3.1 ALGEBRA OF POLYNOMIALS

3.1.1 DEFINITION

A polynomial function, $P(x)$, is an algebraic expression that takes the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0,\ a_n \neq 0$$

where the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers, and the power, $n, n-1, n-2, \ldots$ are integers.

The **degree** of a polynomial, deg$P(x)$, is the highest power of x in the expression.

Examples:

1. $P(x) = -5x^4 + 2x^2 + x - 7$ is a polynomial of degree 4, i.e., deg$P(x) = 4$, with a leading coefficient of –5.

2. $P(x) = 4x^3 + 6x^5 - x^2 + \sqrt{2}$ is a polynomial of degree 5, i.e., deg$P(x) = 5$ with a leading coefficient of 6.

Polynomials can be rewritten in descending powers of x, $P(x) = 6x^5 + 4x^3 - x^2 + \sqrt{2}$.

3. $P(x) = 3x^3 - x^2 + 2\sqrt{x} - 1$ is **not** a polynomial, because not all terms are raised to an integer power.

Some standard polynomials are:

Degree	Name	General form
0	constant	$P(x) = a$
1	linear	$P(x) = ax + b$
2	quadratic	$P(x) = ax^2 + bx + c$
3	cubic	$P(x) = ax^3 + bx^2 + cx + d$
4	quartic	$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

3.1.2 ADDITION AND MULTIPLICATION OF POLYNOMIALS

The standard laws of algebra are readily applied to polynomials. We consider a number of examples to demonstrate the process of addition and multiplication of polynomials.

EXAMPLE 3.1

Consider the polynomials $P(x) = 2x^3 - x + 3$ and $T(x) = x^2 - 3$.

Find (a) $P(x) - 3T(x)$

(b) $P(x) \times T(x)$

(c) $[P(x)]^2$

Solution

(a) $P(x) - 3T(x) = 2x^3 - x + 3 - 3(x^2 - 3)$
$= 2x^3 - x + 3 - 3x^2 + 9$
$= 2x^3 - 3x^2 - x + 12$

(b) $P(x) \times T(x) = (2x^3 - x + 3)(x^2 - 3)$
$= x^2(2x^3 - x + 3) - 3(2x^3 - x + 3)$
$= 2x^5 - x^3 + 3x^2 - 6x^3 + 3x - 9$
$= 2x^5 - 7x^3 + 3x^2 + 3x - 9$

(c) $[P(x)]^2 = (2x^3 - x + 3)^2 = (2x^3 - x + 3)(2x^3 - x + 3)$
$= 2x^3(2x^3 - x + 3) - x(2x^3 - x + 3) + 3(2x^3 - x + 3)$
$= 4x^6 - 2x^4 + 6x^3 - 2x^4 + x^2 - 3x + 6x^3 - 3x + 9$
$= 4x^6 - 4x^4 + 12x^3 + x^2 - 6x + 9$

3.1.3 DIVISION OF POLYNOMIALS

We start by recalling how we can set out a division that involves real numbers. If we consider the problem of dividing 10 by 3, we can quote the results in two ways:

1. $\frac{11}{4} = 2 + \frac{3}{4}$ **2.** $11 = 4 \times 2 + 3$

In either case, '11' is the **dividend**, '4' is the **divisor**, '2' is the **quotient** and '3' is the **remainder**.

We can extend this to include division of polynomials

If $P(x)$ and $D(x)$ are two polynomials over a given field with $\deg P(x) \geq \deg D(x)$ there exist two polynomials $Q(x)$ and $R(x)$ such that

$P(x)$	$=$	$D(x)$	$\times$	$Q(x)$	$+$	$R(x)$
↑		↑		↑		↑
dividend		**divisor**		**quotient**		**remainder**

or $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ where $0 \leq \deg R(x) < \deg D(x)$.

Nb: If $R(x) = 0$ then $D(x)$ is a *factor* of $P(x)$.

The *process* of polynomial division is the same as that of long division of numbers. For example, when dividing 4 into 11, the long division process is set out as follows:

```
       2   (How many 4s go into 11)
  4  | 11
        8  (4 × 2  = 8)
        3  (11 – 8 = 3) [i.e., remainder is 3]
```

Therefore, we have that $\frac{11}{4} = 2 + \frac{3}{4}$ or $11 = 4 \times 2 + 3$

We now consider an example that involves division of two polynomials.

EXAMPLE 3.2

Divide $P(x) = x^3 - 4x^2 + 5x - 1$ by $(x - 2)$

Solution

1 "How many times does x go into x^3, i.e., $x^3 \div x = x^2$"

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3 - 4x^2 + 5x - 1 \end{array}$$

2 "Multiply $(x - 2)$ by x^2, i.e., $x^2 \times (x-2) = x^3 - 2x^2$"

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3 - 4x^2 + 5x - 1 \\ & x^3 - 2x^2 \\ \hline \end{array}$$

3 "Subtract $x^3 - 2x^2$ from $x^3 - 4x^2 + 5x - 1$,
i.e., $(x^3 - 4x^2 + 5x - 1) - (x^3 - 2x^2) = -2x^2 + 5x - 1$"

$$\begin{array}{r|l} & x^2 \\ \hline x-2 & x^3 - 4x^2 + 5x - 1 \\ & x^3 - 2x^2 \\ \hline & \quad -2x^2 + 5x - 1 \end{array}$$

4 "How many times does x go into $-2x^3$, i.e., $-2x^2 \div x = -2x$" (i.e., repeat step 1)

$$\begin{array}{r|l} & x^2 - 2x \\ \hline x-2 & x^3 - 4x^2 + 5x - 1 \\ & x^3 - 2x^2 \\ \hline & \quad -2x^2 + 5x - 1 \end{array}$$

5 "Multiply $(x - 2)$ by $-2x$, i.e., $-2x \times (x-2) = -2x^2 + 4x$" (i.e., repeat step 2)

$$\begin{array}{r|l} & x^2 - 2x \\ \hline x-2 & x^3 - 4x^2 + 5x - 1 \\ & x^3 - 2x^2 \\ \hline & \quad -2x^2 + 5x - 1 \\ & \quad -2x^2 + 4x \\ \hline \end{array}$$

6 "Subtract $-2x^2 + 4x$ from $-2x^2 + 5x - 1$, i.e., $(-2x^2 + 5x - 1) - (-2x^2 + 4x) = x - 1$"
(i.e., repeat step 3)

$$\begin{array}{r|l} & x^2 - 2x \\ \hline x-2 & x^3 - 4x^2 + 5x - 1 \\ & x^3 - 2x^2 \\ \hline & \quad -2x^2 + 5x - 1 \\ & \quad -2x^2 + 4x \\ \hline & \qquad\qquad x - 1 \end{array}$$

7 "How many times does x go into x, i.e., $x \div x = 1$" (i.e., repeat step 1)

$$
\begin{array}{r|l}
 & x^2 - 2x + 1 \\
\hline
x - 2 & x^3 - 4x^2 + 5x - 1 \\
 & x^3 - 2x^2 \\
\hline
 & \quad - 2x^2 + 5x - 1 \\
 & \quad - 2x^2 + 4x \\
\hline
 & \qquad\qquad x - 1
\end{array}
$$

8 "Multiply $(x - 2)$ by 1, i.e., $1 \times (x - 2) = x - 2$" (i.e., repeat step 2)

$$
\begin{array}{r|l}
 & x^2 - 2x + 1 \\
\hline
x - 2 & x^3 - 4x^2 + 5x - 1 \\
 & x^3 - 2x^2 \\
\hline
 & \quad - 2x^2 + 5x - 1 \\
 & \quad - 2x^2 + 4x \\
\hline
 & \qquad\qquad x - 1 \\
 & \qquad\qquad x - 2 \\
\hline
\end{array}
$$

6 "Subtract $(x - 2)$ from $(x - 1)$, i.e., $(x - 1) - (x - 2) = 1$" (i.e., repeat step 3)

$$
\begin{array}{r|l}
 & x^2 - 2x + 1 \\
\hline
x - 2 & x^3 - 4x^2 + 5x - 1 \\
 & x^3 - 2x^2 \\
\hline
 & \quad - 2x^2 + 5x - 1 \\
 & \quad - 2x^2 + 4x \\
\hline
 & \qquad\qquad x - 1 \\
 & \qquad\qquad x - 2 \\
\hline
 & \qquad\qquad\quad 1
\end{array}
$$

And so, we have that $P(x) = (x - 2)(x^2 - 2x + 1) + 1$ or $\frac{P(x)}{x - 2} = x^2 - 2x + 1 + \frac{1}{x - 2}$.

That is, we have a quotient $x^2 - 2x + 1$ and remainder 1.

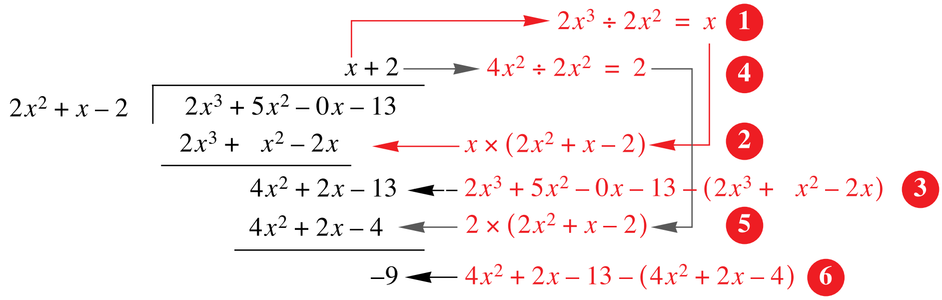
Note that we are always dividing the current expression by the highest degree term in the divisor, $(x - 2)$. Also note that the process is stopped when the degree of the current dividend becomes less than that of the divisor, *i.e.* $\deg(1) < \deg(x - 2)$.

