Proving trigonometric identities

Consider the following identity:

$$\tan\theta + \cot\theta \equiv \sec\theta \operatorname{cosec}\theta$$

Substituting different values of θ into the LHS and RHS will show this identity to be true for those particular values of θ. However, this does not **prove** the identity for all values of θ. Identities can be proved by using other simpler identities which we know to be true for all values of θ. For example, we know that the following identities are true for all values of θ.

- $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta \equiv 1$
- $1 + \tan^2\theta \equiv \sec^2\theta$
- $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$

The general method for proving an identity is to choose either the LHS **or** the RHS (usually whichever is the more complicated) and show, by using known identities, that it can be manipulated into the form of the other. However, two alternative techniques are to show that

- $\text{LHS} - \text{RHS} \equiv 0$ or
- $\dfrac{\text{LHS}}{\text{RHS}} \equiv 1.$

Once the proof is completed, we write QED, which stands for the Latin 'Quod Erat Demonstrandum' – 'Which was to be proved'.

Example 10 Prove the identity $\tan\theta + \cot\theta \equiv \sec\theta \operatorname{cosec}\theta$.

SOLUTION

$$\begin{aligned}\text{LHS} &\equiv \tan\theta + \cot\theta \\ &\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ &\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\ &\equiv \frac{1}{\sin\theta\cos\theta} \quad (\text{since } \sin^2\theta + \cos^2\theta \equiv 1) \\ &\equiv \frac{1}{\sin\theta} \times \frac{1}{\cos\theta} \\ &\equiv \operatorname{cosec}\theta \sec\theta \equiv \text{RHS} \qquad \text{QED}\end{aligned}$$

Example 11 Prove the identity

$$(1 - \sin\theta + \cos\theta)^2 \equiv 2(1 - \sin\theta)(1 + \cos\theta)$$

SOLUTION

$$\begin{aligned}\text{LHS} &\equiv (1 - \sin\theta + \cos\theta)(1 - \sin\theta + \cos\theta) \\ &\equiv 1 - 2\sin\theta + 2\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta \\ &\equiv 2 - 2\sin\theta + 2\cos\theta - 2\sin\theta\cos\theta \quad (\text{since } \sin^2\theta + \cos^2\theta \equiv 1) \\ &\equiv 2(1 - \sin\theta + \cos\theta - \sin\theta\cos\theta) \\ &\equiv 2(1 - \sin\theta)(1 + \cos\theta) \equiv \text{RHS} \qquad \text{QED}\end{aligned}$$

Example 12 Prove the identity

$$\frac{1 + \sin\theta}{1 - \sin\theta} \equiv (\tan\theta + \sec\theta)^2$$

SOLUTION

$$\begin{aligned}\text{RHS} &\equiv \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 \\ &\equiv \left(\frac{\sin\theta + 1}{\cos\theta}\right)^2 \\ &\equiv \frac{(1 + \sin\theta)^2}{\cos^2\theta} \\ &\equiv \frac{(1 + \sin\theta)^2}{1 - \sin^2\theta} \\ &\equiv \frac{(1 + \sin\theta)^2}{(1 + \sin\theta)(1 - \sin\theta)} \\ &\equiv \frac{1 + \sin\theta}{1 - \sin\theta} \equiv \text{LHS} \qquad \text{QED}\end{aligned}$$

Exercise 14C

Prove each of the following identities.

1 $\sin\theta\tan\theta + \cos\theta \equiv \sec\theta$

2 $\operatorname{cosec}\theta + \tan\theta\sec\theta \equiv \operatorname{cosec}\theta\sec^2\theta$

3 $\operatorname{cosec}\theta - \sin\theta \equiv \cot\theta\cos\theta$

4 $(\sin\theta + \cos\theta)^2 - 1 \equiv 2\sin\theta\cos\theta$

5 $(\sin\theta - \operatorname{cosec}\theta)^2 \equiv \sin^2\theta + \cot^2\theta - 1$

6 $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) \equiv 1$

7 $\tan^2\theta + \sin^2\theta \equiv (\sec\theta + \cos\theta)(\sec\theta - \cos\theta)$

8 $\sec^2\theta + \cot^2\theta \equiv \operatorname{cosec}^2\theta + \tan^2\theta$

9 $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) \equiv \sin^3\theta + \cos^3\theta$

10 $\tan^4\theta + \tan^2\theta \equiv \sec^4\theta - \sec^2\theta$

11 $\cos^4\theta - \sin^4\theta \equiv \cos^2\theta - \sin^2\theta$

12 $\sin\theta + \cos\theta \equiv \dfrac{1 - 2\cos^2\theta}{\sin\theta - \cos\theta}$

13 $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} \equiv 2\operatorname{cosec}\theta$

14 $\dfrac{\operatorname{cosec}\theta}{\cot\theta + \tan\theta} \equiv \cos\theta$

15 $\dfrac{1}{1 + \tan^2\theta} + \dfrac{1}{1 + \cot^2\theta} \equiv 1$

16 $\dfrac{1 - \sin\theta}{\cos\theta} \equiv \dfrac{1}{\sec\theta + \tan\theta}$

17 $\dfrac{\tan\theta + \cot\theta}{\sec\theta + \operatorname{cosec}\theta} \equiv \dfrac{1}{\sin\theta + \cos\theta}$

18 $\sec^4\theta - \operatorname{cosec}^4\theta \equiv \dfrac{\sin^2\theta - \cos^2\theta}{\sin^4\theta\cos^4\theta}$

19 $\sqrt{(\sec^2\theta - 1)} + \sqrt{(\operatorname{cosec}^2\theta - 1)} \equiv \dfrac{1}{\sin\theta\cos\theta}$

20 $\dfrac{\sin\theta}{\sqrt{(1 + \cot^2\theta)}} + \dfrac{\cos\theta}{\sqrt{(1 + \tan^2\theta)}} \equiv 1$

21 $\sqrt{\dfrac{1 - \sin\theta}{1 + \sin\theta}} \equiv \sec\theta - \tan\theta$

22 $\dfrac{1 + \sin\theta + \cos\theta}{\cos\theta} \equiv \dfrac{1 - \sin\theta + \cos\theta}{1 - \sin\theta}$

***23** $\sqrt{\dfrac{\tan\theta + \sin\theta}{\cot\theta - \cos\theta}} \equiv \tan^2\theta\sqrt{\dfrac{1 + \sin\theta}{1 - \cos\theta}}$

***24** $\dfrac{\tan^3\theta}{1 + \tan^2\theta} + \dfrac{\cot^3\theta}{1 + \cot^2\theta} \equiv \dfrac{1 - 2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$

***25** $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} \equiv 1 + \sin\theta\cos\theta$

***26** $\dfrac{\cot^2\theta(\sec\theta - 1)}{1 + \sin\theta} \equiv \dfrac{\sec^2\theta(1 - \sin\theta)}{1 + \sec\theta}$