8 Roots of polynomial equations

And the equation will come at last.
LOUIS MACNEICE

Roots of a quadratic equation

If α and β are the roots of a quadratic equation, $f(x) \equiv ax^2 + bx + c = 0$, then the equation must be of the form

$$f(x) = k(x - \alpha)(x - \beta) \quad \text{for some constant } k$$

Therefore, we have

$$k(x - \alpha)(x - \beta) \equiv ax^2 + bx + c$$
$$\Rightarrow \quad k(x^2 - [\alpha + \beta]x + \alpha\beta) \equiv ax^2 + bx + c$$

Equating the coefficients of x^2 gives: $k = a$

Equating the coefficients of x gives: $-k(\alpha + \beta) = b$

And equating the constants gives: $k\alpha\beta = c$

Therefore, we obtain

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Or

The **sum** of the roots is $-\frac{b}{a}$ and the **product** of the roots is $\frac{c}{a}$.

Example 1 In the equation $3x^2 - 7x + 11 = 0$, find

a) the sum of the roots
b) the product of the roots.

SOLUTION

a) Using $\alpha + \beta = -\frac{b}{a}$, we have

$$\text{Sum of the roots, } \alpha + \beta = -\frac{-7}{3} = +\frac{7}{3}$$

b) Using $\alpha\beta = \frac{c}{a}$, we have

$$\text{Product of the roots, } \alpha\beta = \frac{11}{3}$$

Conversely, we may write the quadratic equation as

$$\mathbf{x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0}$$

Example 2 Find the equation whose roots have a sum of $\frac{1}{2}$ and a product of $-\frac{5}{2}$.

SOLUTION

Using $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$, we have

$$x^2 - \tfrac{1}{2}x - \tfrac{5}{2} = 0 \quad \text{or} \quad 2x^2 - x - 5 = 0$$

Example 3 The equation $3x^2 + 9x - 11 = 0$ has roots α and β. Find the equation whose roots are $\alpha + \beta$ and $\alpha\beta$.

SOLUTION

From $3x^2 + 9x - 11 = 0$, we have

$$\alpha + \beta = -3 \quad \text{and} \quad \alpha\beta = -\frac{11}{3}$$

The sum of the **new roots** is: $\alpha + \beta + \alpha\beta = -3 - \dfrac{11}{3} = -\dfrac{20}{3}$

The product of the **new roots** is: $(\alpha + \beta) \times \alpha\beta = -3 \times -\dfrac{11}{3} = 11$

Therefore, the new equation is

$$x^2 + \frac{20}{3}x + 11 = 0 \quad \text{or} \quad 3x^2 + 20x + 33 = 0$$

Example 4 The equation $4x^2 + 7x - 5 = 0$ has roots α and β. Find the equation whose roots are α^2 and β^2.

SOLUTION

From $4x^2 + 7x - 5 = 0$, we have

$$\alpha + \beta = -\frac{7}{4} \quad \text{and} \quad \alpha\beta = -\frac{5}{4}$$

The sum of the new roots is

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

Substituting the above values in the RHS, we obtain

$$\alpha^2 + \beta^2 = \left(-\frac{7}{4}\right)^2 - 2 \times -\frac{5}{4} = \frac{89}{16}$$

The product of the new roots is $\alpha^2\beta^2 = (\alpha\beta)^2$. Substituting the value for $\alpha\beta$, we obtain

$$(\alpha\beta)^2 = \left(-\frac{5}{4}\right)^2 = \frac{25}{16}$$

Therefore, the new equation is

$$x^2 - \frac{89}{16}x + \frac{25}{16} = 0 \quad \text{or} \quad 16x^2 - 89x + 25 = 0$$

Roots of a cubic equation

In a similar manner, if α, β and γ are the roots of a cubic equation, $ax^3 + bx^2 + cx + d = 0$, then we have

$$ax^3 + bx^2 + cx + d \equiv k(x - \alpha)(x - \beta)(x - \gamma)$$

$$\Rightarrow \quad ax^3 + bx^2 + cx + d \equiv k[x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma]$$

Equating coefficients of x^2 gives: $\alpha + \beta + \gamma = -\dfrac{b}{a}$

Equating coefficients of x gives: $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

And equating the constants gives: $\alpha\beta\gamma = -\dfrac{d}{a}$

Example 5 Find the cubic equation in x which has roots 4, 3 and -2.