Although the process seems to be very long, after a little practice you will be able to carry out the process efficiently.

EXAMPLE 3.3

Divide $2x^3 + 5x^2 - 13$ by $2x^2 + x - 2$.

Solution



We note that deg(–9) = 0 and deg($2x^2 + x - 2$) = 2. Then, as deg(–9) < deg($2x^2 + x - 2$) we stop the division process at this stage.

$$\therefore\ 2x^3 + 5x^2 - 13 = (2x^2 + x - 2)(x + 2) - 9 \text{ or } \frac{2x^3 + 5x^2 - 13}{2x^2 + x - 2} = x + 2 + \frac{-9}{2x^2 + x - 2}.$$

That is, the quotient is $(x + 2)$ and the remainder is –9.

Notice also how we have created a 'place holder' by including '$0x$' in the dividend term of our division process. This helps avoid arithmetic mistakes when subtracting terms.

Notice that it is not always the case that there is a constant remainder (although that will be true whenever we divide a polynomial by a linear polynomial). For example, dividing the polynomial $2x^4 + 3x^3 + 5x^2 - 13$ by the quadratic $2x^2 + x - 2$ we obtain

$$\frac{2x^4 + 3x^3 + 5x^2 - 13}{2x^2 + x - 2} = x^2 + x + 3 - \frac{x + 7}{2x^2 + x - 2}$$

So that this time the remainder is a linear polynomial, $-x - 7$.

EXERCISES 3.1

1. Given the polynomials $P(x) = x^4 - x^3 + 2x - 1$, $Q(x) = 3 - x + 2x^3$ and $T(x) = 3x^2 - 2$ evaluate

(a) $2T(x) - Q(x)$ (b) $P(x) + 4T(x)$ (c) $T(x) \times Q(x)$

(d) $P(x)Q(x)$ (e) $[Q(x)]^2$ (f) $[T(x)]^2 - 9P(x)$

2. Divide $3x^2 - 2x + 1$ by $(x - 1)$

3. Divide $4x^3 - 8x^2 + 25x - 19$ by $(2x - 1)$

4. Divide $x^4 - 3x^3 + x + 4$ by $x^2 - 2x + 3$

5. Divide $2x^3 - 5x^2 - 10$ by $(x + 1)$

6. Divide $x^4 + 2x^2 - x$ by $x^2 + 3$

7. Divide $12 + 19x - 7x^2 - 6x^3$ by $(3x - 1)$

8. When $x^3 - 2x + k$ is divided by $(x - 2)$ it leaves a remainder of 5, find k.

9. When $x^3 - 2x + k$ is divided by $(x + 1)$ it leaves a remainder of 0, find k.

10. When $2x^3 - x^2 + kx - 4$ is divided by $(x + 2)$ it leaves a remainder of 0, find k.

11. When $2x^3 - x^2 + kx - 4$ is divided by $(x - 1)$ it leaves a remainder of 2, find k.

3.2 SYNTHETIC DIVISION

Suppose that the polynomial, $A(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is divided by $(x - k)$ so that it results in a quotient $Q(x) = b_2x^2 + b_1x + b_0$ and a constant remainder R. Then, we can write this result as $\frac{A(x)}{x-k} = Q(x) + \frac{R}{x-k}$ or $A(x) = (x-k)Q(x) + R$

Substituting the polynomial terms into the second form we have:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = (x-k)(b_2x^2 + b_1x + b_0) + R$$

$$\Leftrightarrow a_3x^3 + a_2x^2 + a_1x + a_0 = b_2x^3 + (b_1 - kb_2)x^2 + (b_0 - kb_1)x + (R - kb_0)$$

Then, equating the coefficients on both sides of the equation, we have:

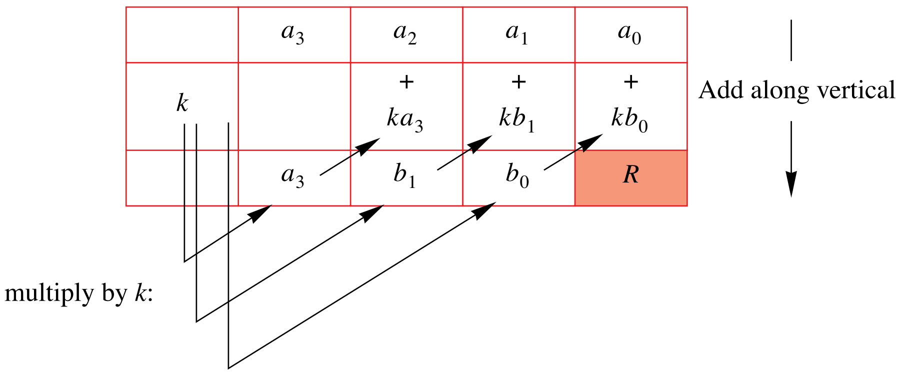
$$a_3 = b_2 \qquad \Rightarrow b_2 = a_3$$

$$a_2 = b_1 - kb_2 \qquad \Rightarrow b_1 = a_2 + kb_2 = a_2 + ka_3$$

$$a_1 = b_0 - kb_1 \qquad \Rightarrow b_0 = a_1 + kb_1$$

$$a_0 = R - kb_0 \qquad \Rightarrow R = a_0 + kb_0$$

Given the 'recursive' nature of this result we can set it up in a table form as follows



Although we have only shown the process for a polynomial of degree 3, this works for any polynomial. This quick method of dividing polynomials by $(x - k)$ is known as ***synthetic division***. This method relies on the relationships between the coefficients of x in the product of the quotient and divisor.

EXAMPLE 3.4

Divide $x^3 + 2x^2 - 3x + 4$ by $(x - 2)$

Solution

The coefficients of the polynomial are: 1 2 –3 4 and the value of k is 2.
Setting up the table to make use of synthetic division we have:

	1	**2**	**–3**	**4**
2				

The next step is to place the leading coefficient in the last row:

	1	**2**	**–3**	**4**
2				
	1			

Then, we multiply 1 by 2 (i.e., k) giving an answer of 2 and placing as shown:

	1	**2**	**–3**	**4**
2		**2**		
	1			

Next, add 2 + 2 = 4 as shown:

	1	**2**	**–3**	**4**
2		**2**		
	1	**4**		

Multiply 4 by 2 (i.e., k) giving an answer of 8 and placing as shown:

	1	**2**	**–3**	**4**
2		**2**	**8**	
	1	**4**		

Next, add –3 + 8 = 5 as shown:

	1	**2**	**–3**	**4**
2		**2**	**8**	
	1	**4**	**5**	

Multiply 5 by 2 (i.e., k) giving an answer of 10 and placing as shown:

	1	**2**	**–3**	**4**
2		**2**	**8**	**10**
	1	**4**	**5**	

Next, add 4 + 10 = 14 as shown:

	1	**2**	**–3**	**4**
2		**2**	**8**	**10**
	1	**4**	**5**	**14**

Then, the coefficients in the last row are the coefficients of the quotient and the last number corresponds to the remainder. This means that the quotient has a constant term of 5, the coefficient of x is 4, the coefficient of x^2 is 1 and has a remainder of 14.

Therefore, we have that $\dfrac{x^3 + 2x^2 - 3x + 4}{x - 2} = x^2 + 4x + 5 + \dfrac{14}{x - 2}$.

EXAMPLE 3.5

Divide $3x^5 - 8x^4 + x^2 - x + 3$ by $(x - 2)$.

Solution

We can now 'fast track' the process into one table. Using the coefficients of the polynomial in the top row we have:

	3	–8	0	1	–1	3
2		6	–4	–8	–14	–30
	3	–2	–4	–7	–15	–27

And so $\frac{3x^5-8x^4+x^2-x+3}{x-2} = 3x^4-2x^3-4x^2-7x-15-\frac{27}{x-2}$

or $3x^5-8x^4+x^2-x+3=(x-2)(3x^4-2x^3-4x^2-7x-15)-27$

EXAMPLE 3.6

Divide $2x^4+x^3-x^2+5x+1$ by $(x+3)$.

Solution

Constructing the appropriate table with the coefficients 2; 1; –1; 5; 1 in the first row and with $k=-3$, we have:

	2	1	–1	5	1
–3		–6	15	–42	111
	2	–5	14	–37	112

And so $\frac{2x^4+x^3-x^2+5x+1}{x+3} = 2x^3-5x^2+14x-37+\frac{112}{x+3}$

or $2x^4+x^3-x^2+5x+1=(x+3)(2x^3-5x^2+14x-37)+112$

In Example 3.6, had we wanted to divide by $(2x-3)$ then, we would have used $k = \frac{3}{2}$. The reason being that $2x-3 = 2\left(x-\frac{3}{2}\right)$ and to use synthetic division, we must divide by $x-k$. This also means that if we had divided by $(2x+3)$ then $k = -\frac{3}{2}$.

EXERCISES 3.2

Use Synthetic Division to find the quotients and remainders below.

1. $2x^2-5x+1 \div (x-1)$

2. $3x^3+x^2-x+3 \div (x-3)$

3. $2x^4-x^3-2x^2+3 \div (x+3)$

4. $2x^3-5x^2+10x-3 \div (2x-1)$

5. $2x^4+3x^2-x \div (x+2)$

6. $5-2x^2-x^4 \div (4-x)$

3.3 THE REMAINDER THEOREM

For any polynomial $P(x)$, the remainder when divided by $(x - \alpha)$ is $P(\alpha)$.

PROOF

The degree of the remainder $R(x)$ must be less than the degree of the divisor $D(x)$. Therefore if $D(x)$ has degree = 1, $R(x)$ has degree = 0 and is therefore constant.

