 7.5$

To find the 24th term we use the general term $u_n = a + (n-1)d$ with $n = 24$:

$$u_{24} = 7.5 + (24 - 1) \times 1.5 = 42$$

**EXAMPLE 8.4**

A car whose original value was \$25600 decreases in value by \$90 per month. How long will it take before the car's value falls below \$15000?

**Solution**

The values can be seen as a sequence: \$25600, \$25510, \$25420 etc.

In this case $a = 25600$ and $d = 25510 - 25600 = -90$ so that:

$$\begin{aligned} u_n &= 25600 + (n-1) \times (-90) \\ &= 25690 - 90n \\ \therefore 15000 &= 25690 - 90n \\ \Leftrightarrow 90n &= 25690 - 15000 \\ \Leftrightarrow n &= 118.777 \end{aligned}$$

The car will be worth less than \$15000 after 119 months

## Using a Graphics Calculator

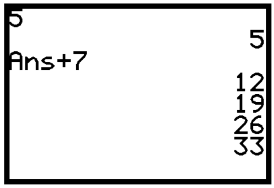
Most graphic calculators have an automatic memory facility (usually called **Ans**) that stores the result of the last calculation as well as an ability to remember the actual calculation. This can be very useful in listing a sequence.

**EXAMPLE 8.5**

List the arithmetic sequence 5, 12, 19, 26,......

**Solution**

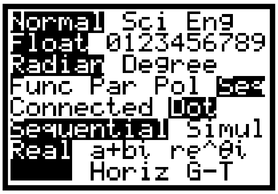
The sequence has a first term of 5. Enter this and press **ENTER** or **EXE**.
The common difference of the sequence is 7 so enter + 7. The display will show Ans + 7 which means 'add 7 to the previous answer.
From here, every time you press **ENTER** (or **EXE**), you will repeat the calculation, generating successive terms of the sequence.




However, the TI–83 is much more sophisticated than this. It is possible to set up a sequence rule on the TI–83. To do this we use the **MODE** key to switch to **Seq** mode and this changes the Equation editor screen from **Y=** to a sequence version (instead of the usual function form).

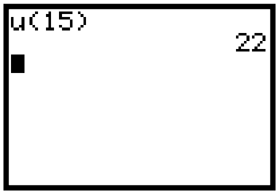
There are three sequence forms; $u(n)$, $v(n)$ and $w(n)$, which can be accessed on the home screen using the 2nd function key with 7, 8 and 9 respectively. Once these equations are defined we can plot their sequence graph.

We now consider Example 8.2, where we obtained the sequence $u_n = 127 - 7n$ and wished to determine the 15th term.



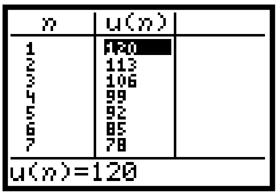
Setting into sequence mode

Define sequence equation

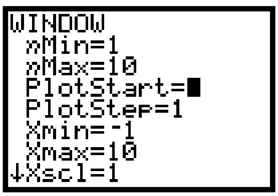


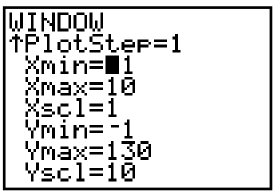
Use 2nd key '7' to call up $u$.

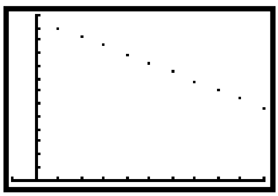
We can also use other features of the TI–83. For example, set up the sequence in a table format:


We can plot the sequence:



The TI–83 has many features that can be used with sequences. Become familiar with all of them.

## EXERCISES 8.1.1

**1.** (a) Show that the following sequences are arithmetic.
(b) Find the common difference.
(c) Define the rule that gives the $n$th term of the sequence.

i. $\{2, 6, 10, 14, \ldots\}$
ii. $\{20, 17, 14, 11, \ldots\}$
iii. $\{1, -4, -9, \ldots\}$
iv. $\{0.5, 1.0, 1.5, 2.0, \ldots\}$
v. $\{y+1, y+3, y+5, \ldots\}$
vi. $\{x+2, x, x-2, \ldots\}$

**2.** Find the 10th term of the sequence whose first four terms are $8, 4, 0, -4$.

**3.** Find the value of $x$ and $y$ in the arithmetic sequence $\{5, x, 13, y, \ldots\}$.

**4.** An arithmetic sequence has 12 as its first term and a common difference of $-5$. Find its 12th term.

**5.** An arithmetic sequence has $-20$ as its first term and a common difference of 3. Find its 10th term.

**6.** The 14th term of an arithmetic sequence is 100. If the first term is 9, find the common difference.

**7.** The 10th term of an arithmetic sequence is $-40$. If the first term is 5, find the common difference.

**8.** If $n+5$, $2n+1$ and $4n-3$ are three consecutive terms of an arithmetic sequences, find $n$.

**9.** The first three terms of an arithmetic sequence are 1, 6, 11.
(a) Find the 9th term.
(b) Which term will equal 151?

**10.** Find $x$ and $y$ given that $4-\sqrt{3}, x, y$ and $2\sqrt{3}-2$ are the first four terms of an arithmetic sequence.

**11.** For each of the following sequences
(a) $u_n = -5 + 2n, n \geq 1$. (b) $u_n = 3 + 4(n+1), n \geq 1$.
determine
i. its common difference
ii. its first term

**12.** The third and fifth terms of an A.P are $x+y$ and $x-y$ respectively. Find the 12th term.

**13.** The sum of the fifth term and twice the third of an arithmetic sequence equals the twelth term. If the seventh term is 25 find an expression for the general term, $u_n$.

**14.** For a given arithmetic sequence, $u_n = m$ and $u_m = n$. Find
(a) the common difference.
(b) $u_{n+m}$.

## 8.1.2 ARITHMETIC SERIES

If the **terms of a sequence are added**, the result is known as a **series**.

The gives the sequence: series: $1, 2, 3, 4, 5, 6, \ldots$
$1 + 2 + 3 + 4 + 5 + 6 + \ldots$

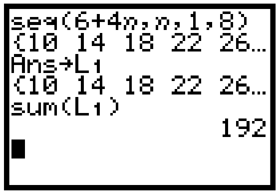
and the gives the sequence: series: $-1, -2, -4, -8, -16 \ldots$
$(-1) + (-2) + (-4) + (-8) + (-16) + \ldots$
[or $- 1 - 2 - 4 - 8 - 16 - \ldots$

The sum of the terms of a series is referred to as $S_n$, the **sum of *n* terms of a series**.
For an arithmetic series, we have

$$\begin{aligned} S_n &= u_1 + u_2 + u_3 + \ldots + u_n \\ &= a + (a+d) + (a+2d) + \ldots\ldots\ldots\ldots + a + (n-1)d \end{aligned}$$

For example, if we have a sequence defined by $u_n = 6 + 4n, n \geq 1$, then the sum of the first 8 terms is given by

$$\begin{aligned} S_8 &= u_1 + u_2 + u_3 + \ldots + u_8 \\ &= 10 + 14 + 18 + \ldots + 38 \\ &= 192 \end{aligned}$$




Again, the screen display of the TI–83 shows how readily we can obtain the sum. Once the sequence has been stored as a **List**, use the **sum(** operation to obtain the answer.

There will be many cases in which we can add the terms of a series in this way. If, however, there are a large number of terms to add, a formula is more appropriate.

There is a story that, when the mathematician Gauss was a child, his teacher was having problems with him because he always finished all his work long before the other students. In an attempt to keep Gauss occupied for a period, the teacher asked him to add all the whole numbers from 1 to 100.

'5050' Gauss replied immediately.

It is probable that Gauss used a method similar to this:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | ......,, | 96 | 97 | 98 | 99 | 100 |
| <u>100</u> | <u>99</u> | <u>98</u> | <u>97</u> | <u>96</u> | <u>95</u> | ......,, | <u>5</u> | <u>4</u> | <u>3</u> | <u>2</u> | <u>1</u> |
| 101 | 101 | 101 | 101 | 101 | 101 | ......,, | 101 | 101 | 101 | 101 | 101 |

Adding each of the pairings gives 100 totals of 101 each. This gives a total of 10100. This is the sum of two sets of the numbers $1 + 2 + 3 + \ldots + 98 + 99 + 100$ and so dividing the full answer by 2 gives the answer 5050, as the young Gauss said, 5050.