SOLUTION

The sum of the roots is

$$\alpha + \beta + \gamma = 4 + 3 + (-2) = 5$$

The sum of the roots taken two at a time is

$$\alpha\beta + \beta\gamma + \gamma\alpha = 4 \times 3 + 3 \times -2 + (-2 \times 4) = -2$$

The product of the roots is

$$\alpha\beta\gamma = 4 \times 3 \times -2 = -24$$

Therefore, the equation is

$$x^3 - 5x^2 - 2x + 24 = 0$$

Example 6 The cubic equation $x^3 + 3x^2 - 7x + 2 = 0$ has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

SOLUTION

From the cubic equation, we have

$$\alpha + \beta + \gamma = -3$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = -7$$
$$\alpha\beta\gamma = -2$$

We now expand $(\alpha + \beta + \gamma)^2$ to obtain

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

Substituting the values, we obtain

$$\alpha^2 + \beta^2 + \gamma^2 = (-3)^2 - 2 \times -7 = 23$$

Therefore, we have

$$\alpha^2 + \beta^2 + \gamma^2 = 23$$

Roots of a polynomial equation of degree n

From the properties of the roots of a quadratic equation and of a cubic equation, we see that in a polynomial equation of degree n, $ax^n + bx^{n-1} + cx^{n-2} + \ldots = 0$, the sum of the roots is $-\frac{b}{a}$ and the product of the roots is given by

$$(-1)^n \frac{\text{Last term}}{\text{First term}}$$

since the last term is the product of $-\alpha, -\beta, -\gamma, -\delta, \ldots$.

Example 7 The roots of $f(x) \equiv 4x^5 + 6x^4 - 3x^3 + 7x^2 - 11x - 3 = 0$ are $\alpha, \beta, \gamma, \delta$ and ε.

a) Find the product of the five roots.

b) **i)** Show that $x = 1$ is a root of the equation.

ii) Hence show that the sum of the roots other than 1 is $-\frac{5}{2}$.

SOLUTION

a) The sum of all five roots, $\alpha, \beta, \gamma, \delta$ and ε, is $-\frac{b}{a} = -\frac{6}{4} = -\frac{3}{2}$.

b) **i)** When $x = 1$, we have

$$f(1) = 4 + 6 - 3 + 7 - 11 - 3 = 0$$

Therefore, from the factor theorem, $x = 1$ is one root of the equation.

ii) The sum of all five roots is $-\frac{3}{2}$ (from part **a**). That is,

$$\alpha + \beta + \gamma + \delta + \varepsilon = -\frac{3}{2}$$

Putting $\varepsilon = 1$, we have

$$\alpha + \beta + \gamma + \delta + 1 = -\frac{3}{2} \quad \Rightarrow \quad \alpha + \beta + \gamma + \delta = -\frac{5}{2}$$

Therefore, the sum of the other four roots is $-\frac{5}{2}$.

Example 8 The equation $z^2 + (3 + i)z + p = 0$ has a root of $2 - i$. Find the value of p and the other root of the equation.

SOLUTION

Since $2 - i$ is a root, $z = 2 - i$ satisfies the equation. Therefore, we have

$$(2 - i)^2 + (3 + i)(2 - i) + p = 0$$
$$\Rightarrow \quad p = -10 + 5i$$

The sum of the roots, $\alpha + \beta = -\frac{b}{a}$, is $-(3 + i)$. Therefore, the other root is

$$-(3 + i) - (2 - i) = -5$$

Exercise 8A

1 Write down the sum and the product of the roots of each of the following equations.

a) $x^2 + 3x - 7 = 0$ **b)** $x^2 - 11x + 5 = 0$ **c)** $x^2 + 5x - 4 = 0$

d) $3x^2 + 11x + 2 = 0$ **e)** $x + 2 = \dfrac{5}{x}$ **f)** $2x^2 = 7 - 4x$

2 Write down the equation whose roots have the sum and the product given below.

a) Sum 7; product 15 **b)** Sum -3; product $+5$

c) Sum -2; product -4 **d)** Sum -5; product -11

3 If α, β, γ are the roots of the equation $x^3 - 5x + 3 = 0$, find the values of

a) $\alpha + \beta + \gamma$ **b)** $\alpha^2 + \beta^2 + \gamma^2$ **c)** $\alpha^3 + \beta^3 + \gamma^3$

