3B Remainder and factor theorems

We saw in the last section that we can factorise polynomials by comparing coefficients. For example, if we know that $(x+2)$ is one factor of x^3+2x^2+x+2, we can write $x^3+2x^2+x+2=(x+2)(ax^2+bx+c)$ and compare coefficients to find that the other factor is (x^2+1).

If we try to factorise x^3+2x^2+x+5 using $(x+2)$ as one factor, we find that it is not possible; $(x+2)$ is not a factor of x^3+2x^2+x+5. However, using the factorisation of x^3+2x^2+x+2 we can write:

$$x^3+2x^2+x+5=(x+2)(x^2+1)+3$$

The number 3 is the **remainder** – it is what is left over when we try to write x^3+2x^2+x+5 as a multiple of $(x+2)$. In the last section we saw that factorising is related to division. In this case, we could say that:

$$\frac{x^3+2x^2+x+5}{x+2}=(x^2+1) \text{ with remainder } 3$$

This is similar to the concept of a remainder when dividing numbers: for example, $25=3\times7+4$, so we would say that 4 is the remainder when 25 is divided by 7.

We can find the reminder by including it as another unknown coefficient. For example, to find the reminder when x^3+2x^2+x+5 is divided by $(x+2)$, we could write

$$x^3+2x^2+x+5=(x+2)(ax^2+bx+c)+R$$

then expand and compare coefficients. This is not a quick task. Luckily there is a shortcut which can help us find the reminder without finding all the other coefficients. If we substitute in a value of x that makes the first bracket equal to zero, in this case $x=-2$, into the above equation, it becomes

$$3=(0)(ax^2+bx+c)+R$$

so $R=3$. This means that R is the value we get when we substitute $x=-2$ into the polynomial expression on the left. Fill-in proof sheet 6 on the CD-ROM, 'Remainder theorem', shows you that the same reasoning can be applied when dividing any polynomial by a linear factor. This leads us to the **Remainder theorem**.

EXAM HINT

Notice that $x = \frac{b}{a}$ is the value which makes $ax - b = 0$.

KEY POINT 3.2

The remainder theorem

The remainder when a polynomial expression is divided by $(ax - b)$ is the value of the expression when $x = \frac{b}{a}$.

Worked example 3.4

Find the remainder when $x^3 + 2x + 7$ is divided by $x + 2$.

Use the remainder theorem by rewriting the divisor in the form $(ax - b)$…

$(x+2) = (x-(-2))$

… then substitute the value of x (obtained from $x = \frac{b}{a}$) into the expression when $x = \frac{b}{a}$

When $x = -2: (-2)^3 + 2\times(-2) + 7 = -5$

So the remainder is -5

If the remainder is zero then $(ax - b)$ is a factor. This is summarised by the **factor theorem**.

KEY POINT 3.3

The factor theorem

If the value of a polynomial expression is zero when $x = \frac{b}{a}$, then $(ax - b)$ is a factor of the expression.

Worked example 3.5

Show that $2x - 3$ is a factor of $2x^3 - 13x^2 + 19x - 6$.

To use the factor theorem we need to substitute in $x = \frac{3}{2}$

When $x = \frac{3}{2}$:

$$2\times\left(\frac{3}{2}\right)^3 - 13\left(\frac{3}{2}\right)^2 + 19\times\left(\frac{3}{2}\right) - 6$$

$$= \frac{27}{4} - \frac{117}{4} + \frac{57}{2} - 6 = 0$$

continued . . .

Therefore, by the factor theorem, $(2x-3)$ is a factor of $2x^3-13x^2+19x-6$.

We can also use the factor theorem to identify a factor of an expression, by trying several different numbers. Once one factor has been found, then comparing coefficients can be used to find the remaining factors. This is the recommended method for factorising cubic expressions on the non-calculator paper.

Worked example 3.6

Fully factorise $x^3+3x^2-33x-35$.

When factorising a cubic with no obvious factors we must put in some numbers and hope that we can apply the factor theorem

When $x=1$ the expression is -64
When $x=2$ the expression is -81
When $x=-1$ the expression is 0
Therefore $x+1$ is a factor.

We can rewrite the expression as $(x+1)\times$ general quadratic and compare coefficients

$$x^3+3x^2-33x-35$$
$$=(x+1)(ax^2+bx+c)$$
$$=ax^3+(a+b)x^2+(b+c)x+c$$
$$a=1, b=2, c=-35$$
$$x^3+3x^2-33x-35=(x+1)(x^2+2x-35)$$

The remaining quadratic also factorises

$$=(x+1)(x+7)(x-5)$$

EXAM HINT

If the expression is going to factorise easily then you only need to try numbers which are factors of the constant term.

A very common type of question asks you to find unknown coefficients in an expression if factors or remainders are given.

Worked example 3.7

$x^3 + 4x^2 + ax + b$ has a factor of $(x - 1)$ and leaves a remainder of 17 when divided by $(x - 2)$. Find the constants a and b.