It is then possible to apply the same approach to such a sequence, bearing in mind that the sequence of numbers must be arithmetic.

Applying this process to the general arithmetic series we have:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$ | $a+d$ | $a+2d$ | ... | $a+(n-3)d$ | $a+(n-2)d$ | $a+(n-1)d$ |
| $a+(n-1)d$ | $a+(n-2)d$ | $a+(n-3)d$ | ... | $a+2d$ | $a+d$ | $a$ |

Each of the pairings comes to the same total.

Here are some examples:

1st pairing: $a+[a+(n-1)d] = 2a+(n-1)d$

2nd pairing: $(a+d)+[a+(n-2)d] = 2a+(n-1)d$

3rd pairing: $(a+2d)+[a+(n-3)d] = 2a+(n-1)d$

$\vdots$

There are $n$ such pairings so $S_n + S_n = n \times [2a+(n-1)d]$

That is, $2S_n = n[2a+(n-1)d]$

Giving the formula, for the sum of $n$ terms of a sequence

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

This formula can now be used to sum large arithmetic series:

**EXAMPLE 8.6**

Find the sum of 20 terms of the series –2 + 1 + 4 + 7 + 10 +. . .

**Solution**

We have the following information: $a = u_1 = -2$ and $d = u_2 - u_1 = 1-(-2) = 3$.

Then, the sum to $n$ terms is given by $S_n = \frac{n}{2}[2a+(n-1)d]$

So that the sum to 20 terms is given by

$$S_{20} = \frac{20}{2}[2\times(-2)+(20-1)\times 3]$$
$$= 10[-4+19\times 3]$$
$$= 530$$

**EXAMPLE 8.7**

Find the sum of 35 terms of the series $-\frac{3}{8}-\frac{1}{8}+\frac{1}{8}+\frac{3}{8}+\frac{5}{8}+\ldots\ldots$

**Solution**

We have the following information: $a = u_1 = -\frac{3}{8}$ and $d = u_2 - u_1 = -\frac{1}{8}-\left(-\frac{3}{8}\right) = \frac{1}{4}$.

Then, with $n = 35$ we have $S_{35} = \frac{35}{2}\left[2\times-\frac{3}{8}+(35-1)\frac{1}{4}\right] = 17.5\left[-\frac{3}{4}+34\times\frac{1}{4}\right]$

$$= 135\frac{5}{8}$$

## EXAMPLE 8.8

An arithmetic series has a third term of 0. The sum of the first 15 terms is –300. What is the first term and the sum of the first ten terms?

### Solution

From the given information we have: $u_3 = a + 2d = 0$ – (1)

&: $S_{15} = \frac{15}{2}[2a + 14d] = -300$

i.e., $15a + 105d = -300$

$\therefore a + 7d = -20$ – (2)

The pair of equations can now be solved simultaneously:

(2) – (1): $5d = -20 \Leftrightarrow d = -4$

Substituting result into (1) we have : $a + 2 \times -4 = 0 \Leftrightarrow a = 8$

This establishes that the series is 8 + 4 + 0 + (–4) + (–8) +. . .

So the first term is 8 and the sum of the first ten terms is $S_{10} = \frac{10}{2}[16 + 9 \times -4] = -100$ .

Using the TI–83 we have, using the general term $u_n = 8 + (n-1) \times -4 = 12 - 4n$

```
seq(12-4n,n,1,10
)
{8 4 0 -4 -8 -1…
Ans→L1
{8 4 0 -4 -8 -1…
sum(L1)
                 -100
```

## EXAMPLE 8.9

A new business is selling home computers. They predict that they will sell 20 computers in their first month, 23 in the second month, 26 in the third and so on, in arithmetic sequence. How many months will pass before the company expects to sell their thousandth computer.

### Solution

The series is: 20 + 23 + 26 + .....

The question implies that the company is looking at the **total** number of computers sold, so we are looking at a series, not a sequence.

The question asks how many terms (months) will be needed before the total sales reach more than 1000. From the given information we have: $a = 20$, $d = 23 - 20 = 3$.

Therefore, we have the sum to $n$ terms given by $S_n = \frac{n}{2}[2 \times 20 + (n-1) \times 3]$

$$= \frac{n}{2}[3n + 37]$$

Next, we determine when $S_n = 1000$ : $\frac{n}{2}[3n + 37] = 1000 \Leftrightarrow 3n^2 + 37n = 2000$

$$\Leftrightarrow 3n^2 + 37n - 2000 = 0$$

We solve for $n$ use either of the following methods:

Method 1: Quadratic formula

$$n = \frac{-37 \pm \sqrt{37^2 - 4 \times 3 \times -2000}}{2 \times 3}$$

$$= 20.37 \text{ or } (-32.7)$$

Method 2: Graphics Calculator **Solve** function

```
solve(3X²+37X-20
00,X,20)
         20.37941486
```

Method 3: Table of values


Notice that we have entered the expression for $S_n$ as the sequence rule for $u(n)$. In fact, the series itself is made up of terms in a sequence of so-called **partial sums**, often called a **sum sequence**. That is, we have that $\{S_1, S_2, S_3, \ldots\} = \{15, 33, 54, \ldots\}$ forms a sequence.

The answer then, is that the company will sell its thousandth computer during the 20th month.

## EXERCISES 8.1.2

**1.** Find the sum of the first ten terms in the arithmetic sequences
(a) $\{1, 4, 7, 10, \ldots\}$ (b) $\{3, 9, 15, 21, \ldots\}$ (c) $\{10, 4, -2, \ldots\}$.

**2.** For the given arithmetic sequences, find the sum, $S_n$, to the requested number of terms.
(a) $\{4, 3, 2, \ldots\}$ for $n = 12$
(b) $\{4, 10, 16, \ldots\}$ for $n = 15$
(c) $\{2.9, 3.6, 4.3, \ldots\}$ for $n = 11$

**3.** Find the sum of the following sequences:
(a) $\{5, 4, 3, \ldots, -15\}$
(b) $\{3, 9, 15, \ldots, 75\}$
(c) $\{3, 5, 7, \ldots, 29\}$

**4.** The weekly sales of washing machines from a retail store that has just opened in a new housing complex increases by 2 machines per week. In the first week of January 1995, 24 machines were sold.
(a) How many are sold in the last week of December 1995?
(b) How many machines did the retailer sell in 1995?
(c) When was the 500th machine sold?

**5.** The fourth term of an arithmetic sequence is 5 while the sum of the first 6 terms is 10. Find the sum of the first nineteen terms.

**6.** Find the sum of the first 10 terms for the sequences defined by
(a) $u_n = -2 + 8n$ (b) $u_n = 1 - 4n$

**7.** The sum of the first eight terms of the sequence $\{\ln x, \ln x^2y, \ln x^3y^2, \ldots\}$ is given by $4(a\ln x + b\ln y)$. Find $a$ and $b$.

## 8.1.3 SIGMA NOTATION

There is a second notation to denote the sum of terms. This other notation makes use the Greek letter $\sum \ldots$ as the symbol to inform us that we are carrying out a summation.
In short, $\sum \ldots$ stands for '**The sum of . . .**'.

This means that the expression $\sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$.

For example, if $u_i = 2 + 5(i-1)$, i.e., an A.P with first term $a = 2$ and common difference $d = 5$, the expression $S_n = \sum_{i=1}^{n} [2 + 5(i-1)]$ would respesent the sum of the first $n$ terms of the sequence. So, the sum of the first 3 terms would be given by

$$S_3 = \sum_{i=1}^{3} [2+5(i-1)] = \underbrace{[2+5(1-1)]}_{i=1} + \underbrace{[2+5(2-1)]}_{i=2} + \underbrace{[2+5(3-1)]}_{i=3}$$
$$= 2 + 7 + 12$$
$$= 21$$

## Properties of $\sum$

**1.** $\sum$ is distributive. That is, $\sum_{i=1}^{n} [u_i + v_i] = \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} v_i$.