4 The equation $2z^2 - (7 - 2\mathrm{i})z + q = 0$ has a root of $1 + \mathrm{i}$. Find **i)** the value of q and **ii)** the other root of the equation.

5 The equation $3z^2 - (1 - \mathrm{i})z + t = 0$ has a root of $3 + 2\mathrm{i}$. Find **i)** the value of t and **ii)** the other root of the equation.

6 Given that α, β, γ are the roots of the equation $x^3 + x^2 + 4x - 5 = 0$, find the cubic equation whose roots are $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. (WJEC)

7 Given the cubic equation $x^3 - 7x + q = 0$ has roots α, 2α and β, find the possible values of q. (WJEC)

8 The equation $3x^2 - 5x + 6 = 0$ has roots α and β. Without solving the given equation, find an equation with integer coefficients whose roots are $(\alpha + \beta)$ and $\alpha\beta$. (EDEXCEL)

9 The roots of the equation $x^3 - 3x^2 - 3x - 7 = 0$ are α, β and γ.

a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.
b) Show that

$$\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} = 0$$ (NEAB)

Equations with related roots

If α and β are the roots of $ax^2 + bx + c = 0$, then we can obtain the equation whose roots are 2α and 2β by making a substitution for x.

First, we express $ax^2 + bx + c = 0$ as

$$a(x - \alpha)(x - \beta) = 0$$

which gives

$$a(2x - 2\alpha)(2x - 2\beta) = 0$$

We obtain the required equation, whose roots are 2α and 2β, by putting $y = 2x$, which gives

$$a(y - 2\alpha)(y - 2\beta) = 0$$

Hence, replacing x by $\frac{y}{2}$ gives an equation whose roots are twice those of the original equation.

Example 9 Find the equation whose roots are 3α and 3β, where α and β are the roots of the equation $2x^2 - 5x + 3 = 0$.

SOLUTION

Replacing x by $\frac{y}{3}$ in $2x^2 - 5x + 3 = 0$, we obtain an equation in y whose roots for $\frac{y}{3}$ are the same as those for x: that is, α and β. Hence, the roots for y will be 3α and 3β.

Therefore, the required equation is

$$2\left(\frac{y}{3}\right)^2 - 5\left(\frac{y}{3}\right) + 3 = 0$$

$$\Rightarrow \quad 2y^2 - 15y + 27 = 0$$

If the equation is to be expressed in terms of x, it would be

$$2x^2 - 15x + 27 = 0$$

Example 10 Find the equation whose roots are $\alpha^2, \beta^2, \gamma^2$, where α, β, γ are the roots of $3x^3 - 7x^2 + 11x - 5 = 0$.

SOLUTION

Replacing x by $\sqrt{y}$ in $3x^3 - 7x^2 + 11x - 5 = 0$, we obtain α, β, γ as the roots for $\sqrt{y}$. Hence, the roots for y are $\alpha^2, \beta^2, \gamma^2$

Therefore, the equation in $\sqrt{y}$ is

$$3(\sqrt{y})^3 - 7(\sqrt{y})^2 + 11(\sqrt{y}) - 5 = 0$$

$$\Rightarrow \quad 3y\sqrt{y} + 11\sqrt{y} = 7y + 5$$

Squaring both sides, we have

$$9y^3 + 66y^2 + 121y = 49y^2 + 70y + 25$$

Therefore, the required equation is

$$9y^3 + 17y^2 + 51y - 25 = 0$$

Exercise 8B

1 The roots of the equation $x^2 + 7x + 11 = 0$ are α and β. Find the equation whose roots are 2α and 2β.

2 The roots of the equation $x^2 - 15x + 7 = 0$ are α and β. Find the equation whose roots are 3α and 3β.

3 The roots of the equation $3x^3 - 4x^2 + 8x - 7 = 0$ are α, β and γ. Find the equation whose roots are 2α, 2β and 2γ.

4 The roots of the equation $x^3 - 3x^2 - 11x + 5 = 0$ are α, β and γ. Find the equation whose roots are $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$.

