# Harmonic form

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The range of $f(x) = 3\cos x + 4\sin x$ is **not** $[-7, 7]$.

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The range of $f(x) = 3\cos x + 4\sin x$ is **not** $[-7, 7]$. The reason the above argument is wrong is that $\cos x$ and $\sin x$ are maximal/minimal for different values of $x$ (there is no $x$ for which $\cos x = 1$ and $\sin x = 1$ simultaneously).

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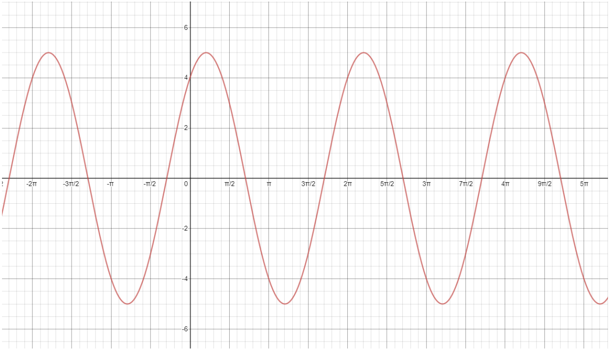

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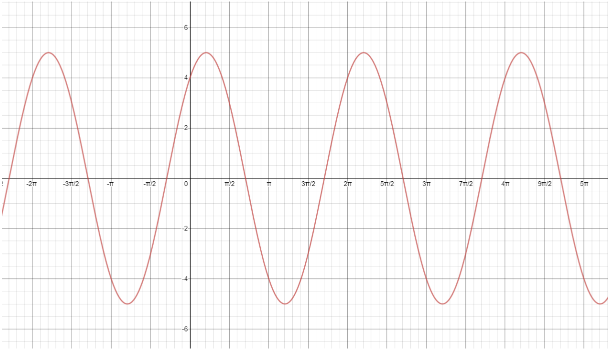


We can actually see what the range is from the graph, but that won't always be possible. What's more important is that the graph is a trigonometric function. So we should be able to write $f(x)$ as a single trigonometric function.

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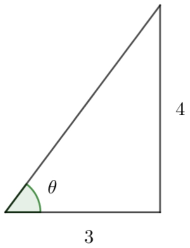

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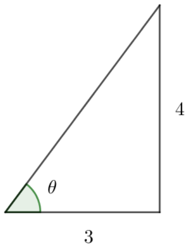


The hypotenuse is then 5 and $\theta = \arctan\left(\frac{4}{3}\right)$.

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where $\theta = \arctan\left(\frac{4}{3}\right)$.
Now we use the compound angle formula to get:

$$5(\cos\theta\cos x + \sin\theta\sin x) = 5\cos(x - \theta)$$

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$\theta$ corresponds to a horizontal shift, so it doesn't influence the range. The amplitude is 5, so the range of $f$ is $[-5, 5]$.

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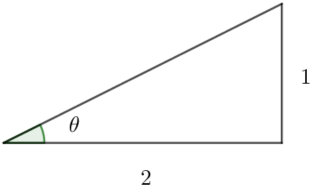
We will try to write $f(x)$ in the form $R\sin(x-\theta)$. So we want to change the 2 into cos and 1 into sin. We can draw a triangle with adjacent side 2 and the opposite side 1.



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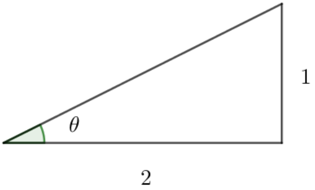


The hypotenuse is $\sqrt{5}$ and $\theta = \arctan\left(\frac{1}{2}\right)$.

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We can now write:

$$\begin{aligned}2\sin x - \cos x =& \sqrt{5}\left(\frac{2}{\sqrt{5}}\sin x - \frac{1}{\sqrt{5}}\cos x\right) = \\ =& \sqrt{5}\left(\cos\theta\sin x - \sin\theta\cos x\right) \\ =& \sqrt{5}\sin(x-\theta)\end{aligned}$$

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where $\theta = \arctan\left(\frac{1}{2}\right)$.

So $f(x) = \sqrt{5}\sin(x-\theta)$, which means that the range of $f(x)$ is $[-\sqrt{5}, \sqrt{5}]$.

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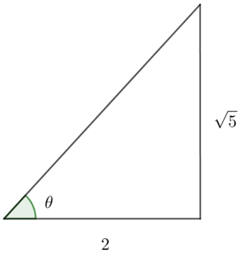
We will try to write $f(x)$ in the form $R\sin(x+\theta)$. So we want to change the 2 into cos and $\sqrt{5}$ into sin. We will draw a triangle with adjacent side 2 and opposite side $\sqrt{5}$:




The hypotenuse is 3 and $\theta = \arctan\left(\frac{\sqrt{5}}{2}\right)$.

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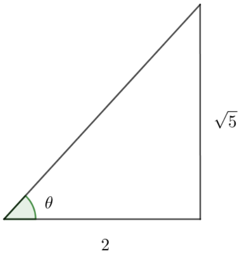


The hypotenuse is 3 and $\theta = \arctan\left(\frac{\sqrt{5}}{2}\right)$.

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We get:

$$
\begin{aligned}
2\sin x + \sqrt{5}\cos x =& 3\left(\frac{2}{3}\sin x + \frac{\sqrt{5}}{3}\cos x\right) = \\
=& 3\left(\cos\theta \sin x + \sin\theta \cos x\right) \\
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We get:

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where $\theta = \arctan\left(\dfrac{\sqrt{5}}{2}\right)$.

So $f(x) = 3\sin(x+\theta)$, which means that the range of $f(x)$ is $[-3, 3]$.

I anticipate many questions, make sure you try to understand the above examples, but we will go back and expand this on Tuesday.