**2.** $\sum_{i=1}^{n} k u_i = k \sum_{i=1}^{n} u_i$, for some constant value $k$.

**3.** $\sum_{i=1}^{n} k = kn$, i.e., adding a constant term, $k$, $n$ times is the same as multiplying $k$ by $n$.

**EXAMPLE 8.10**

Given that $u_i = 5 + 2i$ and that $v_i = 2 - 5i$ find

(a) $\sum_{i=1}^{5} u_i$ (b) $\sum_{i=1}^{5} [2u_i - v_i]$ (c) $\sum_{i=1}^{1000} [5u_i + 2v_i]$

**Solution**

(a) $\sum_{i=1}^{5} u_i = u_1 + u_2 + u_3 + u_4 + u_5 = [5+2] + [5+4] + [5+6] + [5+8] + [5+10]$
$= 7 + 9 + 11 + 13 + 15$
$= 55$

(b) $\sum_{i=1}^{5} [2u_i - v_i] = \sum_{i=1}^{5} (2u_i) + \sum_{i=1}^{5} (-v_i) = 2 \sum_{i=1}^{5} u_i - \sum_{i=1}^{5} v_i$

Now, $2\sum_{i=1}^{5} u_i = 2 \times 55 = 110$

and $\sum_{i=1}^{5} v_i = \sum_{i=1}^{5}(2-5i) = \sum_{i=1}^{5}(2) - 5\sum_{i=1}^{5} i = 2 \times 5 - 5[1+2+3+4+5]$ [Using properties]

$= -65$

Therefore, $\sum_{i=1}^{5}[2u_i - v_i] = 110 - (-65) = 175$

(c) $\sum_{i=1}^{1000}[5u_i + 2v_i] = \sum_{i=1}^{1000}[5(5+2i) + 2(2-5i)]$

$= \sum_{i=1}^{1000}[25 + 10i + 4 - 10i]$

$= \sum_{i=1}^{1000} 29$

$= 29\,000$ [ i.e., $29 \times 1000$]

In this example we have tried to show that there are a number of ways to obtain a sum. It is not always necessary to enumerate every term and then add them. Often, an expression can first be simplified.

## EXERCISES 8.1.3 – MISCELLANEOUS QUESTIONS

**1.** Find the twentieth term in the sequence 9, 15, 21, 27, 33, . . .

**2.** Fill the gaps in this arithmetic sequence: –3, _, _, _, _, _, 12.

**3.** An arithmetic sequence has a tenth term of 17 and a fourteenth term of 30. Find the common difference.

**4.** If $u_{59} = \frac{1}{10}$ and $u_{100} = -1\frac{19}{20}$ for an arithmetic sequence, find the first term and the common difference.

**5.** Find the sum of the first one hundred odd numbers.

**6.** An arithmetic series has twenty terms. The first term is –50 and the last term is 83, find the sum of the series.

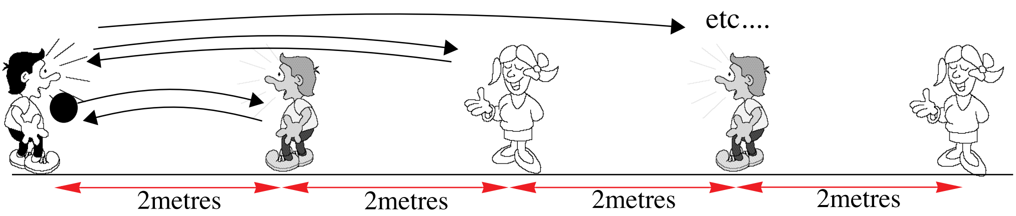
**7.** Thirty numbers are in arithmetic sequence. The sum of the numbers is 270 and the last number is 38. What is the first number?

**8.** How many terms of the arithmetic sequence: 2, 2.3, 2.6, 2.9, . . . must be taken before the terms exceed 100?

**9.** Brian and Melissa save \$50 in the first week of a savings program, \$55 in the second week, \$60 in the third and so on, in arithmetic progression. How much will they save in ten weeks? How long will they have to continue saving if their target is to save \$5000?

**10.** A printing firm offers to print business cards on the following terms:
\$45 for design and typesetting and then \$0.02 per card.
(i) What is the cost of 500 cards from this printer?
(ii) How many cards can a customer with \$100 afford to order?

**11.** A children's game consists of the players standing in a line with a gap of 2 metres between each. The child at the left hand end of the line has a ball which s/he throws to the next child in the line, a distance of 2 metres. The ball is then thrown back to the first child who then throws the ball to the third child in the line, a distance of 4 metres. The ball is then returned to the first child, and so on until all the children have touched the ball at least once.




(a) If a total of five children play and they make the least number of throws so that only the leftmost child touches the ball more than once:
i. What is the largest single throw?
ii. What is the total distance travelled by the ball?
(b) If seven children play, what is the total distance travelled by the ball?
(c) If $n$ children play, derive a formula for the total distance travelled by the ball.
(d) Find the least number of children who need to play the game before the total distance travelled by the ball exceeds 100 metres.
(e) The children can all throw the ball 50 metres at most
i. What is the largest number of children that can play the game?
ii. What is the total distance travelled by the ball?

**12.** Find each sum,

(a) $\sum_{k=1}^{100} k$ (b) $\sum_{k=1}^{100} (2k+1)$ (c) $\sum_{k=1}^{51} (3k+5)$

**13.** If $u_i = -3+4i$ and $v_i = 12-3i$ find

(a) $\sum_{i=1}^{10} (u_i+v_i)$ (b) $\sum_{i=1}^{10} (3u_i+4v_i)$ (c) $\sum_{i=1}^{10} u_i v_i$

**14.** (a) Show that for an arithmetic sequence, $u_n = S_n - S_{n-1}$, where $u_n$ is the $n$th term and $S_n$ is the sum of the first $n$ terms.

(b) Find the general term, $u_n$, of the A.P given that $\sum_{i=1}^{n} u_i = \frac{n}{2}(3n-1)$.

# 8.2 GEOMETRIC SEQUENCES & SERIES

## 8.2.1 GEOMETRIC SEQUENCES

Sequences such as 2, 6, 18, 54, 162, . . . and 200, 20, 2, 0.2, . . . in which each term is obtained by multiplying the previous one by a fixed quantity are known as **geometric sequences**.

The sequence: 2, 6, 18, 54, 162, . . . is formed by starting with 2 and then multiplying by 3 to get the second term, by 3 again to get the third term, and so on.

For the sequence 200, 20, 2, 0.2, . . . , begin with 20 and multiplied by 0.1 to get the second term, by 0.1 again to get the third term and so on.

The constant multiplier of such a sequence is known as the **common ratio**.

The common ratio of 2, 6, 18, 54, 162,.... is 3 and of 200, 20, 2, 0.2,...... it is 0.1.

The ***n*th term of a geometric sequence** is obtained from the first term by multiplying by $n$–1 common ratios. This leads to the formula for the

$n$th term of a geometric sequence: $u_n = a \times r^{n-1}$ where $r = \frac{u_2}{u_1} = \ldots = \frac{u_n}{u_{n-1}}$

and $n$ is the term number, $a$ the first term and $r$ is the common ratio.

**EXAMPLE 8.11**

Find the tenth term in the sequence 2, 6, 18, 54, 162, . . .

**Solution**

The first term is $a = 2$. The common ratio $r = 3 = \frac{6}{2} = \frac{18}{6}$ and $n$, the required term, is 10.

Use the formula to solve the problem:

$$\begin{aligned} u_n &= a \times r^{n-1} \\ u_{10} &= 2 \times 3^{(10-1)} \\ &= 2 \times 3^9 \\ &= 39366 \end{aligned}$$

**EXAMPLE 8.12**

Find the fifteenth term in the sequence 200, 20, 2, 0.2, . . .

**Solution**

In this case, $a = 20$, $r = \frac{20}{200} = \frac{1}{10} = 0.1$ and $n = 15$.

