Exercise 3E

Throughout this exercise, the domain of each function is the set of real numbers unless specifically stated otherwise.

1 Given $f(x) = 2x + 1$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$ evaluate each of the following.

a) f(3) **b)** g(2) **c)** hg(2) **d)** fg(−3) **e)** gf(1) **f)** gh(−2)
g) hf(4) **h)** ff(5) **i)** gg(−3) **j)** hh(12) **k)** fgh(2) **l)** hfg(4)

2 Given $f: x \longrightarrow 3x - 1$, $g: x \longrightarrow x^2$ and $h: x \longrightarrow \frac{2}{x}$, write down and simplify expressions for each of the following.

a) fg(x) **b)** gf(x) **c)** fh(x) **d)** hg(x) **e)** gg(x) **f)** ff(x)

3 Functions f and g are defined by

$$f: x \longrightarrow x^2 + 3 \qquad g: x \longrightarrow x + 5$$

a) Write down and simplify expressions for **i)** fg(x), **ii)** gf(x).
b) Hence solve the equation fg(x) = gf(x).

4 Functions h and k are defined by

$$h: x \longrightarrow \frac{3}{x} \qquad k: x \longrightarrow x + 5$$

a) Write down an expression for hk(x), and hence solve the equation hk(x) = 1.
b) Write down an expression for kh(x), and hence solve the equation kh(x) = 6.

5 Given $f(x) = x^2$ and $g(x) = 2x + 5$, solve the following equations.

a) fg(x) = 9 **b)** gg(x) = 21

6 Given

$$f(x) = x^2,\ x \in \mathbb{R},\ 1 \leqslant x \leqslant 5 \quad \text{and} \quad g(x) = 2x + 5,\ x \in \mathbb{R}$$

find an expression for the composite function gf(x). State the domain and range of gf(x).

7 Functions p and q are defined by

$$p: x \longrightarrow 3x^2 + 1,\ x \in \mathbb{R},\ 0 \leqslant x \leqslant 2 \qquad \text{and} \qquad q: x \longrightarrow x^2 - 2,\ x \in \mathbb{R}$$

Find the composite function qp(x) and state its range.

8 Given

$$f: x \longrightarrow x^2 + 4,\ x \in \mathbb{R} \quad \text{and} \quad g: x \longrightarrow \frac{1}{x - 3},\ x \in \mathbb{R},\ x \geqslant 4$$

find an expression for the composite function gf(x) and state its range.

9 Functions g and h are defined by

$$g: x \longrightarrow x^2 + 3,\ x \in \mathbb{R} \qquad \text{and} \qquad h: x \longrightarrow |x| - 5,\ x \in \mathbb{R}$$

a) Write down an expression for $hg(x)$ and state its range.
b) Write down an expression for $gg(x)$ and state its range.

10 Given

$$f(x) = \sqrt{x+1},\ x \in \mathbb{R},\ x > 0 \quad \text{and} \quad g(x) = x^2,\ x \in \mathbb{R}$$

a) find an expression for $fg(x)$ and state its range
b) find an expression for $gf(x)$ and state its range.

11 Functions h and k are defined by $h(x) = 3x + 5$, and $k(x) = 2 - x$.

a) Write down and simplify expressions for $hh(x)$ and $kk(x)$.
b) Hence solve the equation $hh(x) = kk(x)$.

12 Functions f and g are defined by

$$f: x \longrightarrow x + 1,\ x \in \mathbb{R} \qquad \text{and} \qquad g: x \longrightarrow x^2 - 3,\ x \in \mathbb{R}$$

a) Show that $fg(x) + gf(x) = 2x^2 + 2x - 4$.
b) Hence solve the equation $fg(x) + gf(x) = 0$.

13 Given $f(x) = x^2 + 3$, $g(x) = 2x + a$ and $fg(x) = 4x^2 - 8x + 7$, calculate the value of the constant a.

14 Functions p, q and r are defined by

$$p: x \longrightarrow \frac{3}{x+1} \qquad \text{and} \qquad q: x \longrightarrow \frac{b}{x^2} \qquad \text{and} \qquad r: x \longrightarrow \frac{3x^2}{2+x^2}$$

Given $pq(x) = r(x)$, find the value of the constant b.