$\therefore$ if $\quad P(x) = D(x) \times Q(x) + R$ and $D(x) = (x - \alpha)$ then

$P(x) = (x - \alpha)\, Q(x) + R \quad$ (where R is a constant)

when $x = \alpha$, $\quad P(\alpha) = (\alpha - \alpha)\, Q(x) + R$

$\therefore \quad P(\alpha) = R$

i.e. the remainder on division of $P(x)$ by $(x - \alpha)$ is $P(\alpha)$.

We start by considering Examples 3.2 and 3.6.

Example 3.2: With $P(x) = x^3 - 4x^2 + 5x - 1$ and the divisor $(x - 2)$, we have

$P(2) = (2)^3 - 4(2)^2 + 5(2) - 1 = 8 - 16 + 10 - 1 = 1$

So, $P(2) = 1 \Rightarrow$ remainder is 1

Which agrees with the remainder we had obtained.

Example 3.6: With $P(x) = 2x^4 + x^3 - x^2 + 5x + 1$ and the divisor $(x + 3)$, we have

$P(-3) = 2(-3)^4 + (-3)^3 - (-3)^2 + 5(-3) + 1$

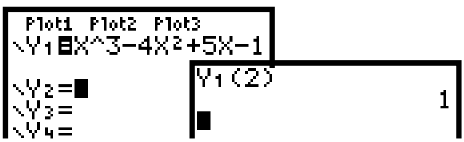
$= 112$

So, $P(-3) = 112 \Rightarrow$ remainder is 112

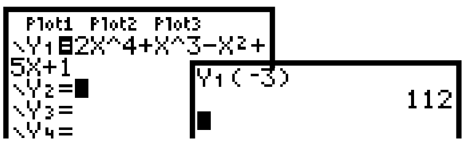
Which agrees with the remainder we had obtained.

Because of the nature of the arithmetic involved in evaluating such problems it is a good idea to make use of a graphics calculator. Using the TI–83 we first enter the equation and then evaluate the polynomial using the required value of k:

Example 3.2



Example 3.6



EXAMPLE 3.7

Find the remainder when $P(x) = x^3 + x^2 - x + 1$ is divided by $(x + 1)$.

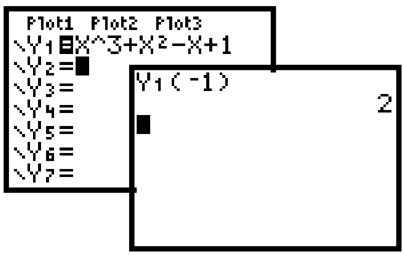
Solution

$P(x) = x^3 + x^2 - x + 1$

$P(-1) = (-1)^3 + (-1)^2 - (-1) + 1$

$= 2$

$\therefore$ the remainder when $P(x)$ is divided by $(x + 1)$ is 2.



EXAMPLE 3.8

Find the remainders when $P(x) = 3x^3 - 2x^2 + 7x - 4$ is divided by:

(a) $(x-1)$ (b) $(x+3)$ (c) $(3x-2)$ (d) $(2x+1)$

Solution

Let R be the remainder in each case.

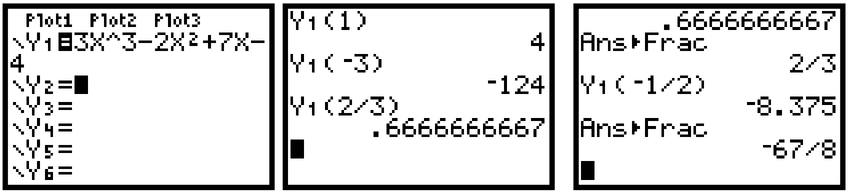
(a) $R = P(1) = 3(1)^3 - 2(1)^2 + 7(1) - 4 = 4$

(b) $R = P(-3) = 3(-3)^3 - 2(-3)^2 + 7(-3) - 4 = -124$

(c) $R = P\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 + 7\left(\frac{2}{3}\right) - 4 = \frac{2}{3}$

(d) $R = P\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right)^2 + 7\left(-\frac{1}{2}\right) - 4 = -\frac{67}{8}$

Using the TI–83 we have:



EXERCISES 3.3

1. Find the remainder when

(a) $P(x) = -x^3 + 2x^2 + 3$ is divided by $(x-3)$.

(b) $Q(x) = 2x^3 - 12x + 7$ is divided by $(x+2)$.

(c) $P(x) = 6 + x - x^2 + 2x^4$ is divided by $(x-1)$.

(d) $P(x) = 4x^3 + 3x^2 - 2x + 1$ is divided by $(2x-1)$.

(e) $Q(x) = x^3 - 4x^2 + x - 3$ is divided by $(3x+2)$.

2. Find the value of k if the remainder of $x^3 + kx^2 - x + 2$ when divided by $x+2$ is 20.

3. Find the remainder when $4x^4 + 3x^2 + 2x - 2$ is divided by

(a) $x-1$

(b) $x+2$

(c) $(x-1)(x+2)$

4. When $2x^3 + ax^2 + bx + 1$ is divided by $(x-1)$ and $(x-2)$, the remainders are 4 and 15 respectively. Find the remainder when it is divided by $(x+1)$.

5. Find the value of k if $P(x)$ is exactly divisible by $d(x)$, where

(a) $P(x) = x^3 - 6x^2 + kx - 6$ and $d(x) = x - 3$

(b) $P(x) = 2x^3 - kx^2 + 1$ and $d(x) = 2x + 1$.

3.4 THE FACTOR THEOREM

$(x - \alpha)$ is a factor of $P(x)$ if and only if $P(\alpha) = 0$

That is, if $(x - \alpha)$ is a **factor** of $P(x)$ then the **remainder** $R = P(\alpha) = 0$.
And, if $P(\alpha) = 0$ then $(x - \alpha)$ is a **factor** of $P(x)$.

PROOF

By the remainder theorem, $P(x) = (x - \alpha) \times Q(x) + R$ for all real x

$\therefore P(\alpha) = R$

but if $P(\alpha) = 0$ *i.e.* $R = 0$ then $P(x) = (x - \alpha)Q(x) + 0$

$= (x - \alpha)Q(x)$

i.e. $(x - \alpha)$ is a factor of $P(x)$

EXAMPLE 3.9

Determine which of $(x - 3)$, $(x - 1)$, $(x + 2)$ are factors of $P(x) = 2x^3 + 7x^2 + 7x + 2$, and hence factorise $P(x)$ completely.

Solution

For $(x - 3)$, $P(3) = 2(3)^3 + 7(3)^2 + 7(3) + 2 \neq 0$, thus $(x - 3)$ is not a factor of $P(x)$.
For $(x - 1)$, $P(1) = 2(1)^3 + 7(1)^2 + 7(1) + 2 \neq 0$, thus $(x - 1)$ is not a factor of $P(x)$
For $(x + 2)$, $P(-2) = 2(-2)^3 + 7(-2)^2 + 7(-2) + 2 = 0$, thus $(x + 2)$ is a factor of $P(x)$
Next, we divide (x + 2) into $2x^3 + 7x^2 + 7x + 2$:

	2	7	7	2
–2		–4	–6	–2
	2	3	1	0

Giving a quotient of $2x^2 + 3x + 1$.

Therefore, $2x^3 + 7x^2 + 7x + 2 = (x + 2)(2x^2 + 3x + 1)$

$= (x + 2)(2x + 1)(x + 1)$

EXAMPLE 3.10

Determine m and n so that $3x^3 + mx^2 - 5x + n$ is divisible by both $(x - 2)$ and $(x + 1)$. Factorise the resulting polynomial completely.

Solution

By the factor theorem, $(x - \alpha)$ is a factor of $P(x)$ if $P(\alpha) = 0$

$\therefore P(2) = 24 + 4m - 10 + n$

As $P(2) = 0$, we have that $24 + 4m - 10 + n = 0$

i.e. $4m + n = -14$ – (1)

$P(-1) = -3 + m + 5 + n$

As $P(-1) = 0$, we have that $-3 + m + 5 + n = 0$

i.e. $m + n = -2$ – (2)

Solving (1) and (2) simultaneously we have:

(1) – (2): $3m = -12 \therefore m = -4$

Substituting into (2): $-4 + n = -2 \therefore n = 2$
Hence $P(x) = 3x^3 - 4x^2 - 5x + 2$.

As we already know that $(x + 1)$ and $(x - 2)$ are factors of $P(x)$, we also have that $(x + 1)(x - 2)$ is a factor of $P(x)$. All that remains then is to find the third factor. As the third factor is also a linear term, we use the general linear term $(ax + b)$ and determine the value of a and b.

We write our polynomial as $P(x) = (x + 1)(x - 2)(ax + b)$.
To find a and b we look at the coefficient of x^3 and the constant term.
i.e., $P(x) = (x^2 - x - 2)(ax + b) = (ax^3 + \ldots - 2b)$.
Then, as $P(x) = 3x^3 - 4x^2 - 5x + 2$ we have that $3x^3 - 4x^2 - 5x + 2 \equiv ax^3 + \ldots - 2b$.
Meaning that, $3 = a$ and $2 = -2b \Leftrightarrow b = -1$.

$$\therefore 3x^3 - 4x^2 - 5x + 2 = (x + 1)(x - 2)(3x - 1)$$

In the last Example we made use of the 'identically equivalent to' notation, i.e., '≡'. You need to be aware that this is not the same as when we use the equality sign, '='.

For example, we have that $(x+2)^2 \equiv x^2 + 4x + 4$, because this statement will be true for any value of x. However, we **cannot** have that $(x+2)^2 \equiv x^2 + 3x + 4$, because this will not be true for all values of x. In fact it will only be true for one value of x (namely, $x = 0$). In such situations what we really want to know is 'For what value(s) of x will $(x+2)^2 = x^2 + 3x + 4$?' Meaning that we need to solve for x. Notice then that when we use the '=' sign we really want to solve for the unknown whereas when we use the '≡' sign we are making a statement.

