Homework

Advanced exercise

(a) Show that, if $\tan^2\phi = 2\tan\phi + 1$, then $\tan 2\phi = -1$.

(b) Find all solutions of the equation

$$\tan\theta = 2 + \tan 3\theta$$

which satisfy $0 < \theta < 2\pi$, expressing your answers as rational multiples of π.

(c) Find all solutions of the equation the equation

$$\cot\theta = 2 + \cot 3\theta$$

which satisfy

$$-\frac{3\pi}{2} < \theta < \frac{\pi}{2}.$$

Before you proceed you should give this question a go.

This is an advanced question (good prep for paper 3 and for students whose aim is to get 7 on the finals).

You're probably not yet familiar with questions of this level of difficulty, so don't feel discouraged. A good time limit for this questions is 30 minutes. Even if you look at the solutions, try to do the steps yourselves.

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The second step is the formula for $\tan 2\phi$ and the third step is the use of the assumption that $\tan^2\phi = 2\tan\theta + 1$.

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We will start by deriving the formula for $\tan 3\theta$:

$$\begin{aligned}
\tan 3\theta &= \tan(\theta + 2\theta) = \\
&= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta} = \\
&= \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\frac{2\tan\theta}{1-\tan^2\theta}} = \\
&= \frac{\tan\theta(1 - \tan^2\theta) + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}
\end{aligned}$$

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$$t^3 - 3t^2 + t + 1 = 0$$

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and one obvious solution to this polynomial is $t = 1$.

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The first equation has solutions $\theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$.

Part (b)

The second equation reduces to $\tan 2\theta = -1$ by part (a) and this has solutions:

$$\theta = \frac{3\pi}{8} \quad \text{or} \quad \theta = \frac{7\pi}{8} \quad \text{or} \quad \theta = \frac{11\pi}{8} \quad \text{or} \quad \theta = \frac{15\pi}{8}$$

So finally the solutions to

$$\tan\theta = 2 + \tan 3\theta$$

are

$$\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{5\pi}{4}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$$

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for θ satisfying $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$.

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for θ satisfying $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$.

Everything is aligned perfectly if we just set $\theta = \frac{\pi}{2} - \phi$. We get the equation:

$$\cot\left(\frac{\pi}{2} - \phi\right) = 2 + \cot\left(3\left(\frac{\pi}{2} - \phi\right)\right)$$

with $0 < \phi < 2\pi$.

Part (c)

Now we of course have $\cot\left(\frac{\pi}{2} - \phi\right) = \tan\phi$ (after all this is the reason we've used the substitution $\theta = \frac{\pi}{2} - \phi$).

Now let's look at $\cot\left(3\left(\frac{\pi}{2} - \phi\right)\right)$:

$$\cot\left(3\left(\frac{\pi}{2} - \phi\right)\right) = \cot\left(\frac{\pi}{2} - (3\phi + \pi)\right) = \tan(3\phi + \pi) = \tan 3\phi$$

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with $0 < \phi < 2\pi$.

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So we end up having to solve:

$$\tan\phi = 2 + \tan 3\phi$$

with $0 < \phi < 2\pi$. We already did this in part (b), so we know that:

$$\phi \in \left\{\frac{\pi}{4}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{5\pi}{4}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$$

Part (c)

Remember that we set $\theta = \frac{\pi}{2} - \phi$, so of course $\phi = \frac{\pi}{2} - \theta$, which gives:

$$\frac{\pi}{2} - \theta \in \left\{ \frac{\pi}{4}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{5\pi}{4}, \frac{11\pi}{8}, \frac{15\pi}{8} \right\}$$

and we get that:

$$\theta \in \left\{ -\frac{11\pi}{8}, -\frac{7\pi}{8}, -\frac{3\pi}{4}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{\pi}{4} \right\}$$

and these are the solutions to the equation

$$\cot \theta = 2 + \cot 3\theta$$

for θ satisfying $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$.

In case of any questions you can email me at T.J.Lechowski@gmail.com.