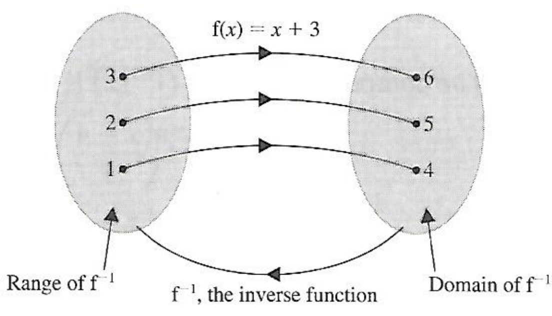
15 Functions f and g are defined for all real numbers and are such that $g(x) = x^2 + 7$, and $gf(x) = 9x^2 + 6x + 8$. Find possible expressions for $f(x)$.

16 **a)** Given $f(x) = ax + b$, and $f^{(3)}(x) = 64x + 21$, find the values of the constants a and b.
b) Suggest a rule for $f^{(n)}(x)$.
[Note: for $f^{(3)}(x)$ read $fff(x)$.]

Inverse functions

Consider the function f defined by $f(x) = x + 3$ with domain $\{1, 2, 3\}$. The range of f is $\{4, 5, 6\}$. We now want a function f^{-1}, called the **inverse function**, which has domain $\{4, 5, 6\}$ and range $\{1, 2, 3\}$ such that

$$f^{-1}(4) = 1 \qquad f^{-1}(5) = 2 \quad \text{and} \quad f^{-1}(6) = 3$$



Eliminating y gives

$$x = \frac{2x+1}{x+2}$$

$$\therefore \quad x(x+2) = 2x+1$$

$$\therefore \quad x^2 - 1 = 0$$

$$\therefore \quad (x-1)(x+1) = 0$$

Solving gives $x = 1$ or $x = -1$.

When $x = 1$, $y = 1$ and when $x = -1$, $y = -1$. Therefore, the coordinates of the points of intersection of the graphs of f and f^{-1} are $(1, 1)$ and $(-1, -1)$.

Two useful techniques for sketching the graph of an inverse function are as follows.

- Reflect the graph of the function f in the line $y = x$.
- Sketch the graph of $y = \mathrm{f}(x)$, turn the page over and then turn it through 90° clockwise. What you see through the page is the graph of the inverse function. (Note that (i) a reflection in the y-axis followed by a rotation through 90° clockwise is equivalent to a reflection in the line $y = x$, and (ii) a reflection in the x-axis followed by a rotation through 90° anticlockwise is equivalent to a reflection in the line $y = x$.)

Exercise 3F

Throughout this exercise, the domain of each function is the set of real numbers unless specifically stated otherwise.

1 Find the inverse of each of the following functions.

a) $\mathrm{f}: x \longrightarrow 3x + 2$
b) $\mathrm{f}: x \longrightarrow 5x - 1$
c) $\mathrm{f}: x \longrightarrow 4 - 3x$

d) $\mathrm{f}: x \longrightarrow \dfrac{2}{x}, \ x \neq 0$
e) $\mathrm{f}: x \longrightarrow \dfrac{3}{x-1}, \ x \neq 1$
f) $\mathrm{f}: x \longrightarrow \dfrac{5}{2-3x}, \ x \neq \frac{3}{2}$

g) $\mathrm{f}: x \longrightarrow \dfrac{x}{2+x}, \ x \neq -2$
h) $\mathrm{f}: x \longrightarrow \dfrac{2x}{5-x}, \ x \neq 5$
i) $\mathrm{f}: x \longrightarrow \dfrac{3x}{2x+1}, \ x \neq -\frac{1}{2}$

j) $\mathrm{f}: x \longrightarrow 1 + \dfrac{1}{x}, \ x \neq 0$
k) $\mathrm{f}: x \longrightarrow 3 + \dfrac{x}{1+x}, \ x \neq -1$
l) $\mathrm{f}: x \longrightarrow 2 - \dfrac{3}{4+x}, \ x \neq -4$