Having said this, the distinction between '≡' and '=' is not always adhered to when presenting a mathematical argument. So, when expanding the term $(x+2)^2$, rather than writing $(x+2)^2 \equiv x^2 + 4x + 4$, more often than not, it will be written as $(x+2)^2 = x^2 + 4x + 4$. In these situations, the meaning attached to the '=' sign will be clear from the context of the problem.

A useful extension of the factor theorem allows us to find factors of any polynomial $P(x)$, if they exist.

> Given a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, then $P(x)$ has a factor $(px - q)$ if and only if p is a factor of a_n and q is a factor of a_0.

This result is useful in helping us guess potential factors of a given polynomial. For example, given a polynomial $P(x) = 2x^3 - 3x^2 + 1$, we would try (as potential factors) the following $(2x+1)$, $(2x-1)$, $(x-1)$ and $(x+1)$.

EXAMPLE 3.11

Factorise the polynomial $T(x) = x^3 - 3x^2 + 4$.

Solution

$T(x)$ can be factorised if we can find a factor $(px - q)$ where p is a factor of 1 and q is a factor of 4.

Factors of 1 are 1×1 and factors of 4 are $\pm 1 \times \pm 4$ and $\pm 2 \times \pm 2$, so *possible* factors of $T(x)$ are $(x \pm 1)$, $(x \pm 2)$, and $(x \pm 4)$.

Using the factor theorem;

for $(x - 1)$, $T(1) = 1 - 3 + 4 \neq \mathbf{0}$, thus $(x - 1)$ is **not a factor** of $T(x)$

for $(x + 1)$, $T(-1) = -1 - 3 + 4 = \mathbf{0}$, thus $(x + 1)$ **is a factor** of $T(x)$.

Having found one factor it is now possible to divide $T(x)$ by $(x + 1)$ to find all other factors. Using synthetic division we have:

	1	–3	0	4
–1		–1	4	–4
	1	–4	4	0

Therefore, $x^3 - 3x^2 + 4 = (x+1)(x^2 - 4x + 4)$

$= (x+1)(x-2)^2$

EXAMPLE 3.12

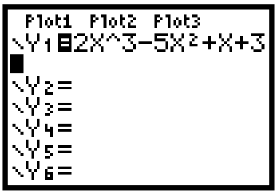
Factorise $P(x) = 2x^3 - 5x^2 + x + 3$.

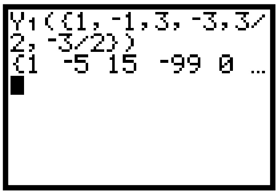
Solution

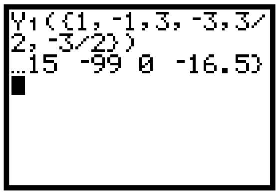
Let $(px - q)$ be a factor of $P(x)$ where p is a factor of 2, and q is a factor of 3.

Factors of 2 are 1×2, and factors of 3 are 1×3 or -1×-3 thus possible factors of $P(x)$ are $(x \pm 1)$, $(x \pm 3)$, $(2x \pm 1)$, $(2x \pm 3)$.

Using the factor theorem with each of these leads to $(2x - 3)$ as a factor of $P(x)$. Of course, using the TI–83 makes life easier:







From the above screens we see that $P\left(\frac{3}{2}\right) = 0 \Leftrightarrow (2x-3)$ is a factor.

Then dividing $P(x)$ by $(2x - 3)$ gives: $2x^3 - 5x^2 + x + 3 = (2x - 3)(x^2 - x - 1)$.

Next, $x^2 - x - 1 = \left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} - 1 = \left(x - \frac{1}{2}\right)^2 - \frac{5}{4} = \left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$.

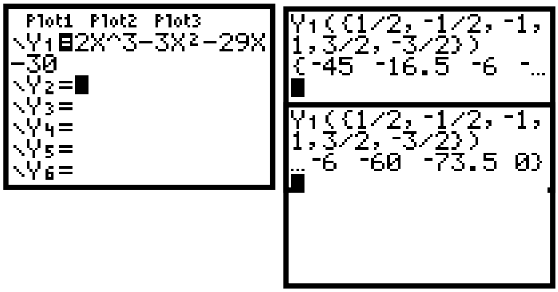
Therefore, $P(x) = (2x-3)\left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

EXAMPLE 3.13

Factorise $P(x) = 2x^3 - 3x^2 - 29x - 30$.

Solution

We start by 'guessing' factors:
Note that we have not tried all possible combinations, just enough to get the first factor.



Then, as $P\left(-\frac{3}{2}\right) = 0$, $(2x+3)$ is a factor.

Using synthetic division we next have:

	2	–3	–29	–30
$-\frac{3}{2}$		–3	9	30
	2	–6	–20	0

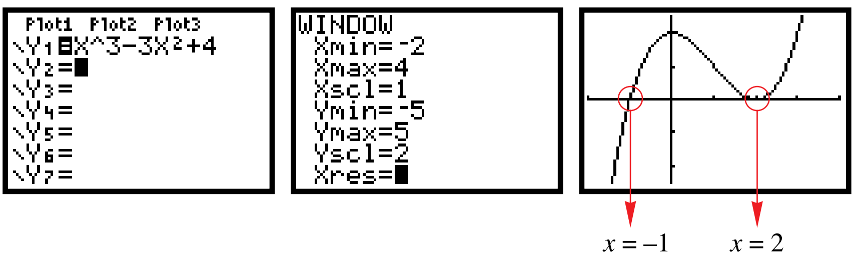
Therefore, $P(x) = \frac{1}{2}(2x+3)(2x^2-6x-20)$ Nb: we have $\frac{1}{2}$, because of the $2\left(x+\frac{3}{2}\right)$ term.

$$= \frac{1}{2}(2x+3)(2x-10)(x+2)$$

$$= (2x+3)(x-5)(x+2)$$

Although we will look at sketching polynomials later on in this chapter, it is worthwhile mentioning at this point that we can reduce the 'guessing game' when looking for factors by looking at where the graph of the polynomial cuts the x–axis.

In Example 3.11, using a graphics calculator to sketch the graph of $T(x) = x^3 - 3x^2 + 4$ gives:



This tells us that $T(x) = 0$ when $x = -1$ and $x = 2$, which in turn means that $(x + 1)$ and $(x - 2)$ are both factors of $T(x)$. In fact, given that the curve touches at $x = 2$, it also tells us that we have a repeated factor, i.e., there are two factors $(x - 2)$. From this information and the fact that the leading coefficient is one, we can then write $T(x) = (x+1)(x-2)^2$.

If the leading coefficient was not one, we would have to write $T(x) = k(x+1)(x-2)^2$. Why?

EXERCISES 3.4

1. Factorise fully, the following polynomials

(a)	$x^3 - 19x + 30$	(b)	$x^3 + x^2 - x - 10$
(c)	$x^3 - x^2 - 4x + 4$	(d)	$3x^3 + x^2 - 12x - 4$
(e)	$2x^3 - x^2 - 18x + 9$	(f)	$x^4 - 3x^2 - 6x + 8$
(g)	$x^3 - x^2 - 8x + 12$	(h)	$5x^3 - 24x^2 + 36x - 16$
(i)	$40 - 19x^2 + 94x - 10x^3$	(j)	$-5x^3 - 9x^2 - 3x + 1$

MISCELLANEOUS EXERCISES

2. The polynomial $P(x)$ is divided by $2x^2 - 1$, resulting in a quotient $3x - 2$ and a remainder $(x + 1)$. Find $P(x)$.

3. Find the remainder when $2x^4 - 2x^2 + x - 5$ is divided by $(2x + 1)$.

4. Find the remainder when $p(x) = x^3 + 2x^2 - 11x - 12$ is divided by $(x + 4)$. Hence factorise $p(x)$.

5. Factorise $g(x) = 2x^3 + 9x^2 + 12x + 4$.

6. Factorise $m(x) = x^3 - 4x^2 - 3x - 10$.

7. Factorise $f(x) = 6x^4 - 11x^3 + 2x^2 + 5x - 2$. Noting that $f(x) = 0$ represents the points where $f(x)$ crosses or touches the x-axis, sketch the graph of $f(x)$.

8. (a) Factorise the polynomial $2 - 13x + 23x^2 - 3x^3 - 9x^4$.
(b) Find all values of x for which $2 - 13x + 23x^2 - 3x^3 - 9x^4 = 0$.
(c) Sketch the graph of the polynomial $q(x) = 2 - 13x + 23x^2 - 3x^3 - 9x^4$.

9. Find the values of a and b if $6x^3 + 7x^2 + ax + b$ is divisible by $(2x - 1)$ and $(x + 1)$.

10. Show that the graph of $y = 2x^3 - 3x^2 + 6x + 4$ cuts the x-axis at only one point.

11. $x^3 + ax^2 - 2x + b$ has $(x + 1)$ as a factor, and leaves a remainder of 4 when divided by $(x - 3)$. Find a and b.

12. Show that $p(x) = 2x^3 - 5x^2 - 9x - 1$ has no factors of the form $(x - k)$, where k is an integer.

13. Given that $(x - 1)$ and $(x - 2)$ are factors of $6x^4 + ax^3 - 17x^2 + bx - 4$, find a and b, and any remaining factors.

14. A cubic polynomial gives remainders $(5x + 4)$ and $(12x - 1)$ when divided by $x^2 - x + 2$ and $x^2 + x - 1$ respectively. Find the polynomial.