5 The roots of the equation $2x^2 + 3x + 17 = 0$ are α and β. Find the equation whose roots are α^2 and β^2.

6 The roots of the equation $3x^2 - 7x + 15 = 0$ are α and β. Find the equation whose roots are α^2 and β^2.

7 The equation $2x^2 + 7x + 3 = 0$ has roots α and β. Find the equation whose roots are

a) $2\alpha, 2\beta$　**b)** $\frac{\alpha}{3}, \frac{\beta}{3}$　**c)** α^2, β^2　**d)** $\alpha + 2, \beta + 2$

8 The equation $3x^2 + 9x - 2 = 0$ has roots α and β. Find the equation whose roots are

a) $4\alpha, 4\beta$　**b)** $\frac{\alpha}{2}, \frac{\beta}{2}$　**c)** α^2, β^2　**d)** $\alpha - 3, \beta - 3$

9 The roots of the equation $x^3 + 3x^2 + 5x + 7 = 0$ are α, β and γ. Find the equation whose roots are

a) $3\alpha, 3\beta, 3\gamma$　**b)** $\alpha^2, \beta^2, \gamma^2$　**c)** $\alpha + 3, \beta + 3, \gamma + 3$

10 The roots of the equation $x^4 + 3x^3 + 7x^2 - 11x + 1 = 0$ are α, β, γ and δ. Find the equation whose roots are 3α, 3β, 3γ and 3δ.

11 The equation $x + 2 + \frac{3}{x} = 0$ has roots α and β. Find the equation whose roots are 5α and 5β.

12 The roots of the quadratic equation $x^2 - 3x + 4 = 0$ are α and β. Without solving the equation, find a quadratic equation, with integer coefficients, whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. (EDEXCEL)

Complex roots of a polynomial equation

If $z \equiv x + iy$ is a root of a polynomial equation with **real coefficients**, then $\bar{z} \equiv x - iy$ is also a root of the polynomial equation, where $\bar{z}$ is the conjugate of z (see page 3).

Proof

Suppose z is a root of the polynomial

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0 = 0$$

Then, taking the conjugate of both sides, we have

$$\overline{a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_0} = 0$$

Using $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, we obtain

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \overline{a_{n-2} z^{n-2}} + \ldots + \overline{a_0} = 0$$

And using $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, we obtain

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \overline{a_{n-2}}\,\overline{z^{n-2}} + \ldots + \overline{a_0} = 0$$

which gives

$$\overline{a_n}\,(\bar{z})^n + \overline{a_{n-1}}\,(\bar{z})^{n-1} + \overline{a_{n-2}}\,(\bar{z})^{n-2} + \ldots + \overline{a_0} = 0$$

Since all the a_i are real, $\overline{a_i} = a_i$. Therefore, we have

$$a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + a_{n-2}(\bar{z})^{n-2} + \ldots + a_0 = 0$$

Hence, $\bar{z}$ is also a root of the polynomial.

The complex roots of a polynomial with **real coefficients** always occur in **conjugate complex pairs**.

Note We found in Example 8 (page 150) that when a quadratic equation does **not have real coefficients**, the roots are **not conjugate complex pairs**. (In Example 8, they are $2 - i$ and -5.)

Example 11 Show that $4 - i$ is a root of the polynomial equation

$$f(z) \equiv z^3 - 6z^2 + z + 34 = 0$$

Hence find the other roots.

SOLUTION

To prove that $z = 4 - i$ is a root, we prove that $f(4 - i) = 0$. If $z = 4 - i$ is a root, then $z = 4 + i$ is also a root, since the roots occur as conjugate complex pairs.