Using the general term $u_n = a \times r^{n-1}$, the 15th term is given by

$$\begin{aligned} u_{15} &= 200 \times 0.1^{(15-1)} \\ &= 200 \times 0.1^{14} \\ &= 2 \times 10^{-12} \end{aligned}$$

**EXAMPLE 8.13**

Find the eleventh term in the sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

**Solution**

The sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$ has a common ratio of $r = \frac{-1/2}{1} = -\frac{1}{2}$.

Using the general term $u_n = a \times r^{n-1}$, we have $u_{11} = 1 \times \left(-\frac{1}{2}\right)^{(11-1)}$

$$= \left(-\frac{1}{2}\right)^{10}$$

$$\approx 0.000977$$

Many questions will be more demanding in terms of the way in which you use this formula. You should also recognise that the formula can be applied to a range of practical problems.

Many of the practical problems involve growth and decay and can be seen as similar to problems studied in Chapter 7.

**EXAMPLE 8.14**

A geometric sequence has a fifth term of 3 and a seventh term of 0.75. Find the first term, the common ratio and the tenth term.

**Solution**

From the given information we can set up the following equations:

$$u_5 = a \times r^4 = 3 \quad \text{– (1)}$$

$$\& \qquad u_7 = a \times r^6 = 0.75 \quad \text{– (2)}$$

As with similar problems involving arithmetic sequences, the result is a pair of simultaneous equations. In this case these can best be solved by dividing (2) by (1) to get:

$$\frac{a \times r^6}{a \times r^4} = \frac{0.75}{3} \Leftrightarrow r^2 = 0.25 \Leftrightarrow r = \pm 0.5$$

Substituting results into (1) we have: $\quad a\left(\pm\frac{1}{2}\right)^4 = 3 \Leftrightarrow a = 48$

Therefore, the 10th term is given by $\quad u_{10} = 48 \times (\pm 0.5)^9 = \pm\frac{3}{32}$

There are two solutions: 48, 24, 12, 6, . . . (for the case $r = 0.5$) & 48, –24, 12, –6, . . . ($r = -0.5$).

**EXAMPLE 8.15**

Find the number of terms in the geometric sequence:
0.25, 0.75, 2.25, . . . , 44286.75.

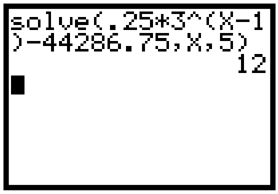
**Solution**

The sequence $0.25, 0.75, 2.25, \ldots, 44286.75$ has a first term $a = 0.25$ and a common ratio $r = \frac{0.75}{0.25} = 3$. In this problem it is $n$ that is unknown. Substitution of the data into the formula gives: $u_n = 0.25 \times 3^{(n-1)} = 44286.75$

The equation that results can be solved using logarithms (see Chapter 7).

$$\begin{aligned} 0.25 \times 3^{(n-1)} &= 44286.75 \\ \therefore 3^{(n-1)} &= 177147 \\ \Leftrightarrow \log_{10} 3^{(n-1)} &= \log_{10} 177147 \\ \Leftrightarrow (n-1)\log_{10} 3 &= \log_{10} 177147 \\ \Leftrightarrow n - 1 &= \frac{\log_{10} 177147}{\log_{10} 3} \\ \therefore n - 1 &= 11 \\ \Leftrightarrow n &= 12 \end{aligned}$$

Or, by making use of the TI–83




## EXAMPLE 8.16

A car originally worth \$34000 loses 15% of its value each year.

(a) Write a geometric sequence that gives the year by year value of the car.
(b) Find the value of the car after 6 years.
(c) After how many years will the value of the car fall below \$10000?

**Solution**

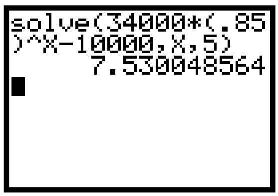
(a) If the car loses 15% of its value each year, its value will fall to 85% (100% – 15%) of its value in the previous year. This means that the common ratio is 0.85 (the fractional equivalent of 85%). Using the formula, the sequence is: $u_n = 34000 \times 0.85^{(n-1)}$
i.e. \$34000, \$28900, \$24565, \$20880.25, . . .

(b) The value after six years have passed is the **seventh** term of the sequence. This is because the first term of the sequence is the value after **no** years have passed.
$u_7 = 34000 \times 0.85^6 \approx 12823$ or \$12823.

(c) This requires solution of the equation $10000 = 34000 \times 0.85^n$:

$$\begin{aligned} 10000 &= 34000 \times 0.85^n \\ 0.85^n &= 0.2941 \\ \log_{10}(0.85^n) &= \log_{10} 0.2941 \\ n\log_{10} 0.85 &= \log_{10} 0.2941 \\ n - 1 &= \frac{\log_{10} 0.2941}{\log_{10} 0.85} \\ n &\approx 7.53 \end{aligned}$$

Or, by making use of the TI–83




This means that the car's value will fall to \$10000 after about 7 years 6 months.

**EXAMPLE 8.17** The number of people in a small country town increases by 2% per year. If the population at the start of 1970 was 12500, what is the predicted population at the start of the year 2010?

**Solution**

A quantity can be increased by 2% by multiplying by 1.02. Note that this is different from finding 2% of a quantity which is done by multiplying by 0.02.

The sequence is: 12500, 12500 × 1.02, 12500 × $1.02^2$ etc. with $a = 12500$, $r = 1.02$.

It is also necessary to be careful about which term is required. In this case, the population at the start of 1970 is the first term, the population at the start of 1971 the second term, and so on. The population at the start of 1980 is the **eleventh** term and at the start of 2010 we need the forty-first term:

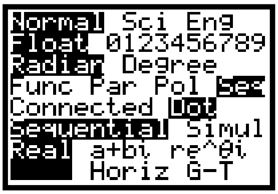
$$u_{41} = 12500 \times 1.02^{40}$$
$$\approx 27600$$

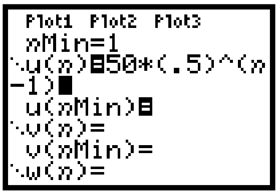
In all such cases, you should round your answer to the level given in the question or, if no such direction is given, round the answer to a reasonable level of accuracy. In this question, the original population appears to have been given to the nearest 100 and so it is hardly reasonable to give a higher level of accuracy in the answer.

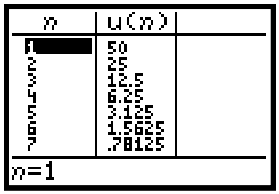
## Using a Graphics Calculator

As with arithmetic sequences, geometric sequences such as 50, 25, 12.5, . . . can be listed using a graphics calculator. For this sequence we have $a = 50$ and $r = 0.5$, so, $u_n = 50(0.5)^{n-1}$

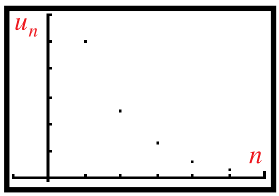
We first set the **MODE** to **Seq** and then enter the sequence rule:




Plot of sequence:




**EXERCISES 8.2.1**

**1.** Find the common ratio, the 5th term and the general term of the following sequences

(a) 3, 6, 12, 24, . . .  (b) $3, 1, \frac{1}{3}, \frac{1}{9}, \ldots$  (c) $2, \frac{2}{5}, \frac{2}{25}, \frac{2}{125}, \ldots$

(d) −1, 4, −16, 64, …  (e) $ab, a, \frac{a}{b}, \frac{a}{b^2}, \ldots$  (f) $a^2, ab, b^2, \ldots$

**2.** Find the value(s) of $x$ if each of the following are in geometric sequence

(a) $3, x, 48$

(b) $\frac{5}{2}, x, \frac{1}{2}$

**3.** The third and seventh terms of a geometric sequence are $\frac{3}{4}$ and 12 respectively.

(a) Find the 10th term.

(b) What term is equal to 3072?

**4.** A rubber ball is dropped from a height of 10 m and bounces to reach $\frac{5}{6}$ of its previous height after each rebound. Let $u_n$ is the ball's maximum height **before** its $n$th rebound.

(a) Find an expression for $u_n$ .

(b) How high will the ball bounce **after** its 5th rebound.