15. A cubic polynomial gives remainders $(13x - 2)$ and $(-1 - 7x)$ when divided by $x^2 - x - 3$ and $x^2 - 2x + 5$ respectively. Find the polynomial.

16. Given that $x + 1$ is a factor of $ax^3 + bx^2 + cx + d$, find the relationship between a, b, c and d.

17. Given that $x - 3$ is a factor of $P(x) = x^3 + kx^2 - x - 6$, find the remainder when $P(x)$ is divided by $x - 2$.

18. Given that $x^3 \equiv a(x+1)^3 + b(x+1)^2 + c(x+1) + d$, find the values of a, b, c and d.

19. Factorise the polynomial $T(x) = ax^3 - 5x^2 + bx - 4$ given that when divided by $x + 1$ it leaves a remainder of –3 and that $x - 2$ is a factor.

20. Given that $P(x) = 2x^4 + mx^3 - nx^2 - 7x + k$ is divisible by $(x - 2)$ and $(x + 3)$ and leaves a remainder of –18 when divided by $(x + 1)$.

(a) Solve for m, n and k.

(b) Hence, find all linear factors of $P(x)$.

21. $P(x) = x^3 + mx^2 + nx + k$ is divisible by $x^2 - 4$ and leaves a remainder of 30 when divided by $(x - 3)$. Solve for m, n and k and hence fully factorise $P(x)$ into its three linear factors.

22. The remainders when $T(x) = kx^n - 3x^2 + 6$ is divided by $(x - 1)$ and $(x + 2)$ are 1 and 10 respectively. Find k and n.

23. Show that if $P(x) = x^4 + mx^2 + nx + k^2$ is divisible by $x^2 - 1$, then $P(x)|x^2 - k^2$.

24. If the polynomial $P(x) = x^2 + ax + 1$ is a factor of $T(x) = 2x^3 - 16x + b$, find the values of a and b.

25. The polynomial $P(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are all integers. If p and q are two relatively prime integers, show that if $(qx - p)$ is a factor of $P(x)$, then p is a factor of d and q is a factor of a.

26. When a polynomial, $P(x)$, is divided by $x - \alpha$, it leaves a remainder of α^3 and when it is divided by $x - \beta$ it leaves a remainder of β^3. Find the remainder when $P(x)$ is divided by $(x - \alpha)(x - \beta)$.

27. (a) Prove that if $x^3 + mx + n$ is divisible by $(x - k)^2$, then $\left(\frac{m}{3}\right)^3 + \left(\frac{n}{2}\right)^2 = 0$

(b) Prove that if $x^3 + mx + n$ and $3x^2 + m$ have a common factor $(x - k)$ then $4m^3 + 27n^2 = 0$.

28. Prove that $P(x) = x^n - a^n$ is divisible by $(x - a)$ for all integer values of n.

3.5 EQUATIONS & INEQUATIONS

3.5.1 POLYNOMIAL EQUATIONS

The factor theorem has some very useful consequences, one of which allows us to solve equations of the form $P(x) = 0$.

> The expression
>
> $$P(x) = 0 \text{ where } P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$
>
> is called a **polynomial equation**.
>
> The **roots** or **solutions** of this equation are the zeros of $P(x)$.

We have already seen how to solve linear equations, $ax + b = 0$ and quadratic equations, $ax^2 + bx + c = 0$, in Chapter 2. Making use of the factor theorem we can solve (where solutions exist) for a polynomial using the following steps:

Step 1: Factorise the polynomial using the factor theorem (if necessary).
Step 2: Use the null factor law.
Step 3: Solve for the unknown.

EXAMPLE 3.14

Solve $x^3 + 4x^2 + 3x = 0$.

Solution

In this case we start by factoring the 'x' out:

$$x^3 + 4x^2 + 3x = 0 \Leftrightarrow x(x^2 + 4x + 3) = 0$$
$$\Leftrightarrow x(x+3)(x+1) = 0 \text{ [factorised polynomial]}$$
$$\Leftrightarrow x = 0 \text{ or } x + 3 = 0 \text{ or } x + 1 = 0 \text{ [Using null factor law]}$$
$$\Leftrightarrow x = 0 \text{ or } x = -3 \text{ or } x = -1 \text{ [Solving for } x\text{]}$$

EXAMPLE 3.15

Solve $2x^3 + 9x^2 + 7x - 6 = 0$.

Solution

Let $P(x) = 2x^3 + 9x^2 + 7x - 6$.
Then, $P(1) = 2 + 9 + 7 - 6 \neq 0$
$P(-1) = -2 + 9 - 7 - 6 \neq 0$
$P(-2) = -16 + 36 - 14 - 6 = 0$. Therefore, $x + 2$ is a factor of $P(x)$.
Using synthetic division we have:

	2	9	7	–6
–2		–4	–10	6
	2	5	–3	0

Therefore, $P(x) = (x+2)(2x^2+5x-3)$

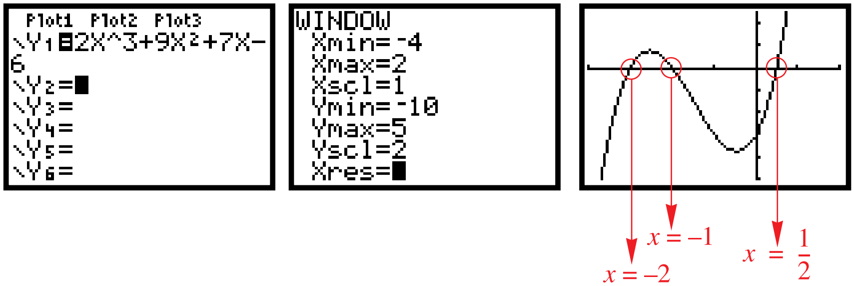
$= (x+2)(2x-1)(x+3)$

Then, $P(x) = 0 \Leftrightarrow (x+2)(2x-1)(x+3) = 0$

$\Leftrightarrow x+2 = 0$ or $2x-1 = 0$ or $x+3 = 0$

$\Leftrightarrow x = -2$ or $x = \frac{1}{2}$ or $x = -3$

Again, we can take some of the guess work out of problems such as these by making use of the graphics calculator. In Example 3.15 we could have sketched the graph of $2x^3 + 9x^2 + 7x - 6$ and referred to where it meets the x–axis:



From the graph we have that $P(x) = 0$ at $x = -2$ so that $(x+2)$ is a factor of $P(x)$. From here we can then proceed to use synthetic division, fully factorise $P(x)$ and then solve for x as we did in the example.

In fact, having judiciously selected the settings on the Window screen of the graphics calculator, we have managed to obtain all three solutions without the need of further work!

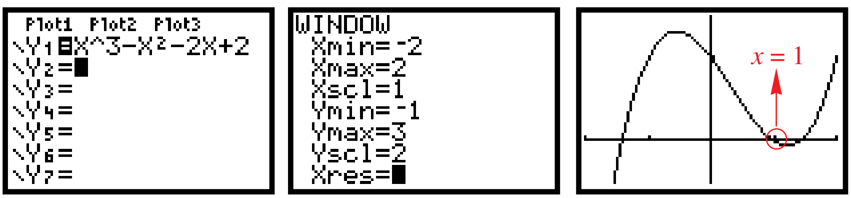
However, as the next example will show, we cannot always rely on the graphics calculator to determine all the solutions to a polynomial equation.

EXAMPLE 3.16

Solve $x^3 - x^2 - 2x + 2 = 0$.

Solution

We start by using the graphics calculator:



This time we can only obtain one obvious solution to $x^3 - x^2 - 2x + 2 = 0$, namely $x = 1$.

That is, if $P(x) = x^3 - x^2 - 2x + 2$ then $P(1) = 0 \Leftrightarrow x - 1$ is a factor.

Using synthetic or long division we have $P(x) = (x-1)(x^2-2)$

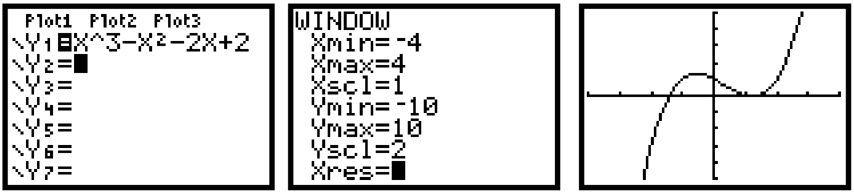
$= (x-1)(x-\sqrt{2})(x+\sqrt{2})$

Therefore, $P(x) = 0 \Leftrightarrow (x-1)(x-\sqrt{2})(x+\sqrt{2}) = 0$

$$\Leftrightarrow x = 1 \text{ or } x = \sqrt{2} \text{ or } x = -\sqrt{2}.$$

Note then that the reason we couldn't obtain obvious solutions using the graphics calculator is that the other solutions are irrational.

Also, we need to be careful when setting the Window screen. The settings below could lead to a false assumption (namely that there are repeated roots):



EXAMPLE 3.17

Solve $x^2 + 11 = 6x + \frac{6}{x}$.

Solution

Solving an equation such as this requires that we first eliminate the fractional part. So, multiplying both sides by x gives:

$$x(x^2+11) = x\left(6x + \frac{6}{x}\right) \Leftrightarrow x^3 + 11x = 6x^2 + 6$$

$$\Leftrightarrow x^3 - 6x^2 + 11x - 6 = 0$$

Next we let $P(x) = x^3 - 6x^2 + 11x - 6$, then, either using the graphics calculator or guessing a zero of $P(x)$ we have that $P(1) = 0$ and so, $(x-1)$ is a factor. Then, using synthetic or long division we have,

$$P(x) = (x-1)(x^2-5x+6)$$
$$= (x-1)(x-3)(x-2)$$

Therefore, $x^2 + 11 = 6x + \frac{6}{x} \Leftrightarrow (x-1)(x-3)(x-2) = 0 \therefore x = 1$ or $x = 3$ or $x = 2$.