Next, we find the quadratic with **real** coefficients which is a factor. We then divide $f(z)$ by this quadratic to find the other factor.

Substituting $z = 4 - \mathrm{i}$ in $\mathrm{f}(z) \equiv z^3 - 6z^2 + z + 34 = 0$, we have

$$\begin{aligned} \mathrm{f}(4 - \mathrm{i}) &= (4 - \mathrm{i})^3 - 6(4 - \mathrm{i})^2 + (4 - \mathrm{i}) + 34 \\ &= 52 - 47\mathrm{i} - 90 + 48\mathrm{i} + 4 - \mathrm{i} + 34 \\ &= 0 \end{aligned}$$

Therefore, $4 - \mathrm{i}$ is a root of $\mathrm{f}(z) \equiv z^3 - 6z^2 + z + 34 = 0$. Hence, $4 + \mathrm{i}$ is also a root.

If $z - (4 + \mathrm{i})$ and $z - (4 - \mathrm{i})$ are factors of the polynomial, so is

$$[z - (4 + \mathrm{i})][z - (4 - \mathrm{i})] = z^2 - 8z + 17$$

Dividing $z^3 - 6z^2 + z + 34 = 0$ by $z^2 - 8z + 17$, we obtain

$$\mathrm{f}(z) = (z^2 - 8z + 17)(z + 2)$$

Therefore, the three roots of $\mathrm{f}(z) \equiv z^3 - 6z^2 + z + 34 = 0$ are $4 + \mathrm{i}$, $4 - \mathrm{i}$ and -2.

Example 12 Show that $2 + \mathrm{i}$ is a root of the polynomial equation

$$\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$$

Hence find the other roots.

SOLUTION

As in Example 11, to prove that $z = 2 + \mathrm{i}$ is a root, we prove that $\mathrm{f}(2 + \mathrm{i}) = 0$. If $z = 2 + \mathrm{i}$ is a root, then $z = 2 - \mathrm{i}$ is also a root.

Next, we find the quadratic with **real** coefficients which is a factor. We then divide $\mathrm{f}(z)$ by this quadratic to find the other factors.

Substituting $z = 2 + \mathrm{i}$ in $\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$, we have

$$\begin{aligned} \mathrm{f}(2 + \mathrm{i}) &= (2 + \mathrm{i})^4 - 12(2 + \mathrm{i})^3 + 62(2 + \mathrm{i})^2 - 140(2 + \mathrm{i}) + 125 \\ &= -7 + 24\mathrm{i} - 24 - 132\mathrm{i} + 186 + 248\mathrm{i} - 280 - 140\mathrm{i} + 125 \\ &= 0 \end{aligned}$$

Therefore, $(2 + \mathrm{i})$ is a root of $\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$. Hence, $(2 - \mathrm{i})$ is also a root.

If $z - (2 + \mathrm{i})$ and $z - (2 - \mathrm{i})$ are factors of the polynomial, so is

$$[z - (2 + \mathrm{i})][z - (2 - \mathrm{i})] = z^2 - 4z + 5$$

Dividing $z^4 - 12z^3 + 62z^2 - 140z + 125$ by $z^2 - 4z + 5$, we obtain

$$\mathrm{f}(z) = (z^2 - 4z + 5)(z^2 - 8z + 25)$$

Using the quadratic formula, we find that the roots of $z^2 - 8z + 25 = 0$ are $4 \pm 3\mathrm{i}$.

Therefore, the four roots of $\mathrm{f}(z) \equiv z^4 - 12z^3 + 62z^2 - 140z + 125 = 0$ are $2 + \mathrm{i}$, $2 - \mathrm{i}$, $4 + 3\mathrm{i}$ and $4 - 3\mathrm{i}$.