(c) How many times has the ball bounced by the time it reaches a maximum height of $\frac{6250}{1296}$ m.

**5.** The terms $k + 4, 5k + 4, k + 20$ are in a geometric sequence. Find the value(s) of $k$.

**6.** A computer depreciates each year to 80% of its value from the previous year. When bought the computer was worth $8000.

(a) Find its value after

i. 3 years

ii. 6 years

(b) How long does it take for the computer to depreciate to a quarter of its purchase price.

**7.** The sum of the first and third terms of a geometric sequence is 40 while the sum of its second and fourth terms is 96. Find the sixth term of the sequence.

**8.** The sum of three successive terms of a geometric sequence is $\frac{35}{2}$ while their product is 125. Find the three terms.

**9.** The population in a town of 40,000 increases at 3% per annum. Estimate the town's population after 10 years.

**10.** Following new government funding it is expected that the unemployed workforce will decrease by 1.2% per month. Initially there are 120,000 people unemployed. How large an unemployed workforce can the government expect to report in 8 months time.

**11.** The cost of erecting the ground floor of a building is $44,000, for erecting the first floor it costs $46,200, to erect the second floor costs $48,510 and so on.
Using this cost structure

(a) How much will it cost to erect the 5th floor?

(b) What will be to total cost of erecting a building with six floors?

## 8.2.2 GEOMETRIC SERIES

When the terms of a geometric sequence are added, the result is a **geometric series**.
For example:
The sequence 3, 6, 12, 24, 48, . . . gives rise to the series: 3 + 6 + 12 + 24 + 48 + . . .

and, the sequence $24, -16, 10\frac{2}{3}, -7\frac{1}{9}, \ldots$ leads to the series $24 - 16 + 10\frac{2}{3} - 7\frac{1}{9} + \ldots$

Geometric series can be summed using the formula that is derived by first multiplying the series by $r$:

$$\begin{aligned}
S_n &= a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \\
r \times S_n &= \quad ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n \\
S_n - r \times S_n &= a - ar^n \quad \text{(subtracting the second equation from the first)} \\
S_n(1-r) &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

This formula can also be written as: $S_n = \dfrac{a(r^n-1)}{r-1}$, $r \neq 1$ . It is usual to use the version of the formula that gives a positive value for the denominator. And so, we have:

> The sum of the first $n$ terms of a geometric series, $S_n$ , where $r \neq 1$ is given by
>
> $S_n = \dfrac{a(1-r^n)}{1-r}, |r| < 1$ or $S_n = \dfrac{a(r^n-1)}{r-1}, |r| > 1$

**EXAMPLE 8.18**

Sum the following series to the number of terms indicated.

(a) $2 + 4 + 8 + 16 + \ldots$ 9 terms.

(b) $5 - 15 + 45 - 135 + \ldots$ 7 terms.

(c) $24 + 18 + \frac{27}{2} + \frac{81}{8} + \ldots$ 12 terms.

(d) $20 - 30 + 45 - 67.5 + \ldots$ 10 terms.

**Solution**

(a) In this case $a = 2$, $r = 2$ and $n = 9$.
Because $r = 2$ it is more convenient to use:

$$\begin{aligned}
S_n &= \frac{a(r^n-1)}{r-1} \\
S_9 &= \frac{2(2^9-1)}{2-1} \\
&= 1022
\end{aligned}$$

Using this version of the formula gives positive values for the numerator and denominator. The other version is correct but gives negative numerator and denominator and hence the same answer.

(b) $a = 5, r = -3$ and $n = 7$.

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \text{or} \qquad S_n = \frac{a(r^n-1)}{r-1}$$

$$S_7 = \frac{5(1-(-3)^7)}{1-(-3)} \qquad \text{or} \qquad S_7 = \frac{5((-3)^7-1)}{(-3)-1}$$

$$= 2735 \qquad\qquad = 2735$$

(c) $a = 24, r = 0.75$ and $n = 12$.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{12} = \frac{24\left(1-\left(\frac{3}{4}\right)^{12}\right)}{1-\left(\frac{3}{4}\right)}$$ This version gives the positive values.

$$= 92.95907$$

(d) $a = 20, r = -1.5$ and $n = 10$.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{20(1-(-1.5)^{10})}{1-(-1.5)}$$

$$= -453.32031$$

When using a calculator to evaluate such expressions, it is advisable to use brackets to ensure that correct answers are obtained. For both the graphics and scientific calculator, the negative common ratio must be entered using the +/– or (–) key.

```
20(1-(-1.5)^10)/
(1-(-1.5))
        -453.3203125
```

Other questions that may be asked in examinations could involve using both formulas. A second possibility is that you may be asked to apply sequence and series theory to some simple problems.

**EXAMPLE 8.19**

The 2nd term of a geometric series is –30 and the sum of the first two terms is –15. Find the first term and the common ratio.

**Solution**

From the given information we have: $u_2 = -30 \therefore ar = -30$ – (1)

$$S_2 = -15 \therefore \frac{a(r^2-1)}{r-1} = -15 \quad \text{– (2)}$$

The result is a pair of simultaneous equations in the two unknowns. The best method of solution is substitution:

From (1): $a = \dfrac{-30}{r}$. Substituting into (2): $\dfrac{\frac{-30}{r}(r^2-1)}{r-1} = -15 \Leftrightarrow \dfrac{(-30)(r^2-1)}{r(r-1)} = -15$

$$\therefore \frac{-30(r+1)\cancel{(r-1)}}{r\cancel{(r-1)}} = -15$$
$$\Leftrightarrow -30(r+1) = -15r$$
$$\Leftrightarrow -30r - 30 = -15r$$
$$\Leftrightarrow r = -2$$
$$\therefore a = \frac{-30}{r} = \frac{-30}{-2} = 15$$

The series is 15 – 30 + 60 – 120 + 240 – . . . which meets the conditions set out in the question.

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**EXAMPLE 8.20**

A family decide to save some money in an account that pays 9% annual compound interest calculated at the end of each year. They put \$2500 into the account at the beginning of each year. All interest is added to the account and no withdrawals are made. How much money will they have in the account on the day after they have made their tenth payment?

**Solution**

The problem is best looked at from the last payment of \$2500 which has just been made and which has not earned any interest.

The previous payment has earned one lot of 9% interest and so is now worth $2500 \times 1.09$.

The previous payment has earned two years' worth of compound interest and is worth $2500 \times 1.09^2$.

This process can be continued for all the other payments and the various amounts of interest that each has earned. They form a geometric series:

Last payment First payment

$$2500 + 2500 \times 1.09 + 2500 \times 1.09^2 + \ldots\ldots + 2500 \times 1.09^9$$

The total amount saved can be calculated using the series formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{2500(1.09^{10} - 1)}{1.09 - 1}$$
$$= 37982.32$$

The family will save about \$37,982.

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## EXERCISES 8.2.2

**1.** Find the common ratios of these geometric sequences:

(a) $7, 21, 63, 189, \ldots$ (b) $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$ (c) $1, -1, 1, -1, 1, \ldots$

(d) $9, -3, 1, -\frac{1}{3}, \frac{1}{9}, \ldots$ (e) $64, 80, 100, 125, \ldots$ (f) $27, -18, 12, -8, \ldots$

**2.** Find the term indicated for each of these geometric sequences:

(a) $11, 33, 99, 297, \ldots$ 10th term.
(b) $1, 0.2, 0.04, 0.008, \ldots$ 15th term.
(c) $9, -6, 4, -\frac{8}{3}, \ldots$ 9th term.
(d) $21, 9, \frac{27}{7}, \frac{81}{49}, \ldots$ 6th term.
(e) $-\frac{1}{3}, -\frac{1}{4}, -\frac{3}{16}, -\frac{9}{64}, \ldots$ 6th term.