2 Find the inverse of each of the following functions, and state the domain on which each inverse is defined.

a) $\mathrm{f}(x) = x^2, \ x \in \mathbb{R}, \ x > 2$
b) $\mathrm{f}(x) = \dfrac{1}{2+x}, \ x \in \mathbb{R}, \ x > 0$

c) $\mathrm{f}(x) = \sqrt{x-2}, \ x \in \mathbb{R}, \ x > 3$
d) $\mathrm{f}(x) = 3x^2 - 1, \ x \in \mathbb{R}, \ 1 < x < 4$

e) $\mathrm{f}(x) = \sqrt{2x+3}, \ x \in \mathbb{R}, \ x \geqslant 11$
f) $\mathrm{f}(x) = \dfrac{1}{x} - 3, \ x \in \mathbb{R}, \ 2 < x < 5$

g) $\mathrm{f}(x) = (x+2)^2 + 3, \ x \in \mathbb{R}, \ x \geqslant -2$
h) $\mathrm{f}(x) = x^3 + 1, \ x \in \mathbb{R}$

i) $f(x) = \frac{1}{x-2}$, $x \in \mathbb{R}$, $x > 3$

j) $f(x) = \sqrt{3-x}$, $x \in \mathbb{R}$, $x \leqslant 2$

k) $f(x) = (x-3)^2 + 5$, $x \in \mathbb{R}$, $4 \leqslant x \leqslant 6$

l) $f(x) = 5 - \sqrt{x+3}$, $x \in \mathbb{R}$, $x \geqslant -3$

3 Given $f: x \longrightarrow 3x - 4$, $x \in \mathbb{R}$,

a) find an expression for the inverse function $f^{-1}(x)$
b) sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same set of axes
c) solve the equation $f(x) = f^{-1}(x)$.

4 a) Sketch the graph of the function defined by

$$f(x) = 10 - 2x, \quad x \in \mathbb{R}, \quad x \geqslant 0$$

b) Find an expression for the inverse function $f^{-1}(x)$, and sketch the graph of $f^{-1}(x)$ on the same set of axes.
c) Calculate the value of x for which $f(x) = f^{-1}(x)$.

5 A function is defined by $f(x) = x^2 - 6$, $x \in \mathbb{R}$, $x > 0$.

a) Find an expression for the inverse function $f^{-1}(x)$.
b) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same set of axes.
c) Calculate the value of x for which $f(x) = f^{-1}(x)$.

6 a) Sketch the graph of the function defined by

$$f: x \longrightarrow (x-2)^2 \quad x \in \mathbb{R}, \quad x \geqslant 2$$

b) Find an expression for the inverse function $f^{-1}(x)$, and sketch the graph of $f^{-1}(x)$ on the same set of axes.
c) Calculate the value of x for which $f(x) = f^{-1}(x)$.

7 The functions f and g are defined by

$$f: x \longrightarrow 2x - 5, \quad x \in \mathbb{R} \qquad \text{and} \qquad g: x \longrightarrow 7 - 4x, \quad x \in \mathbb{R}$$

a) Solve the equation $f(x) = g(x)$.
b) Write down expressions for $f^{-1}(x)$ and $g^{-1}(x)$.
c) Solve the equation $f^{-1}(x) = g^{-1}(x)$, and comment on your answer.

8 The function h with domain $\{x: x \geqslant 0\}$ is defined by $h(x) = \frac{4}{x+3}$.

a) Sketch the graph of h and state its range.
b) Find an expression for $h^{-1}(x)$.
c) Calculate the value of x for which $h(x) = h^{-1}(x)$.

9 Functions f and g are defined by

$$f: x \longrightarrow 3x + 1, \quad x \in \mathbb{R} \qquad \text{and} \qquad g: x \longrightarrow x - 2, \quad x \in \mathbb{R}$$

a) Write down and simplify an expression for the composite function $fg(x)$.
b) Find expressions for each of these inverse functions.
i) $f^{-1}(x)$ **ii)** $g^{-1}(x)$ **iii)** $(fg)^{-1}(x)$
c) Verify that $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$.

10 The function g is defined by $g(x) = 2x^2 - 3,\ x \in \mathbb{R},\ x \geqslant 0$.

a) State the range of g and sketch its graph.
b) Explain why the inverse function g^{-1} exists and sketch its graph.
c) Given also that h is defined by $h(x) = \sqrt{5x+2},\ x \in \mathbb{R},\ x \geqslant -\frac{2}{5}$, solve the inequality $gh(x) \geqslant x$.

11 Functions f and g are defined by

$$f: x \to 2x + 3,\ x \in \mathbb{R} \qquad \text{and} \qquad g: x \to \frac{1}{x-1},\ x \in \mathbb{R},\ x \neq 1$$

a) Find an expression for the inverse function $f^{-1}(x)$.
b) Find an expression for the composite function $gf(x)$.
c) Solve the equation $f^{-1}(x) = gf(x) - 1$.