Example 13 The roots of the equation $f(x) \equiv 2x^3 - 3x^2 + 7x - 19 = 0$ are α, β and γ. Show that

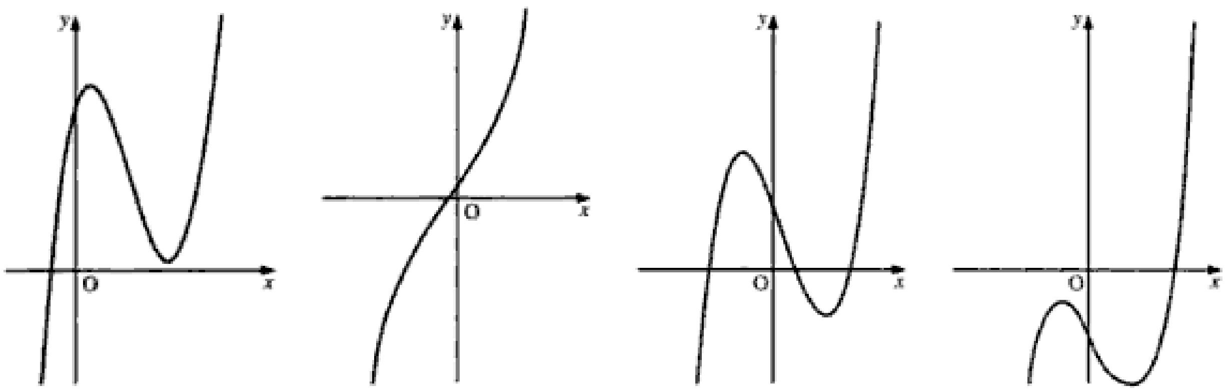
a) there is only one real root

b) the real root lies between $x = 2$ and $x = 3$

c) the real part of the two complex roots lies between $-\frac{1}{4}$ and $-\frac{3}{4}$.

SOLUTION

To show that a cubic equation has only one real root, we find the values of $f(x)$ at its turning points. Hence, we will be able to see which of the following curves is $f(x)$.



Note When the values of $f(x)$ at its turning points are of opposite sign, $f(x) = 0$ has three real roots.

a) To find the values of $f(x)$ at its turning points, we differentiate $f(x)$:

$$f(x) \equiv 2x^3 - 3x^2 + 7x - 19$$
$$f'(x) = 6x^2 - 6x + 7$$

Hence, we have

$$6x^2 - 6x + 7 = 0$$
$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36 - 168}}{12}$$

That is, $f'(x) = 0$ has no real roots. Hence, the cubic $f(x)$ has no turning points, which means that $f(x) = 0$ has only one real root.

b) We find that

$$f(2) = -1 \quad \text{and} \quad f(3) = +29$$

So, $f(x)$ has opposite signs at $x = 2$ and $x = 3$ and is continuous for $2 \leqslant x \leqslant 3$. Therefore, the real root of $f(x) = 0$ lies between $x = 2$ and $x = 3$.

c) Let the three roots of the equation be α, β, γ, where α is a real number between 2 and 3, and β and γ are complex numbers.

Since the roots of a polynomial with real coefficients occur in conjugate complex pairs, β and γ are conjugate complex numbers, which we will represent by $p + \mathrm{i}q$ and $p - \mathrm{i}q$.

Using $\alpha + \beta + \gamma = -\frac{b}{a}$, we find

$$\alpha + \beta + \gamma = \frac{3}{2}$$

which gives

$$\alpha + p + iq + p - iq = \frac{3}{2}$$

$$\Rightarrow \quad 2p = \frac{3}{2} - \alpha$$

Since $2 < \alpha < 3$, we therefore have

$$\frac{3}{2} - 3 < 2p < \frac{3}{2} - 2$$

$$\Rightarrow \quad -\frac{3}{2} < 2p < -\frac{1}{2}$$

$$\Rightarrow \quad -\frac{3}{4} < p < -\frac{1}{4}$$

Hence, the real part of each complex root lies between $-\frac{1}{4}$ and $-\frac{3}{4}$.