**3.** Find the number of terms in each of these geometric sequences and the sum of the numbers in each sequence:

(a) $4, 12, 36, \ldots, 236196$
(b) $11, -22, 44, \ldots, 704$
(c) $100, -10, 1, \ldots, -10^{-5}$
(d) $48, 36, 27, \ldots, \frac{6561}{1024}$
(e) $\frac{1}{8}, -\frac{9}{32}, \frac{81}{128}, \ldots, \frac{6561}{2048}$
(f) $100, 10, 1, \ldots, 10^{-10}$

**4.** Write the following in expanded form and evaluate.

(a) $\sum_{k=1}^{7} \left(\frac{1}{2}\right)^k$
(b) $\sum_{i=1}^{6} 2^{i-4}$
(c) $\sum_{j=1}^{4} \left(\frac{2}{3}\right)^j$
(d) $\sum_{s=1}^{4} (-3)^s$
(e) $\sum_{t=1}^{6} 2^{-t}$

**5.** The third term of a geometric sequence is 36 and the tenth term is 78,732. Find the first term in the sequence and the sum of these terms.

**6.** A bank account offers 9% interest compounded annually. If \$750 is invested in this account, find the amount in the account at the end of the twelfth year.

**7.** When a ball is dropped onto a flat floor, it bounces to 65% of the height from which it was dropped. If the ball is dropped from 80 cm, find the height of the fifth bounce.

**8.** A computer loses 30% of its value each year.
(a) Write a formula for the value of the computer after $n$ years.
(b) How many years will it be before the value of the computer falls below 10% of its original value?

**9.** A geometric sequence has a first term of 7 and a common ratio of 1.1. How many terms must be taken before the value of the term exceeds 1000?

**10.** A colony of algae increases in size by 15% per week. If 10 grams of the algae are placed in a lake, find the weight of algae that will be present in the lake after 12 weeks. The lake will be considered 'seriously polluted' when there is in excess of 10000 grams of algae in the lake. How long will it be before the lake becomes seriously polluted?

**11.** A geometric series has nine terms, a common ratio of 2 and a sum of 3577. Find the first term.

**12.** A geometric series has a third term of 12, a common ratio of $-\frac{1}{2}$ and a sum of $32\frac{1}{16}$. Find the number of terms in the series.

**13.** A geometric series has a first term of 1000, seven terms and a sum of $671\frac{7}{8}$. Find the common ratio.

**14.** A geometric series has a third term of 300, and a sixth term of 37500. Find the common ratio and the sum of the first fourteen terms (in scientific form correct to two significant figures).

**15.** A \$10000 loan is offered on the following terms: 12% annual interest on the outstanding debt calculated monthly. The required monthly repayment is \$270. How much will still be owing after nine months.

**16.** As a prize for inventing the game of chess, its originator is said to have asked for one grain of wheat to be placed on the first square of the board, 2 on the second, 4 on the third, 8 on the fourth and so on until each of the 64 squares had been covered. How much wheat would have been the prize?

## 8.2.3 COMBINED A.Ps AND G.Ps

There will be occasions on which questions will be asked that relate to both arithmetic and geometric sequences and series.

**EXAMPLE 8.21** A geometric sequence has the same first term as an arithmetic sequence. The third term of the geometric sequence is the same as the tenth term of the arithmetic sequence with both being 48. The tenth term of the arithmetic sequence is four times the second term of the geometric sequence. Find the common difference of the arithmetic sequence and the common ratio of the geometric sequence.

**Solution**

When solving these sorts of question, write the data as equations, noting that $a$ is the same for both sequences. Let $u_n$ denote the general term of the A.P and $v_n$ the general term of the G.P. We then have:

$$u_{10} = a + 9d = 48,\ v_3 = ar^2 = 48,$$

i.e.,

$$a + 9d = ar^2 = 48 \quad -(1)$$

$$u_{10} = 4v_2 \Rightarrow a + 9d = 4ar \quad -(2)$$

(1) represents the information 'The third term of the geometric sequence is the same as the tenth term of the arithmetic sequence with both being 48'.
(2) represents 'The tenth term of the arithmetic sequence is four times the second term of the geometric sequence'.

There are three equations here and more than one way of solving them. One of the simplest is:

From (1) $a + 9d = 48$ and so substituting into (2) : $\quad 48 = 4ar \Leftrightarrow ar = 12 \quad -(3)$

Also from (1) we have: $\quad ar^2 = 48 \Leftrightarrow (ar)r = 48 \quad -(4)$

Substituting (3) into (4): $12r = 48 \Leftrightarrow r = 4$

Substituting result into (1): $a \times 16 = 48 \Leftrightarrow a = 3$

Substituting result into (1): $3 + 9d = 48 \Leftrightarrow d = 5$

The common ratio is 4 and the common difference is 5.

It is worth checking that the sequences are as specified:

Geometric sequence: 3, 12, 48
Arithmetic sequence: 3, 8, 13, 18, 23, 28, 33, 38, 43, 48

## EXERCISES 8.2.3

**1.** Consider the following sequences:
Arithmetic: 100, 110, 120, 130, . . .
Geometric: 1, 2, 4, 8, 16, . . .
Prove that:
The terms of the geometric sequence will exceed the terms of the arithmetic sequence after the 8th term.
The sum of the terms of the geometric sequence will exceed the sum of the terms of the arithmetic after the 10th term.

**2.** An arithmetic series has a first term of 2 and a fifth term of 30. A geometric series has a common ratio of –0.5. The sum of the first two terms of the geometric series is the same as the second term of the arithmetic series. What is the first term of the geometric series?

**3.** An arithmetic series has a first term of –4 and a common difference of 1. A geometric series has a first term of 8 and a common ratio of 0.5. After how many terms does the sum of the arithmetic series exceed the sum of the geometric series?

**4.** The first and second terms of an arithmetic and a geometric series are the same and are equal to 12. The sum of the first two terms of the arithmetic series is four times the first term of the geometric series. Find the first term of each series. If the A.P has $d = 4$.

**5.** Bo-Youn and Ken are to begin a savings program. Bo-Youn saves \$1 in the first week \$2 in the second week, \$4 in the third and so on, in geometric progression. Ken saves \$10 in the first week, \$15 in the second week, \$20 in the third and so on, in arithmetic progression. After how many weeks will Bo-Youn have saved more than Ken?

**6.** Ari and Chai begin a training program. In the first week Chai will run 10km, in the second he will run 11km and in the third 12km, and so on, in arithmetic progression. Ari will run 5km in the first week and will increase his distance by 20% in each succeeding week.
(a) When does Ari's weekly distance first exceed Chai's?
(b) When does Ari's total distance first exceed Chai's?

**7.** The Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, . . . in which each term is the sum of the previous two terms is neither arithmetic nor geometric. However, after the eighth term (21) the sequence becomes approximately geometric. If we assume that the sequence is geometric:

(a) What is the common ratio of the sequence (to four significant figures)?

(b) Assuming that the Fibonacci sequence can be approximated by the geometric sequence after the eighth term, what is the approximate sum of the first 24 terms of the Fibonacci sequence?

## 8.2.4 CONVERGENT SERIES

If a geometric series has a common ratio between –1 and 1, the terms get smaller and smaller as $n$ increases.

The sum of these terms is still given by the formula: $S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$

For $-1 < r < 1, r^n \to 0$ as $n \to \infty \Rightarrow S_n = \frac{a}{1-r}$.

> If $|r| < 1$, the infinite sequence has a sum given by $S_\infty = \frac{a}{1-r}$.

This means that if the common ratio of a geometric series is between –1 and 1, the sum of the series will approach a value of $\frac{a}{1-r}$ as the number of terms of the series becomes large i.e., the **series is convergent**.

**EXAMPLE 8.22**

Find the sum to infinity of the series

(a) $16 + 8 + 4 + 2 + 1 + \ldots$

(b) $9 - 6 + 4 - \frac{8}{3} + \frac{16}{9} - \ldots$

**Solution**

(a) $16 + 8 + 4 + 2 + 1 + \ldots$

In this case $a = 16, r = \frac{1}{2} \Rightarrow S_\infty = \frac{a}{1-r} = \frac{16}{1-\frac{1}{2}} = 32$

(b) $9 - 6 + 4 - \frac{8}{3} + \frac{16}{9} -$

$a = 9, r = -\frac{2}{3} \Rightarrow S_\infty = \frac{a}{1-r} = \frac{9}{1-\left(-\frac{2}{3}\right)} = 5.4$

There are a variety of applications for convergent geometric series. The following examples illustrate two of these.