EXERCISES 3.5.1

1. Solve the following over the real number field.

(a)	$x^3 + 2x^2 - 5x - 6 = 0$	(b)	$2x^3 - 5x^2 + x + 2 = 0$
(c)	$-x^3 + 7x + 6 = 0$	(d)	$6x^3 - 31x^2 + 25x + 12 = 0$
(e)	$2x^3 + 3x^2 + 4x - 3 = 0$	(f)	$-x^3 + x^2 + 5x + 3 = 0$
(g)	$x^3 + 2x^2 - 7x + 4 = 0$	(h)	$x^3 - 2x^2 - 14x - 12 = 0$
(i)	$2x^3 + 9x^2 + 8x + 2 = 0$	(j)	$3x^3 - x^2 - 18x + 6 = 0$

2. Solve the equation $2x^3 - 13x^2 + 16x - 5 = 0$.

3. Solve the equation $4x^3 - 4x^2 - 11x + 6 = 0$.

4. Solve the equation $x^4 - 3x^3 - 3x^2 + 7x + 6 = 0$.

5. Solve the equation $x^4 + x^3 - 9x^2 + 11x - 4 = 0$.

6. (a) Write down the equation of a polynomial with zeros –2, 3 and –4.
(b) Write down the equation of a polynomial with zeros 0.5, 2 and –1.
(c) Write down the equation of a polynomial with zeros 0.5, 2 and –0.5 and a leading coefficient of 8.

7. Solve the equations

(a) $6x^2 + x = 19 - \frac{12}{x}$ (b) $x^2 + 7 = x - \frac{1}{x} + \frac{8}{x^2}$

(c) $\frac{x^2 + 3x}{3x + 4} = \frac{2}{x}$ (d) $\frac{x^2 + 9}{11x^2 + 9} = \frac{1}{2x}$

8. Solve the following to three significant figures

(a) $2x^3 - 5x + 2 = 0$ (b) $x^3 - 6x^2 + 10x - 6 = 0$

(c) $x^3 - 2x^2 + 7x - 2 = 0$ (d) $-2x^4 + 6x^2 - 1 = 0$

9. Solve $2x^3 + kx^2 - 11x - 6 = 0$ given that one solution is $x = -3$.

10. Two solutions to the equation $2x^4 + ax^3 + x^2 + 6x + b = 0$ are $x = -3$ and $x = 2$. Find the other two solutions, if they exist.

11. Given that $mx^4 - 5x + n$ and $x^4 - 2x^3 - mx^2 - nx - 8$ are both divisible by $(x - 2)$, solve the equation $x^4 - 9x^3 - 3nx^2 + 4mx + 4mn = 0$.

12. Prove that if the roots of $x^3 - ax^2 + bx - c = 0$ are in arithmetic sequence then $2a^3 - 9ab + 27c = 0$. Hence, find $\{x : x^3 - 12x^2 + 39x - 28 = 0\}$.

3.5.2 POLYNOMIAL INEQUATIONS

As with quadratic inequalities (Chapter 2), we may analogously create polynomial inequations such as $P(x) > 0$, $P(x) \geq 0$, $P(x) < 0$ and $P(x) \leq 0$.

The solution of these inequations is relatively straight forward. After sketching the graph of the given polynomial, we note for which values of x the curve lies above, on or below the x–axis.

The other method is to make use of a **sign diagram**. We can use simple sign diagrams from factorised expressions of polynomials. These are somewhat less useful now because of the ease with which the graphics calculator displays all the relevant information. However, if the polynomial is already factorised then its use is appropriate.

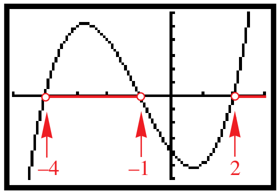
EXAMPLE 3.18

Find (a) $\{x : x^3 + 3x^2 - 6x - 8 > 0\}$.

(b) $\{x : (2x-1)(x+2)(x-1) \le 0\}$.

Solution

(a) Let $P(x) = x^3 + 3x^2 - 6x - 8$.
Sketching its graph using the TI–83 we have:



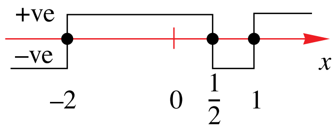
From the graph we see that the curve lies above the x–axis for values of x such that $-4 < x < -1$ and $x > 2$.

$\therefore \{x : x^3 + 3x^2 - 6x - 8 > 0\} = \{x : -4 < x < -1\} \cup \{x : x > 2\}$.

(b) As the polynomial is already factorised, we make use of a sign diagram to solve the inequality.

The zeros of the polynomial $P(x) = (2x-1)(x+2)(x-1)$ are $1, -2$ and $\frac{1}{2}$.

Based on the zeros we can construct our sign diagram:



Test point: Use $x = 0$, $P(0) = (-1)(2)(-1) = 2$.
i.e., at $x = 0$ $P(x)$ is positive.

Then, $\{x : (2x-1)(x+2)(x-1) \le 0\}$ is given by the values of x for which the sign diagram lies below or on the x–axis.

That is, $\{x : (2x-1)(x+2)(x-1) \le 0\} = \{x : x \le -2\} \cup \left\{x : \frac{1}{2} \le x \le 1\right\}$.

EXERCISES 3.5.2

1. Solve the following inequalities

(a)	$(x-1)(x+1)(x-2) > 0$	(b)	$(2x-4)(x+2)(x-3) \le 0$
(c)	$(2-x)(x+2)(x+3) \le 0$	(d)	$x(x-1)^2 > 0$
(e)	$(x+2)^2(2x+1) \ge 0$	(f)	$(x^2-4)(x+4) < 0$
(g)	$(x-2)(1+x)^2(1-x) < 0$	(h)	$2(x-3)^2(2-x) \ge 0$

2. Solve the following inequalities

(a)	$x^3 + 2x^2 - 5x - 6 > 0$	(b)	$2x^3 - 5x^2 + x + 2 \le 0$
(c)	$-x^3 + 7x + 6 \le 0$	(d)	$6x^3 - 31x^2 + 25x + 12 > 0$
(e)	$2x^3 + 3x^2 + 4x - 3 \ge 0$	(f)	$-x^3 + x^2 + 5x + 3 < 0$
(g)	$x^3 + 2x^2 + 4 < 7x$	(h)	$x^3 - 2x^2 - 14x \ge 12$
(i)	$2x^3 + 9x^2 + 8x + 2 \le 0$	(j)	$3x^3 - x^2 + 6 < 18x$
(k)	$4x^3 - 5x^2 + 3x \le 2$	(l)	$x^4 + 5x > 3x^3 + 3$

3.6 SKETCHING POLYNOMIALS

3.6.1 GRAPHICAL SIGNIFICANCE OF ROOTS

We have already been making use of the graphs of polynomial functions to help us during this chapter. We are now in a position where we can sketch the graphs of polynomial functions as well as give meaning to the geometrical relationship between the polynomial expression and its graph. In particular we are interested in the geometrical significance of the roots of a polynomial.

The relationship between the roots of a polynomial and its graph can be summarised as follows:

If the polynomial $P(x)$ is factorised into unique (single) factors, $(x-a)$, $(x-b)$, $(x-c)$, . . . so that

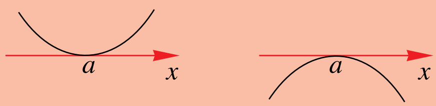
$$P(x) = (x-a)(x-b)(x-c)\ldots, \text{ where } a \neq b \neq c \neq \ldots,$$

the curve will **cut the *x*–axis** at each of the points $x = a, x = b, x = c, \ldots$.
That is, at each of these points the curve will look like one of

If the polynomial $P(x)$ is factorised and has a repeated (squared) factor , $(x-a)^2$, and unique factors $(x-b)$, $(x-c)$, . . . so that

$$P(x) = (x-a)^2(x-b)(x-c)\ldots, \text{ where } a \neq b \neq c \neq \ldots,$$

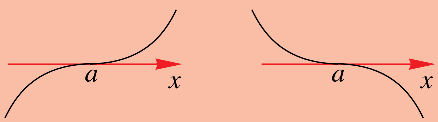
the curve will **touch the *x*–axis** at $x = a$ and cut the x–axis at each of the other points $x = b, x = c, \ldots$.
That is, at $x = a$ the curve will look like one of



If the polynomial $P(x)$ is factorised and has a repeated (cubed) factor , $(x-a)^3$, and unique factors $(x-b)$, $(x-c)$, . . . so that

$$P(x) = (x-a)^3(x-b)(x-c)\ldots, \text{ where } a \neq b \neq c \neq \ldots,$$

the curve will **cut the *x*–axis** at $x = a$ but with a change in concavity, i.e., there will be a **stationary point of inflection** at $x = a$ and it will cut the x–axis at each of the other points $x = b, x = c, \ldots$.
That is, at $x = a$ the curve will look like one of

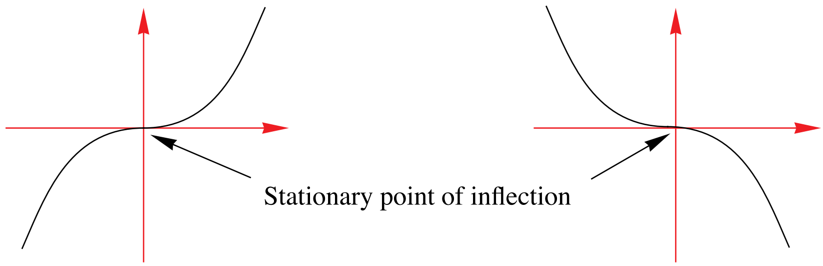


3.6.2 CUBIC FUNCTIONS

A cubic function has the general form $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0,\ a, b, c, d \in \mathbb{R}$.