12 Given $f(x) = \dfrac{1}{1-x},\ x \in \mathbb{R},\ x \neq 0,\ x \neq 1$

a) find expressions for **i)** $ff(x)$ **ii)** $fff(x)$ **iii)** $f^{(4)}(x)$
[Note: $f^{(4)}(x) = ffff(x)$]
b) Hence write down expressions for **i)** $f^{-1}(x)$ **ii)** $f^{(13)}(x)$ **iii)** $f^{(360)}(x)$

13 Functions g and h are defined by

$$g(x) = \frac{5}{x-3},\ x \in \mathbb{R},\ x \neq 3 \qquad \text{and} \qquad h(x) = x^2 + 4,\ x \in \mathbb{R},\ x > 0$$

Find

a) an expression for the inverse function $g^{-1}(x)$
b) an expression for the composite function $gh(x)$
c) the solutions to the equation $3g^{-1}(x) = 10gh(x) + 9$.

14 Functions f and g are defined by

$$f: x \to \frac{1}{x+3},\ x \in \mathbb{R},\ x \neq -3 \qquad \text{and} \qquad g: x \to \frac{2}{x-4},\ x \in \mathbb{R},\ x \neq 4$$

a) Show that $fg: x \to \dfrac{x-4}{3x-10},\ x \in \mathbb{R},\ x \neq \dfrac{10}{3}$.
b) Find an expression for $(fg)^{-1}(x)$.

15 Given $f: x \to \dfrac{a}{x} + b,\ x \in \mathbb{R},\ x \neq 0,\ x \neq b,\ x \neq -\dfrac{a}{b}$, and $ff(x) = f^{-1}(x)$, prove that the constants a and b satisfy the equation $a + b^2 = 0$.

16 The function f is defined by $f(x) = \dfrac{ax+b}{cx+d},\ x \in \mathbb{R},\ x \neq -\dfrac{d}{c},\ b \neq 0,\ c \neq 0$.

a) Prove that if $a + d = 0$, then $f(x) = f^{-1}(x)$.
b) Prove that if $a + d \neq 0$ and $(a-d)^2 + 4bc = 0$, then the graph of $y = f(x)$ intersects the graph of $y = f^{-1}(x)$ in exactly one point.

Exercise 3E

1 a) 7 **b)** 4 **c)** $\frac{1}{4}$ **d)** 19 **e)** 9 **f)** $\frac{1}{4}$ **g)** $\frac{1}{9}$ **h)** 23 **i)** 81 **j)** 12 **k)** $\frac{3}{2}$ **l)** $\frac{1}{33}$ **2 a)** $3x^2-1$ **b)** $(3x-1)^2$ **c)** $\frac{6}{x}-1$ **d)** $\frac{2}{x^2}$

2 e) x^4 **f)** $9x-4$ **3 a) i)** $x^2+10x+28$ **ii)** x^2+8 **b)** -2 **4 a)** $\frac{3}{x+5}, -2$ **b)** $\frac{3}{x}+5, 3$ **5 a)** $-4, -1$ **b)** $\frac{3}{2}$

6 $2x^2+5,\ 1 \leqslant x \leqslant 5,\ 7 \leqslant \mathrm{gf}(x) \leqslant 55$ **7** $9x^4+6x^2-1,\ -1 \leqslant \mathrm{qp}(x) \leqslant 167$ **8** $\frac{1}{x^2+1},\ 0 < \mathrm{gf}(x) \leqslant 1$

9 a) $x^2-2,\ \mathrm{hg}(x) \in \mathbb{R},\ \mathrm{hg}(x) \geqslant -2$ **b)** $x^4+6x^2+12,\ \mathrm{gg}(x) \in \mathbb{R},\ \mathrm{gg}(x) \geqslant 12$ **10 a)** $\sqrt{x^2+1},\ \mathrm{fg}(x) \in \mathbb{R},\ \mathrm{fg}(x) > 1$

10 b) $x+1,\ \mathrm{gf}(x) \in \mathbb{R},\ \mathrm{gf}(x) > 1$ **11 a)** $9x+20,\ x$ **b)** $-\frac{5}{2}$ **12** $-2, 1$ **13** -2 **14** 2 **15** $\pm(3x+1)$ **16 a)** $a=4,\ b=1$