**EXAMPLE 8.23**

Use an infinite series to express the recurring decimal $0.\dot{4}6\dot{2}$ as a rational number.

**Solution**

$0.\dot{4}6\dot{2}$ can be expressed as the series: $0.462 + 0.000462 + 0.000000462 + \ldots$

or $\frac{462}{1000} + \frac{462}{1000000} + \frac{462}{1000000000} + \ldots$

This is a geometric series with $a = \frac{462}{1000}$, $r = \frac{1}{1000}$

It follows that $S_\infty = \frac{a}{1-r} = \frac{\frac{462}{1000}}{1 - \frac{1}{1000}} = \frac{\frac{462}{1000}}{\frac{999}{1000}} = \frac{462}{999}$

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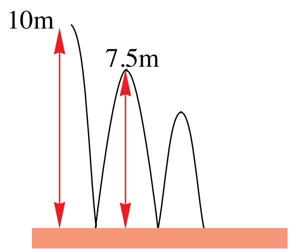
**EXAMPLE 8.24**

A ball is dropped from a height of 10 metres. On each bounce the ball bounces to three quartes of the height of the previous bounce. Find the distance travelled by the ball before it comes to rest (if it does not move sideways).

**Solution**

The ball bounces in a vertical line and does not move sideways. On each bounce after the drop, the ball moves both up and down and so travels twice the distance of the height of the bounce.




Distance $= 10 + 15 + 15 \times \frac{3}{4} + 15 \times \left(\frac{3}{4}\right)^2 + \ldots$

All but the first term of this series are geometric $a = 15$, $r = \frac{3}{4}$

Distance $= 10 + S_\infty = 10 + \frac{15}{1 - \frac{3}{4}} = 70\text{m}$

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## EXERCISES 8.2.4

**1.** Evaluate:

(i) $27 + 9 + 3 + \frac{1}{3} + \ldots$ (ii) $1 - \frac{3}{10} + \frac{9}{100} - \frac{27}{1000} + \ldots$

(iii) $500 + 450 + 405 + 364.5 + \ldots$ (iv) $3 - 0.3 + 0.03 - 0.003 + 0.0003 - \ldots$

**2.** Use geometric series to express the recurring decimal $23.232323\ldots$ as a mixed number.

**3.** Biologists estimate that there are 1000 trout in a lake. If none are caught, the population will increase at 10% per year. If more than 10% are caught, the population will fall. As an approximation, assume that if 25% of the fish are caught per year, the population will fall by 15% per year. Estimate the total catch before the lake is ‘fished out’. If the catch rate is reduced to 15%, what is the total catch in this case? Comment on these results.

**4.** Find the sum to infinity of the sequence 45, –30, 20, . . .

**5.** The second term of a geometric sequence is 12 while the sum to infinity is 64. Find the first three terms of this sequence.

**6.** Express the following as rational numbers

(a) $0.3\dot{6}$ (b) $0.\dot{3}\dot{7}$ (c) $2.1\dot{2}$

**7.** A swinging pendulum covers 32 centimetres in its first swing, 24 cm on its second swing, 18 cm on its third swing and so on. What is the total distance this pendulum swings before coming to rest?

**8.** The sum to infinity of a geometric sequence is $\frac{27}{2}$ while the sum of the first three terms is 13. Find the sum of the first 5 terms.

**9.** Find the sum to infinity of the sequence $1+\sqrt{3}, 1, \frac{1}{\sqrt{3}+1}, \ldots$

**10.** (a) Find i. $\sum_{i=0}^{n}(-t)^i$, $|t|<1$. ii. $\sum_{i=0}^{\infty}(-t)^i$, $|t|<1$.

(b) i. Hence, show that $\ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots$, $|x|<1$

ii. Using the above result, show that $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$.

**11.** (a) Find i. $\sum_{i=0}^{n}(-t^2)^i$, $|t|<1$. ii. $\sum_{i=0}^{\infty}(-t^2)^i$, $|t|<1$.

(b) i. Hence, show that $\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots$, $|x|<1$

ii. Using the above result, show that $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$.

## EXERCISES 8.2.5 – MISCELLANEOUS QUESTIONS

**1.** $2k+2$, $5k+1$ and $10k+2$ are three successive terms of a geometric sequence. Find the value(s) of $k$.

**2.** Evaluate $\dfrac{1+2+3+\ldots+10}{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{512}}$

**3.** Find a number which, when added to each of 2, 6 and 13 gives three numbers in geometric sequence.

**4.** Find the fractional equivalent of

(a) $2.3\dot{8}$ (b) $4.\dot{6}\dot{2}$ (c) $0.41717\ldots$

**5.** Find the sum of all intergers between 200 and 400 that are divisible by 6.

**6.** Find the sum of the first 50 terms of an arithmetic progression given that the 15th term is 34 and the sum of the first 8 terms is 20.

**7.** Find the value of $p$ so that $p + 5$, $4p + 3$ and $8p - 2$ will form successive terms of an arithmetic progression.

**8.** For the series defined by $S_n = 3n^2 - 11n$, find $t_n$ and hence show that the sequence is arithmetic.

**9.** How many terms of the series $6 + 3 + \frac{3}{2} + \ldots$ must be taken to give a sum of $11\frac{13}{16}$?

**10.** If $1 + 2x + 4x^2 + \ldots = \frac{3}{4}$, find the value of $x$.

**11.** Logs of wood are stacked in a pile so that there are 15 logs on the top row, 16 on the next row, 17 on the next and so on. If there are 246 logs in total,

(a) how many rows are there?

(b) how many logs are there in the bottom row?

**12.** The lengths of the sides of a right angled triangle form the terms of an arithmetic sequence. If the hypotenuse is 15 cm in length, what is the length of the other two sides?

**13.** The sum of the first 8 terms of a geometric series is 17 times the sum of its first four terms. Find the common ratio.

**14.** Three numbers $a$, $b$ and $c$ whose sum is 15 are successive terms of a G.P, and $b$, $a$, $c$ are successive terms of an A.P. Find $a$, $b$ and $c$.

**15.** The sum of the first $n$ terms of an arithmetic series is given by $S_n = \frac{n(3n+1)}{2}$.

(a) Calculate $S_1$ and $S_2$.

(b) Find the first three terms of this series.

(c) Find an expression for the $n$th term.

**16.** An ant walks along a straight path. After travelling 1 metre it stops, turns through an angle of 90˚ in an anticlockwise direction and sets off in a straight line covering a distance of half a metre. Again, the ant turns through an angle of 90˚ in an anticlockwise direction and sets off in a straight line covering a quarter of a metre. The ant continues in this manner indefinitely.

(a) How many turns will the ant have made after covering a distance of $\frac{63}{32}$ metres?

(b) How far will the ant 'eventually' travel?

**5.** (a) $I = \frac{a}{n^k}$ **6.** (a) 0.10 (b) $\lambda = \lambda_0 \times 10^{-kx}$ (c) 16.82% (d) $k = -\frac{1}{x}\log\left(\frac{\lambda}{\lambda_0}\right)$

## EXERCISE 8.1.1

**1.** i. (b) 4 (c) $t_n = 4n - 2$ ii. (b) –3 (c) $t_n = -3n + 23$ iii. (b) –5 (c) $t_n = -5n + 6$ iv. (b) 0.5 (c) $t_n = 0.5n$ v. (b) 2 (c) $t_n = y + 2n - 1$ vi. (b) –2 (c) $t_n = x - 2n + 4$ **2.** –28 **3.** 9,17 **4.** –43 **5.** 7 **6.** 7 **7.** –5 **8.** 0 **9.** (a) 41 (b) 31st **10.** 2, $\sqrt{3}$ **11.** (a) i. 2 ii. –3 (b) i. 4 ii. 11 **12.** $x - 8y$ **13.** $t_n = 5 + \frac{10}{3}(n-1)$ **14.** (a) –1 (b) 0

## EXERCISE 8.1.2

**1.** (a) 145 (b) 300 (c) –170 **2.** (a) –18 (b) 690 (c) 70.4 **3.** (a) –105 (b) 507 (c) 224 **4.** (a) 126 (b) 3900 (c) 14th week **5.** 855 **6.** (a) 420 (b) –210 **7.** $a = 9, b = 7$