We first consider the polynomial $f(x) = ax^3$.

For $a > 0$ we have: For $a < 0$ we have:

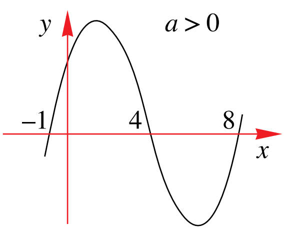


All other cubic polynomials with real coefficients can be factorised into one of the following forms:

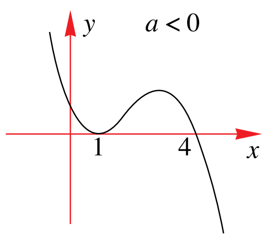
$P(x) = a(x-k)^3$	i.e., 3 identical real zeros, meaning three equal roots, and so, a stationary point of inflection at $x = k$.
$P(x) = a(x-k)^2(x-m)$	i.e., 2 identical real zeros and one other real zero, meaning two equal roots and a third different root, so that at $x = k$ there is a turning point on the x–axis.
$P(x) = a(x-k)(x-m)(x-n)$	i.e., 3 distinct real zeros, meaning three different roots, and so the curve will cut the x–axis at three different points on the x–axis.
$P(x) = a(x-k)(x^2+px+q)$	i.e., 1 real zero and an irreducible real quadratic, meaning that there is only one root and so the curve cuts the x–axis at only one point, $x = k$.

Some examples are shown below.

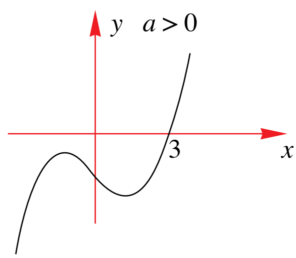
$P(x) = a(x-4)(x-8)(x+1)$



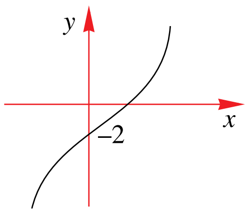
$P(x) = a(x-1)^2(x-4)$



$P(x) = a(x-3)(x^2+4x+5)$



$P(x) = x^3 + 4x - 2$



Notice that in this case we cannot factorise the cubic into rational linear factors. The x–intercept can be found using the TI–83 ($x = 0.4735$ to 4 d.p.)

The key to sketching polynomials is to first express them (where possible) in factored form. Once that is done we can use the results of §3.6.1. Of course, although we have only looked at the cubic function in detail, the results of §3.6.1 hold for polynomials of higher order than three.

EXAMPLE 3.19

Sketch the graph of
(a) $P(x) = (x-1)(x-3)(x+1)$.
(b) $P(x) = (x-1)^2(x+3)$.
(c) $P(x) = (2-x)^3$

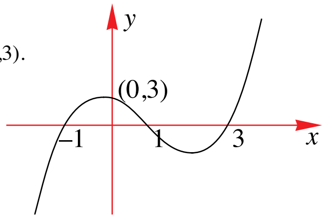
Solution

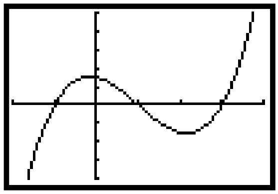
(a) The polynomial $P(x) = (x-1)(x-3)(x+1)$ is already in factored form. As the factors are unique, there are three distinct roots and so the curve cuts the x–axis at $x = 1$, $x = -1$ and $x = 3$.

As the leading coefficient is positive the graph has the basic shape: 

The y–intercept occurs when $x = 0$, i.e., $P(0) = (-1)(-3)(1) = 3$.

Note: Turning point does not occur at (0,3).

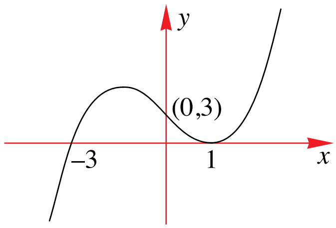


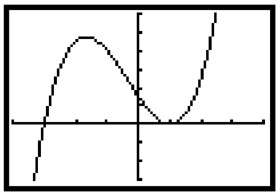


(b) The polynomial $P(x) = (x-1)^2(x+3)$ is in factored form and has a repeated factor $(x-1)^2$ and a unique factor $(x+3)$. That is, it has a **double root** at $x = 1$ and a single root at $x = -3$. This means the curve will have a **turning point** on the x–axis at $x = 1$ and will cut the x–axis at $x = -3$.

As the leading coefficient is positive the graph has the basic shape: 

The y–intercept occurs when $x = 0$, i.e., $P(0) = (-1)^2(3) = 3$.

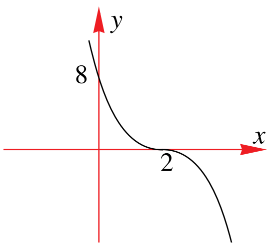


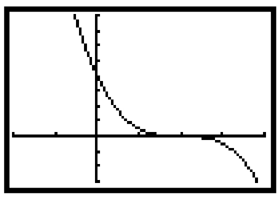


(c) The polynomial $P(x) = (2-x)^3$ is in factored form with repeated factor $(2-x)^3$. This means that there is a **treble root** at $x = 2$ and so, there is a **stationary point of inflection** on the x–axis at $x = 2$.

As the leading coefficient is negative the graph has the basic shape: 

The y–intercept occurs when $x = 0$, i.e., $P(0) = (2)^3 = 8$.





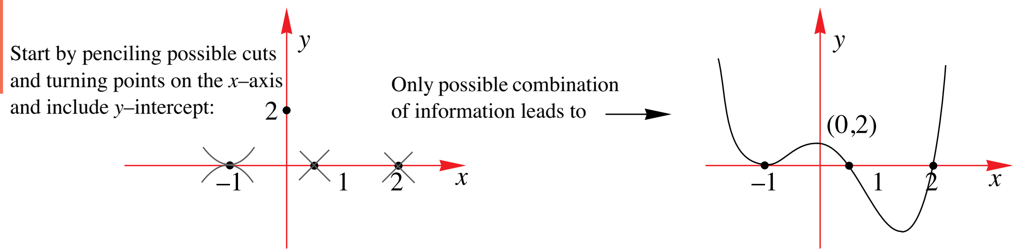
Note: We leave the general discussion of the cubic polynomial function $f(x) = a(x-k)^3 + h$ to Chapter 5, except to state that this curve would look exactly like $f(x) = ax^3$ but with its stationary point of inflection now located at (k, h).

EXAMPLE 3.20

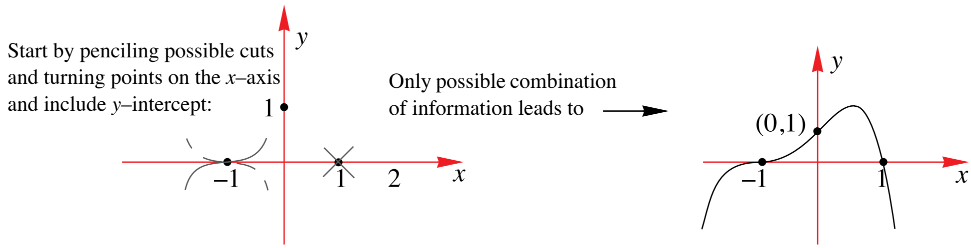
Sketch the graph of (a) $P(x) = (2x-1)(x+1)^2(x-2)$
(b) $P(x) = (1-x)(x+1)^3$
(c) $P(x) = x(x+2)^3(x-1)$.

Solution

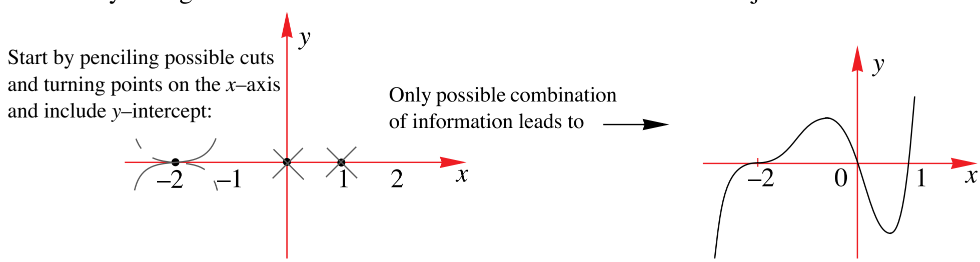
(a) We have single roots at $x = \frac{1}{2}$ and $x = 2$ and a **double root** at $x = -1$. This means that the curve will cut the x–axis at $x = 0.5$ and $x = 2$ but will have a **turning point** at $(-1, 0)$.
The y–intercept is given by $P(0) = (-1)(1)^2(-2) = 2$.
We start by filling in the information on a set of axes and then sort of 'join the dots':



(b) We have a single root at $x = 1$ and a **treble root** at $x = -1$. This means that the curve will cut the x–axis at $x = -1$ and will have a **stationary point of inflection** at $(-1, 0)$.
The y–intercept is given by $P(0) = (1)(1)^3 = 1$.
We start by filling in the information on a set of axes and then sort of 'join the dots':



(c) We have single roots at $x = 0$ and $x = 1$ and a **treble root** at $x = -2$. This means that the curve will cut the x–axis at $x = 0$ and $x = 1$ and will have a **stationary point of inflection** at $(-2, 0)$.
The y–intercept is given by $P(0) = (0)(2)^3(-1) = 0$.
We start by filling in the information on a set of axes and then sort of 'join the dots':



So far we have looked at sketching graphs of polynomials whose equations have been in factored form. So what happens when a polynomial function isn't in factored form? Well, in this case we first factorise the polynomial (if possible) and use the same process as we have used so far. We factorise the polynomial either by 'observation' or by making use of the factor theorem.