16 b) $4^n x + \frac{4^n-1}{3}$

Exercise 3F

1 a) $\frac{x-2}{3}$ **b)** $\frac{1+x}{5}$ **c)** $\frac{4-x}{3}$ **d)** $\frac{2}{x}, x \neq 0$ **e)** $\frac{3+x}{x}, x \neq 0$ **f)** $\frac{2x-5}{3x}, x \neq 0$ **g)** $\frac{2x}{1-x}, x \neq 1$ **h)** $\frac{5x}{2+x}, x \neq -2$

1 i) $\frac{x}{3-2x}, x \neq \frac{3}{2}$ **j)** $\frac{1}{x-1}, x \neq 1$ **k)** $\frac{3-x}{x-4}, x \neq 4$ **l)** $\frac{4x-5}{2-x}, x \neq 2$ **2 a)** $\sqrt{x},\ x \in \mathbb{R},\ x > 4$ **b)** $\frac{1-2x}{x},\ x \in \mathbb{R},\ 0 < x < \frac{1}{2}$

2 c) $2+x^2,\ x \in \mathbb{R},\ x > 1$ **d)** $\sqrt{\frac{x+1}{3}},\ x \in \mathbb{R},\ 2 < x < 47$ **e)** $\frac{x^2-3}{2},\ x \in \mathbb{R},\ x \geqslant 5$ **f)** $\frac{1}{x+3},\ x \in \mathbb{R},\ -\frac{14}{5} < x < -\frac{5}{2}$

2 g) $-2+\sqrt{x-3},\ x \in \mathbb{R},\ x \geqslant 3$ **h)** $\sqrt[3]{x-1},\ x \in \mathbb{R}$ **i)** $\frac{2x+1}{x},\ x \in \mathbb{R},\ 0 < x < 1$ **j)** $3-x^2,\ x \in \mathbb{R},\ x \geqslant 1$

2 k) $3+\sqrt{x-5},\ x \in \mathbb{R},\ 6 \leqslant x \leqslant 14$ **l)** $(x-5)^2-3,\ x \in \mathbb{R},\ x \leqslant 5$ **3 a)** $\frac{x+4}{3},\ x \in \mathbb{R}$ **c)** 2 **4 b)** $\frac{10-x}{2},\ x \in \mathbb{R},\ x \leqslant 10$ **c)** $\frac{10}{3}$

5 a) $\sqrt{x+6},\ x \in \mathbb{R},\ x > -6$ **c)** 3 **6 b)** $2+\sqrt{x},\ x \in \mathbb{R},\ x > 0$ **c)** 4 **7 a)** 2 **b)** $\frac{x+5}{2},\ x \in \mathbb{R};\ \frac{7-x}{4},\ x \in \mathbb{R}$

7 c) -1, from **a)** $\mathrm{f}(2) = \mathrm{g}(2) = -1$ **8 a)** $0 < \mathrm{h}(x) \leqslant \frac{4}{3}$ **b)** $\mathrm{h}^{-1}(x) = \frac{4}{x} - 3,\ x \in \mathbb{R},\ x \neq 0$ **c)** 1 **9 a)** $3x-5$

9 b) i) $\frac{x-1}{3},\ x \in \mathbb{R}$ **ii)** $x+2,\ x \in \mathbb{R}$ **iii)** $\frac{x+5}{3},\ x \in \mathbb{R}$ **10 a)** $\mathrm{g}(x) \geqslant -3$ **b)** g^{-1} exists because g is one-to-one **c)** $x \geqslant -\frac{1}{9}$

11 a) $\frac{x-3}{2},\ x \in \mathbb{R}$ **b)** $\frac{1}{2(x+1)},\ x \in \mathbb{R},\ x \neq -1$ **c)** $\pm\sqrt{2}$ **12 a) i)** $1-\frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$ **ii)** $x,\ x \in \mathbb{R}$ **iii)** $\frac{1}{1-x},\ x \in \mathbb{R},\ x \neq 1$

12 b) i) $1-\frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$ **ii)** $\frac{1}{1-x},\ x \in \mathbb{R},\ x \neq 1$ **iii)** $x,\ x \in \mathbb{R}$ **13 a)** $3+\frac{5}{x},\ x \in \mathbb{R},\ x \neq 0$ **b)** $\frac{5}{x^2+1},\ x \in \mathbb{R},\ x > 0$ **c)** $\frac{1}{3}, 3$

14 b) $\frac{10x-4}{3x-1},\ x \in \mathbb{R},\ x \neq \frac{1}{3}$