Apply factor theorem

when $x = 1$:

$1 + 4 + a + b = 0$

$\Leftrightarrow a + b = -5 \qquad (1)$

Apply remainder theorem

when $x = 2$:

$8 + 16 + 2a + b = 17$

$\Leftrightarrow 2a + b = -7 \qquad (2)$

Two equations with two unknowns can be solved simultaneously

(2) − (1)

$a = -2$

$b = -3$

Exercise 3B

1. Use the remainder theorem to find the remainder when:

(a) (i) $x^2 + 3x + 5$ is divided by $x + 1$

(ii) $x^2 + x - 4$ is divided by $x + 2$

(b) (i) $x^3 - 6x^2 + 4x + 8$ is divided by $x - 3$

(ii) $x^3 - 7x^2 + 11x$ is divided by $x - 1$

(c) (i) $6x^4 + 7x^3 - 5x^2 + 5x + 10$ is divided by $2x + 3$

(ii) $12x^4 - 10x^3 + 11x^2 - 5$ is divided by $3x - 1$

(d) (i) x^3 is divided by $x + 2$

(ii) $3x^4$ is divided by $x - 1$

2. Decide whether each of the following expressions are factors of $2x^3 - 73 - 3x + 2$.

(a) x

(b) $x - 1$

(c) $x + 1$

(d) $x - 2$

(e) $x + 2$

(f) $x - \frac{1}{2}$

(g) $x + \frac{1}{2}$

(h) $2x - 1$

(i) $2x + 1$

(j) $3x - 1$

3. Fully factorise the following expressions:

(a) (i) x^3+2x^2-x-2 (ii) x^3+x^2-4x-4

(b) (i) $x^3-7x^2+16x-12$ (ii) $x^3+6x^2+12x+8$

(c) (i) $x^3-3x^2+12x-10$ (ii) $x^3-2x^2+2x-15$

(d) (i) $6x^3-11x^2+6x-1$ (ii) $12x^3+13x^2-37x-30$

4. $6x^3+ax^2+bx+8$ has a factor $(x+2)$ and leaves a remainder of -3 when divided by $(x-1)$.
Find a and b. *[5 marks]*

5. x^3+8x^2+ax+b has a factor of $(x-2)$ and leaves a remainder of 15 when divided by $(x-3)$.
Find a and b. *[5 marks]*

6. The polynomial $x^2+kx-8k$ has a factor $(x-k)$. Find the possible values of k. *[5 marks]*

7. The polynomial $x^2-(k+1)x-3$ has a factor $(x-k+1)$. Find k. *[6 marks]*

8. $x^3-ax^2-bx+168$ has factors $(x-7)$ and $(x-3)$.

(a) Find a and b.

(b) Find the remaining factor of the expression. *[6 marks]*

9. x^3+ax^2+9x+b has a factor of $(x-11)$ and leaves a remainder of -52 when divided by $(x+2)$.

(a) Find a and b.

(b) Find the remainder when x^3+ax^2+9x+b is divided by $(x-2)$. *[6 marks]*

10. $f(x)=x^3+ax^2+3x+b$.
The remainder when $f(x)$ is divided by $(x+1)$ is 6. Find the remainder when $f(x)$ is divided by $(x-1)$. *[5 marks]*

$f(x)$ is just a name given to the expression. You will learn more about this notation in chapter 5.

11. The polynomial x^2-5x+6 is a factor of $2x^3-15x^2+ax+b$. Find the values of a and b. *[6 marks]*

Exercise 3B

1. (a) (i) 3 (ii) −2
   (b) (i) −7 (ii) 5
   (c) (i) −2 (ii) −4
   (d) (i) −8 (ii) 3

2. (a) No (b) No
   (c) Yes (d) Yes
   (e) No (f) Yes
   (g) No (h) Yes
   (i) No (j) No

3. (a) (i) $(x+1)(x-1)(x+2)$
   (ii) $(x+1)(x-2)(x+2)$
   (b) (i) $(x-2)^2(x-3)$
   (ii) $(x+2)^3$
   (c) (i) $(x-1)(x^2-2x+10)$
   (ii) $(x-3)(x^2+x+5)$
   (d) (i) $(x-1)(2x-1)(3x-1)$
   (ii) $(x+2)(4x+3)(3x-5)$

4. $a=1, b=-18$

5. $a=-44, b=48$

6. $k=0,4$

7. $k=-\frac{1}{2}$

8. (a) $a=2, b=59$ (b) $(x+8)$

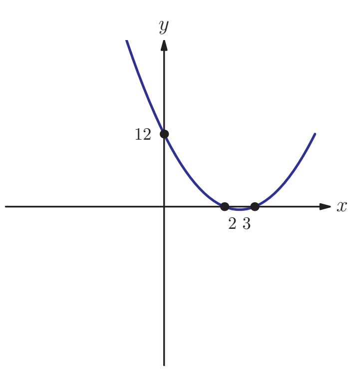
9. (a) $a=-12, b=22$ (b) 0

10. 14

11. $a=37, b=-30$

Exercise 3C

1. (a) (i)



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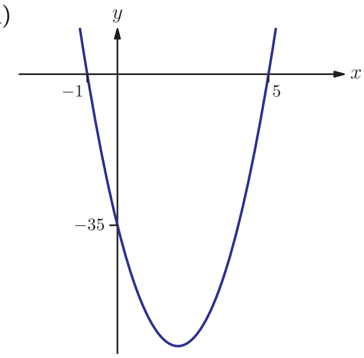
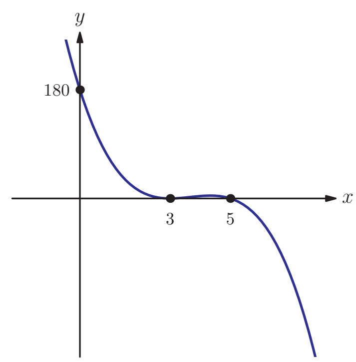
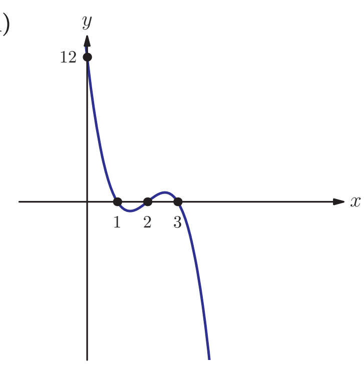


(b) (i)



(ii)



(c) (i)