EXAMPLE 3.21

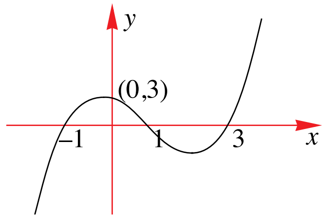
Sketch the graph of $f(x) = x^3 - 3x^2 - x + 3$.

Solution

By observation, we have:
$$x^3 - 3x^2 - x + 3 = x^2(x-3) - (x-3)$$
$$= (x^2 - 1)(x - 3)$$
$$= (x+1)(x-1)(x-3)$$

That is, $f(x) = (x+1)(x-1)(x-3)$.

This is in fact the same function as that in Example 3.19 (a) and so we have:

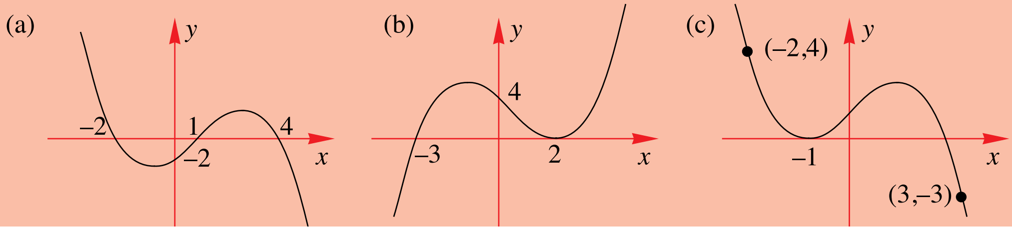


As always, we also have at our disposal the graphics calculator!

We now look at obtaining the equation of a polynomial from a given set of information. If a graph of a polynomial has sufficient information, then it is possible to determine the unique polynomial satisfying all the given information.

EXAMPLE 3.21

Determine the equation of the following cubic graphs



Solution

(a) In this instance we have the curve cutting the x–axis at three distinct points, $x = -2$, $x = 1$ and $x = 4$, meaning that the function will have three distinct factors, namely, $x + 2$, $x - 1$ and $x - 4$.

Therefore, we can write down the equation $f(x) = a(x+2)(x-1)(x-4)$, where a needs to be determined.

Using the point $(0, -2)$ we have $f(0) = a(0+2)(0-1)(0-4) = -2 \therefore 8a = -2 \Leftrightarrow a = -\frac{1}{4}$.

Therefore, $f(x) = -\frac{1}{4}(x+2)(x-1)(x-4)$.

(b) In this instance we have the curve cutting the x–axis at one point, $x = -3$, and having a turning point at $x = 2$. Meaning that the function will have a single linear factor, $x + 3$, and a repeated factor $(x-2)^2$.

Therefore, we can write down the equation $f(x) = a(x+3)(x-2)^2$, where a needs to be determined.

Using the point (0, 4) we have $f(0) = a(0+3)(0-2)^2 = 4 \therefore 12a = 4 \Leftrightarrow a = \frac{1}{3}$.

Therefore, $f(x) = \frac{1}{3}(x+3)(x-2)^2$.

(c) The only obvious information is that there is a turning point at $x = -1$ and so the polynomial will have a repeated factor $(x+1)^2$.

Therefore the polynomial will take on the form $f(x) = (ax+b)(x+1)^2$.

Then, to determine the values of a and b we use the coordinates (–2,4) and (3,–3).

At (–2,4): $4 = (-2a+b)(-2+1)^2 \Leftrightarrow 4 = -2a+b$ – (1)

At (3,–3): $-3 = (3a+b)(3+1)^2 \Leftrightarrow -3 = 48a+16b$ – (2)

Solving for a and b we have:

From (1) $b = 4+2a$ and substituting into (2) we have $-3 = 48a+16(4+2a) \therefore a = -\frac{67}{80}$

Substituting into $b = 4+2a$ we have $b = 4+2\left(-\frac{67}{80}\right) = \frac{93}{40}$.

Therefore, $f(x) = \left(-\frac{67}{80}x+\frac{93}{40}\right)(x+1)^2 = \frac{1}{80}(186-67x)(x+1)^2$.

EXERCISES 3.6

1. Sketch the graphs of the following polynomials

(a) $P(x) = x(x-2)(x+2)$

(b) $P(x) = (x-1)(x-3)(x+2)$

(c) $T(x) = (2x-1)(x-2)(x+1)$

(d) $P(x) = \left(\frac{x}{3}-1\right)(x+3)(x-1)$

(e) $P(x) = (x-2)(3-x)(3x+1)$

(f) $T(x) = (1-3x)(2-x)(2x+1)$

(g) $P(x) = -x^2(x-4)$

(h) $P(x) = (1-4x^2)(2x-1)$

(i) $T(x) = (x-1)(x-3)^2$

(j) $T(x) = \left(1-\frac{x}{2}\right)^2(x+2)$

(k) $P(x) = x^2(x+1)(2x-3)$

(l) $P(x) = 4x^2(x-2)^2$

(m) $P(x) = \frac{1}{2}(x-3)(x+1)(x-2)^2$

(n) $T(x) = -(x-2)(x+2)^3$

(o) $P(x) = (x^2-9)(3-x)^2$

(p) $T(x) = -2x(x-1)(x+3)(x+1)$

(q) $P(x) = x^4+2x^3-3x^2$

(r) $T(x) = \frac{1}{4}(4-x)(x+2)^3$

(s) $T(x) = -x^3(x^2-4)$

(t) $T(x) = (2x-1)\left(\frac{x}{2}-1\right)(x-1)(1-x)$

2. Sketch the graph of the following polynomials

(a) $P(x) = x^3 - 4x^2 - x + 4$ (b) $P(x) = x^3 - 6x^2 + 8x$

(c) $P(x) = 6x^3 + 19x^2 + x - 6$ (d) $P(x) = -x^3 + 12x + 16$

(e) $P(x) = x^4 - 5x^2 + 4$ (f) $P(x) = 3x^3 - 6x^2 + 6x - 12$

(g) $P(x) = -2x^4 + 3x^3 + 3x^2 - 2x$ (h) $P(x) = 2x^4 - 3x^3 - 9x^2 - x + 3$

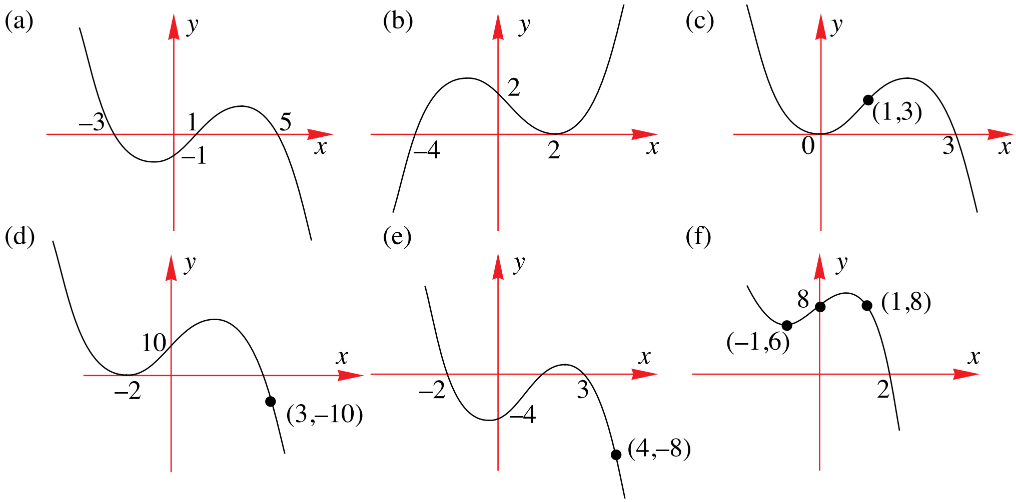
(i) $T(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$ (j) $T(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

3. Sketch the graph of

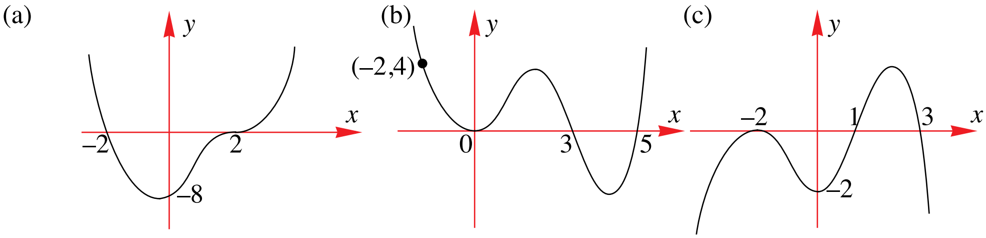
(a) $P(x) = x^3 - kx$ where i. $k = b^2$ ii. $k = -b^2$.

(b) $P(x) = x^3 - kx^2$ where i. $k = b^2$ ii. $k = -b^2$.

4. Determine the equations of the following cubic functions



5. Determine the equation of the following functions



6. Sketch a graph of $f(x) = (x-b)(ax^2 + bx + c)$ if $b > 0$ and

(a) $b^2 - 4ac = 0, a > 0, c > 0$

(b) $b^2 - 4ac > 0, a > 0, c > 0$

(c) $b^2 - 4ac < 0, a > 0, c > 0$

7. (a) On the same set of axes sketch the graphs of $f(x) = (x-a)^3$ and $g(x) = (x-a)^2$. Find $\{(x, y) : f(x) = g(x)\}$.

(b) Hence find $\{x : (x-a)^3 > (x-a)^2\}$.