Exercise 8C

1 Solve the equation $x^4 - 5x^3 + 2x^2 - 5x + 1 = 0$, given that i is a root.

2 Solve the equation $3x^4 - x^3 + 2x^2 - 4x - 40 = 0$, given that 2i is a root.

3 Determine the number of real roots of the equation $2x^3 + x^2 = 3$.

4 Determine the number of real roots of the equation $2x^3 - 7x + 2 = 0$.

5 Determine the range of possible values of k if the equation $x^3 + 3x^2 = k$ has three real roots.

6 One root of the equation $z^4 - 5z^3 + 13z^2 - 16z + 10 = 0$ is $1 + i$. Find the other roots.

7 **a)** Show that one root of the equation $z^3 + 5z^2 - 56z + 110 = 0$ is $3 + i$.
b) Find the other roots of the equation.

8 **a)** Show that one root of the equation $z^4 - 2z^3 + 6z^2 + 22z + 13 = 0$ is $2 - 3i$.
b) **i)** Find the other roots of the equation.
ii) Hence factorise $z^4 - 2z^3 + 6z^2 + 22z + 13$ into two quadratics, each of which has real coefficients.

9 The polynomial f(z) is defined by

$$f(z) \equiv z^4 - 2z^3 + 3z^2 - 2z + 2$$

a) Verify that i is a root of the equation $f(z) = 0$.
b) Find all the other roots of the equation $f(z) = 0$. (EDEXCEL)

10 Given that $2 + i$ is a root of the equation $3x^3 - 14x^2 + 23x - 10 = 0$, find the other roots of the equation. (WJEC)

11 One of the complex roots of $2z^4 - 13z^3 + 33z^2 - 80z - 50 = 0$ is $(1 - 3i)$, where $i^2 = -1$.

i) State one other complex root.
ii) Find the other two roots and plot all four on an Argand diagram. (NICCEA)

12 Given that 3i is a root of the equation $3z^3 - 5z^2 + 27z - 45 = 0$, find the other two roots. (OCR)

13 a) Verify that $z = 2$ is a solution of the equation $z^3 - 8z^2 + 22z - 20 = 0$.
b) Express $z^3 - 8z^2 + 22z - 20$ as a product of a linear factor and a quadratic factor with real coefficients. Hence find **all** the solutions of $z^3 - 8z^2 + 22z - 20 = 0$. (SQA/CSYS)

14 Two of the roots of a cubic equation, in which all the coefficients are real, are 2 and $1 + 3i$.

i) State the third root.
ii) Find the cubic equation, giving it in the form $z^3 + az^2 + bz + c = 0$. (OCR)

15 Verify that $z = 1 + i$ is a solution of the equation $z^3 + 16z^2 - 34z + 36 = 0$.

Write down a second solution of the equation.

Hence find constants α and β such that

$$z^3 + 16z^2 - 34z + 36 = (z^2 - \alpha z + \alpha)(z + \beta) \qquad \text{(SQA/CSYS)}$$

16 The roots of the equation $7x^3 - 8x^2 + 23x + 30 = 0$ are α, β, γ.

a) Write down the value of $\alpha + \beta + \gamma$.
b) Given that $1 + 2i$ is a root of the equation, find the other two roots. (NEAB)

17 Derive expressions for the three cube roots of unity in the form $re^{i\theta}$. Represent the roots on an Argand diagram.

Let ω denote one of the non-real roots. Show that the other non-real root is ω^2. Show also that

$$1 + \omega + \omega^2 = 0$$

Given that

$$\alpha = p + q \qquad \beta = p + q\omega \qquad \gamma = p + q\omega^2$$

where p and q are real,

i) find, in terms of p, $\alpha\beta + \beta\gamma + \gamma\alpha$
ii) show that $\alpha\beta\gamma = p^3 + q^3$
iii) find a cubic equation, with coefficients in terms of p and q, whose roots are α, β, γ.

(NEAB)

18 The polynomial $f(z)$ has real coefficients and one root of the equation $f(z) = 0$ is $5 + 4i$. Show that $z^2 - 10z + 41$ is a factor of $f(z)$.

Given now that

$$f(z) = z^6 - 10z^5 + 41z^4 + 16z^2 - 160z + 656,$$

solve the equation $f(z) = 0$, giving each root exactly in the form $a + ib$. (OCR)