## EXERCISE 8.1.3

**1.** 123 **2.** –3, –0.5, 2, 4.5, 7, 9.5, 12 **3.** 3.25 **4.** $a = 3$ $d = -0.05$ **5.** 10 000 **6.** 330 **7.** –20 **8.** 328 **9.** $725, 37wks **10.** i. $55 ii. 2750 **11.** (a) (i) 8m (ii) 40m (b) 84m (c) Dist $= 2n^2 - 2n = 2n(n-1)$ (d) 8 (e) 26 players, 1300m **12.** (a) 5050 (b) 10200 (c) 4233 **13.** (a) 145 (b) 390 (c) –1845 **14.** (b) $3n - 2$

## EXERCISE 8.2.1

**1.** (a) $r = 2, u_5 = 48, u_n = 3 \times 2^{n-1}$ (b) $r = \frac{1}{3}, u_5 = \frac{1}{27}, u_n = 3 \times \left(\frac{1}{3}\right)^{n-1}$ (c) $r = \frac{1}{5}, u_5 = \frac{2}{625}, u_n = 2 \times \left(\frac{1}{5}\right)^{n-1}$ (d) $r = -4, u_5 = -256, u_n = -1 \times (-4)^{n-1}$ (e) $r = \frac{1}{b}, u_5 = \frac{a}{b^3}, u_n = ab \times \left(\frac{1}{b}\right)^{n-1}$ (f) $r = \frac{b}{a}, u_5 = \frac{b^4}{a^2}, u_n = a^2 \times \left(\frac{b}{a}\right)^{n-1}$ **2.** (a) ±12 (b) $\frac{\pm\sqrt{5}}{2}$ **3.** (a) ±96 (b) 15th **4.** (a) $u_n = 10 \times \left(\frac{5}{6}\right)^{n-1}$ (b) $\frac{15625}{3888} \cong 4.02$ (c) $n = 5$ (4 times) **5.** –2, $\frac{4}{3}$ **6.** (a) i. $4096 ii. $2097.15 (b) 6.2 yrs **7.** $\left(u_n = \frac{1000}{169} \times \left(\frac{12}{5}\right)^{n-1}\right), \frac{1990656}{4225} \cong 471.16$ **8.** 2.5,5,10 or 10,5,2.5 **9.** 53757 **10.** 108 952 **11.** (a) $56 156 (b) $299 284

## EXERCISE 8.2.2

**1.** (a) 3 (b) $\frac{1}{3}$ (c) –1 (d) $-\frac{1}{3}$ (e) 1.25 (f) $-\frac{2}{3}$ **2.** (a) 216513 (b) $1.6384 \times 10^{-10}$ (c) $\frac{256}{729}$ (d) $\frac{729}{2401}$ (e) $-\frac{81}{1024}$ **3.** (a) 11; 354292 (b) 7; 473 (c) 8; 90.90909 (d) 8; 172.778 (e) 5; 2.256 (f) 13; 111.1111111111 **4.** (a) $\frac{127}{128}$ (b) $\frac{63}{8}$ (c) $\frac{130}{81}$ (d) 60 (e) $\frac{63}{64}$ **5.** 4; 118096 **6.** $2109.50 **7.** 9.28cm **8.** (a) $V_n = V_0 \times 0.7^n$ (b) 7 **9.** 54 **10.** 53.5gms; 50 weeks. **11.** 7 **12.** 9 **13.** –0.5, –0.7797 **14.** $r = 5$, $1.8 \times 10^{10}$ **15.** $8407.35 **16.** $1.8 \times 10^{19}$ or about 200 billion tonnes.



## EXERCISE 8.2.3

**1.** Term 9 AP = 180, GP = 256. Sum to 11 terms AP = 1650, GP = 2047. **2.** 18. **3.** 12 **4.** 7, 12 **5.** 8 weeks (Ken $220 & Bo-Youn $255) **6.** (a) week 8 (b) week 12 **7.** (a) 1.618 (b) 121379 [~121400, depends on rounding errors]

## EXERCISE 8.2.4

**1.** (i) $\frac{81}{2}$ (ii) $\frac{10}{13}$ (iii) 5000 (iv) $\frac{30}{11}$ **2.** $23\frac{23}{99}$ **3.** 6667 fish. [Nb: $t_{43} < 1$ . If we use $n = 43$ then ans is 6660 fish]; 20 000 fish. Overfishing means that fewer fish are caught in the long run. [*An alternate estimate for the total catch is* 1665 *fish.*] **4.** 27 **5.** 48,12,3 or 16,12,9 **6.** (a) $\frac{11}{30}$ (b) $\frac{37}{99}$ (c) $\frac{191}{90}$ **7.** 128 cm **8.** $\frac{121}{9}$ **9.** $2 + \frac{4}{3}\sqrt{3}$ **10.** $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$ **11.** $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

## EXERCISE 8.2.5

**1.** 3, –0.2 **2.** $\frac{2560}{93}$ **3.** $\frac{10}{3}$ **4.** (a) $\frac{43}{18}$ (b) $\frac{458}{99}$ (c) $\frac{413}{990}$ **5.** 9900 **6.** 3275 **7.** 3 **8.** $t_n = 6n - 14$ **9.** 6 **10.** $-\frac{1}{6}$ **11.** i. 12 ii. 26 **12.** 9, 12 **13.** ±2 **14.** (5, 5, 5), (5, –10, 20) **15.** (a) 2, 7 (b) 2, 5, 8 (c) $3n - 1$ **16.** (a) 5 (b) 2 m

## EXERCISE 8.3

**1.** $2773.08 **2.** $4377.63 **3.** $1781.94 **4.** $12216 **5.** $35816.95 **6.** $40349.37 **7.** $64006.80 **8.** $276971.93, $281325.41 **9.** $63762.25 **10.** $98.62, $9467.14, interest $4467.14. Flat interest = $6000 **11.** $134.41, $3790.44, 0.602% /month (or 7.22% p.a)

## EXERCISE 9.1

| **1.** | *a* cm | *b* cm | *c* cm | *A* | *B* | *C* |
|---|---|---|---|---|---|---|
| 1 | 3.8 | 4.1 | 1.6 | 67° | 90° | 23° |
| 2 | 81.5 | 98.3 | 55.0 | 56° | 90° | 34° |
| 3 | 32.7 | 47.1 | 33.9 | 44° | 90° | 46° |
| 4 | 1.61 | 30.7 | 30.7 | 3° | 90° | 87° |
| 5 | 2.3 | 2.74 | 1.49 | 57° | 90° | 33° |
| 6 | 48.5 | 77 | 59.8 | 39° | 90° | 51° |
| 7 | 44.4 | 81.6 | 68.4 | 33° | 90° | 57° |
| 8 | 2.93 | 13.0 | 12.7 | 13° | 90° | 77° |
| 9 | 74.4 | 94.4 | 58.1 | 52° | 90° | 38° |
| 10 | 71.8 | 96.5 | 64.6 | 48° | 90° | 42° |
| 11 | 23.3 | 34.1 | 24.9 | 43° | 90° | 47° |
| 12 | 43.1 | 43.2 | 2.3 | 87° | 90° | 3° |
| 13 | 71.5 | 80.2 | 36.4 | 63° | 90° | 27° |
| 14 | 33.5 | 34.1 | 6.5 | 79° | 90° | 11° |
| 15 | 6.1 | 7.2 | 3.82 | 58° | 90° | 32° |
| 16 | 29.1 | 30 | 7.3 | 76° | 90° | 14° |
| 17 | 29.0 | 29.1 | 2.0 | 86° | 90° | 4° |
| 18 | 34.5 | 88.2 | 81.2 | 23° | 90° | 67° |
| 19 | 24.0 | 29.7 | 17.5 | 54° | 90° | 36° |
| 20 | 41.2 | 46.2 | 21.0 | 63° | 90° | 27° |
| 21 | 59.6 | 72.9 | 41.8 | 55° | 90° | 35° |
| 22 | 5.43 | 6.8 | 4.09 | 53° | 90° | 37° |